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CHEBYSHEV CENTERS AND APPROXIMATION IN PRE-HILBERT C^*-MODULES

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ABSTRACT. We extend the study of Chebyshev centers in pre-Hilbert C^*-modules by considering the C^*-algebra valued map defined by $|x| = \langle x, x\rangle^{1/2}$. We prove that if T is a remotal subset of a pre-Hilbert C^*-module M, and $F \subseteq M$ is star-shaped at a relative Chebyshev center c of T with respect to F, then $|x - q_T(x)|^2 \geq |x - c|^2 + |c - q_T(c)|^2 (x \in F)$. The uniqueness of Chebyshev center follows from this inequality. This is a generalization of a well-known result on Hilbert spaces.

1. Introduction

A normed algebra is an algebra A with a norm $\|.\|$ such that $\|xy\| \leq \|x\|\|y\|, \quad x, y \in A$. A complete normed algebra A is called a Banach algebra. An involution $*$ on an algebra A is a mapping $x \longrightarrow x^*$ from A onto A such that $(\lambda x + y)^* = \bar{\lambda}x^* + y^*, (xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in A, \lambda \in \mathbb{C}$. An involutive Banach algebra is called a Banach $*$-algebra. A Banach $*$-algebra A is said to be a C^*-algebra if $\|xx^*\| = \|x\|^2$. An element x in a C^*-algebra A with unit e is called positive if $\mathrm{sp}(x) \subseteq [0, \infty)$, where $\mathrm{sp}(x) = \{\lambda \in \mathbb{C}; \lambda e - x \text{ is not invertible}\}$; we write $x \geq 0$ if x is a positive element.

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Suppose that A is a C^*-algebra and E is a linear space, which is a right A-module and the scalar multiplication satisfies $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in E, a \in A, \lambda \in \mathbb{C}$. The space E is called a pre-Hilbert A-module if there exists an A-valued map $\langle .,. \rangle : E \to A$ with the following properties:
(i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
(ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle, \quad x, y, z \in E, \lambda \in \mathbb{C}$.
(iii) $\langle x, ya \rangle = \langle x, y \rangle a, \qquad x, y \in E$ and $a \in A$.
(iv) $\langle x, y \rangle^* = \langle y, x \rangle, \qquad x, y \in E$.
Such a map $\langle .,. \rangle : E \to A$ is called an A-valued inner product. E is called a (right) Hilbert A-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We note that Hilbert C^*-modules contain both Hilbert spaces and C^*-algebras. In fact, every Hilbert space is a Hilbert $\mathbb{C}$-module and if A is a C^*-algebra, then A is a Hilbert A-module, whenever we define $\langle a, b \rangle = a^*b, \quad a, b \in A$.
We define an A-valued map by $|x| = \langle x, x \rangle^{1/2}$. This is not actually an extension of a norm, in general, since it may happen that the triangle inequality does not hold [7].
The importance of our approach to the theory of approximation in pre-Hilbert C^*-modules is that we do not use the triangle inequality. This may motivate us to study the geometry in case the triangle inequality does not hold.
Hilbert C^*-modules were first introduced and investigated by I. Kaplansky [5], M. Rieffel [13] and W. Paschke [11]. They played an essential role in operator algebras [12], KK-Theory [3], operator spaces [2], quantum group theory [14], Morita equivalence [13] and so on. They are a generalization of Hilbert spaces, but there are some differences between the two classes. For example, each operator on a Hilbert space has an adjoint, but a bounded A-module map on a Hilbert A-module is not adjointable, in general, ([7], page 8). Throughout this paper, we assume that $(M, \langle .,. \rangle)$ is a pre-Hilbert C^*-module over a commutative C^*-algebra A. In particular, the commutative C^*-algebras which are boundedly complete lattices with respect to their natural order structures, i.e., those having the property that each set of functions that has an upper bound has a least upper bound, are of special interest. An easy example is the complex field $\mathbb{C}$. One however shows that if a commutative C^*-algebra $C(X)$ is a boundedly complete lattice with respect to the natural partial ordering of its real-linear subspace $C(X, \mathbb{R})$ of continuous real-valued functions on X, then X is extremely disconnected,

i.e., its open sets have open closures [4].
Let T be a non-empty subset of M. The mapping $Q_T : M \to 2^T$ defined by $Q_T(x) = \{y \in T : |x-y| = \max\{|x-t| : t \in T\}\}$ is called the farthest point map of T. We call T a remotal (uniquely remotal) set, if for each $x \in M$ the set $Q_T(x)$ is non-empty (is a singleton). The element of $Q_T(x)$ is denoted by $q_{_T}(x)$ if it is a singleton. A subset F of M is said to be star-shaped at a vertex $s \in F$ if and only if for each $x \in F$ the line segment $[s, x] = \{\lambda s + (1 - \lambda)x : 0 \le \lambda \le 1\}$ lies in F.
A relative Chebyshev center of $T \subseteq M$ in $F \subseteq M$ is an element c in M that satisfies $|c - q_{_T}(c)| = \min\{|x - q_{_T}(x)| : x \in F\} := r_{_F}(T)$, if the minimum exists. In the case that $F = M$, we call c the Chebyshev center of T and denote $r_{_F}(T)$ by $r(T)$. We represent by $d(T)$, the A-valued diameter $\max\{|t - s| : t, s \in T\}$ of T, if it exists.
One outstanding open problem in the geometry of normed spaces is the Farthest Point Problem [9]. This problem asks whether every uniquely remotal set in a normed space is a singleton. There are some cases such as the finite dimensional spaces and the Banach spaces c_0 and c, in which the problem is solved affirmatively [1]. The problem is related to the problem of proving the convexity of Chebyshev sets in a Hilbert space [6](recall that a subset T of a normed space X is called Chebyshev, if for every $x \in X$ there exists a unique best approximation of x in T). The reader is referred to [7],[8] and [12] for details on Hilbert C^*-modules, on commutative C^*-algebras.

2. **Main results**

Let $(M, \langle .,. \rangle)$ be a pre-Hilbert C^*-module over a commutative C^*-algebra A. We now establish some interesting results similar to those in [10] about Hilbert C^*-modules. We start our work with an applicable example of a remotal set.

Example 1. Let $X = \{a, b\}, A = C(X)$ and $E = \{f \in C(X) : f(a) = 0\}$. Then, E is a maximal ideal of the C^*-algebra A and so can be regarded as a Hilbert A-module. Assume that $T = \{f_1, f_2\} \subseteq E$, where $f_1(b) = 1$ and $f_2(b) = 2$. Then, T is remotal, since for each $f \in E$ there exists a function $q_{_T}(f) \in T$ such that $|f(b) - q_{_T}(f)(b)| = \max\{|f(b) - 1|, |f(b) - 2|\}$. In fact, a straightforward verification shows that for each $f \in E$, if $Ref(b) > \frac{3}{2}$, then $q_{_T}(f) = f_1$; if $Ref(b) = \frac{3}{2}$, then $q_{_T}(f)$ can be chosen to be f_1 or f_2; and if $Ref(b) < \frac{3}{2}$, then $q_T(f) = f_2$ and also $d(T) = |f_1(b) - f_2(b)| = 1$.

Lemma 2.1. *Suppose T is a uniquely remotal subset of M and F is a star-shaped subset of M at a vertex c such that c is a relative center of T with respect to F. Then, 0 is a relative center of $c-T$ with respect to $c-F$.*

Proof. We first prove the identity $c - q_T(x) = q_{c-T}(c-x)$, for all $x \in F$. We know

$$|c - x - q_{c-T}(c-x)| \geq |c - x - (c - q_T(x))|.$$

Since $c - q_{c-T}(c-x) \in T$, $|x - q_T(x)| \geq |x - (c - q_{c-T}(c-x))|$. Hence,

$$|c - x - q_{c-T}(c-x)| = |c - x - (c - q_T(x))|.$$

Therefore, $q_{c-T}(c-x) = c - q_T(x)$, since T is a uniquely remotal set. We now show that $|0 - q_{c-T}(0)| \leq |c_1 - q_{c-T}(c_1)|$, for all $c_1 \in c - F$. We know that $|c - q_T(c)| \leq |x_1 - q_T(x_1)|$ for all $x_1 \in F$. So,

$$|q_{c-T}(0)| = |c - (c - q_{c-T}(0))| \leq |c - x_1 - (c - q_T(x_1))|.$$

It follows therefore that $|0 - q_{c-T}(0)| \leq |c - x_1 - q_{c-T}(c - x_1)|$, and so $|0 - q_{c-T}(0)| \leq |c_1 - q_{c-T}(c_1)|$, for all $c_1 = c - x_1 \in c - F$. □

Theorem 2.2. *Suppose T is a uniquely remotal subset of M and F is a star-shaped subset of M at a vertex c such that c is also a relative center of T with respect to F. Then,*
(i) $Re(\langle c - x, c - q_T(x)\rangle) \leq 0$, for all $x \in F$.
(ii) if $q_T(c) \in F$ is a cluster point of $\bigcup\{Q_T(x) : x \in [c, q_T(c)]\}$, then $T = \{c\}$.

Proof. (i) By lemma 2.2, we may assume, without loss of generality, that $c = 0$. Let $0 < \alpha < 1$. By the definition of the farthest point map q_T, we have

$$|x - q_T(x)|^2 \geq |x - q_T(\alpha x)|^2, |\alpha x - q_T(\alpha x)|^2 \geq |\alpha x - q_T(x)|^2.$$

Therefore,

$$\langle x - q_T(x), x - q_T(x)\rangle \geq \langle x - q_T(\alpha x), x - q_T(\alpha x)\rangle,$$

$$\langle \alpha x - q_T(\alpha x), \alpha x - q_T(\alpha x)\rangle \geq \langle \alpha x - q_T(x), \alpha x - q_T(x)\rangle.$$

By adding both sides of these inequalities, we obtain

$$(1-\alpha)[\langle x, q_{_T}(\alpha x)\rangle + \langle q_{_T}(\alpha x), x\rangle] \geq (1-\alpha)[\langle x, q_{_T}(x)\rangle + \langle q_{_T}(x), x\rangle].$$

Hence,

$$Re(\langle x, q_{_T}(\alpha x)\rangle) \geq Re(\langle x, q_{_T}(x)\rangle). \tag{2.1}$$

On the other hand, $|\alpha x - q_{_T}(\alpha x)| \geq |0 - q_{_T}(0)| \geq |q_{_T}(\alpha x) - 0|$, for all $x \in F$, since 0 is the relative Chebyshev center with respect to F. Hence,

$$\begin{aligned}\langle \alpha x - q_{_T}(\alpha x), \alpha x - q_{_T}(\alpha x)\rangle = |\alpha x - q_{_T}(\alpha x)|^2 &\geq |q_{_T}(\alpha x)|^2 \\ &= \langle q_{_T}(\alpha x), q_{_T}(\alpha x)\rangle.\end{aligned}$$

We have

$$\langle \alpha x, \alpha x\rangle - \langle q_{_T}(\alpha x), \alpha x\rangle - \langle \alpha x, q_{_T}(\alpha x)\rangle + \langle q_{_T}(\alpha x), q_{_T}(\alpha x)\rangle \geq \langle q_{_T}(\alpha x), q_{_T}(\alpha x)\rangle.$$

Therefore,

$$\alpha^2|x|^2 - \langle q_{_T}(\alpha x), \alpha x\rangle - \langle \alpha x, q_{_T}(\alpha x)\rangle \geq 0.$$

Dividing by α, we have

$$\alpha|x|^2 \geq \langle q_{_T}(\alpha x), x\rangle + \langle x, q_{_T}(\alpha x)\rangle.$$

Then,

$$\alpha|x|^2 \geq 2Re(\langle x, q_{_T}(\alpha x)\rangle). \tag{2.2}$$

We have from (2.1) and (2.2) that

$$\alpha|x|^2 \geq 2Re(\langle x, q_{_T}(x)\rangle). \tag{2.3}$$

Since (2.3) holds for each $\alpha(0 < \alpha < 1)$ and by the Gelfand representation of A, we get

$$Re(\langle x, q_{_T}(x)\rangle) \leq 0.$$

(ii) Suppose that there exists a sequence $\{\lambda_n\}$ in $[0,1]$ such that $y_n = q_{_T}(x_n) \to q_{_T}(c)$, where $x_n = \lambda_n c + (1-\lambda_n)q_{_T}(c)$. It follows from (i) that

$$Re(\langle c - x_n, c - q_{_T}(x_n)\rangle) \leq 0.$$

But, $\langle c - x_n, c - y_n\rangle = \langle c - (\lambda_n c + (1-\lambda_n)q_{_T}(c)), c - y_n\rangle = (1-\lambda_n)\langle c - q_{_T}(c), c - y_n\rangle$. Since $1 - \lambda_n \geq 0$, we infer that $Re(\langle c - q_{_T}(c), c - y_n\rangle) \leq 0$. Due to $c - y_n \to c - q_{_T}(c)$ and the continuity of the inner product, we conclude that $Re(\langle c - q_{_T}(c), c - q_{_T}(c)\rangle) \leq 0$. Hence, $|c - q_{_T}(c)|^2 = Re(\langle c - q_{_T}(c), c - q_{_T}(c)\rangle) = 0$. Thus, $|c - q_{_T}(c)| = \max\{|c - t| : t \in T\} = 0$. It follows that $c - t = 0$, for all $t \in T$. So, $T = \{c\}$. □

Theorem 2.3. *Suppose T is a remotal subset of M, $d(T)$ exists, $F \subseteq M$ and c is a relative center of T with respect to F. Then, the followings hold:*
(i) $|x - q_{_T}(x)|^2 \geq |x - c|^2 + {r_{_F}}^2(T), \quad$ for all $x \in F$.
(ii) c is unique and if $F \cap Q_T(c) \neq \phi$, then $d(T) \geq \sqrt{2} r_{_F}(T)$.
(iii) If T is uniquely remotal and $Re(\langle c - x_0, c - q_{_T}(x_0)\rangle) = 0$, for some $x_0 \in F$, then $q_{_T}(x_0) = q_{_T}(c)$, and therefore, if $q_{_T}(c) \in F$, then T is a singleton if and only if $Re(\langle c - q_{_T}(c), c - q_{_T}(q_{_T}(c))\rangle) = 0$.

Proof. (i) By lemma 2.2, we can assume that $c = 0$. By Theorem 2.3(i), we have
$Re(\langle x, q_{_T}(x)\rangle) \leq 0$, for all $x \in F$. Since F is star-shaped,
$\langle \alpha x, q_{_T}(\alpha x)\rangle + \langle q_{_T}(\alpha x), \alpha x\rangle \leq 0$, for all $x \in F, 0 \leq \alpha \leq 1$.
It follows that $Re(\langle x, q_{_T}(\alpha x)\rangle) \leq 0$. We thus obtain:

$$\begin{aligned}
{r_{_F}}^2(T) &\leq |\alpha x - q_{_T}(\alpha x)|^2 \\
&= \langle \alpha x - q_{_T}(\alpha x), \alpha x - q_{_T}(\alpha x)\rangle \\
&= \langle \alpha x - x + x - q_{_T}(\alpha x), \alpha x - x + x - q_{_T}(\alpha x)\rangle \\
&= (\alpha - 1)^2\langle x, x\rangle + \langle(\alpha - 1)x, x - q_{_T}(\alpha x)\rangle \\
&\quad + \langle x - q_{_T}(\alpha x), (\alpha - 1)x\rangle + \langle x - q_{_T}(\alpha x), x - q_{_T}(\alpha x)\rangle \\
&= (\alpha - 1)^2|x|^2 + 2(\alpha - 1)\langle x, x\rangle + (1 - \alpha)[\langle x, q_{_T}(\alpha x)\rangle \\
&\quad + \langle q_{_T}(\alpha x), x\rangle] + |x - q_{_T}(\alpha x)|^2 \\
&\leq (\alpha^2 - 1)|x|^2 + |x - q_{_T}(\alpha x)|^2 \\
&\leq (\alpha^2 - 1)|x|^2 + |x - q_{_T}(x)|^2.
\end{aligned}$$

Therefore, we have $|x - q_{_T}(x)|^2 \geq (1 - \alpha^2)|x|^2 + {r_{_F}}^2(T)$, for all $\alpha \in [0, 1]$.
Therefore, $|x - q_{_T}(x)|^2 \geq |x|^2 + {r_{_F}}^2(T)$.
(ii) If c' is another Chebyshev center with respect to F, then by (i),

$$|c - q_{_T}(c)|^2 = |c' - q_{_T}(c')|^2 \geq |c' - c|^2 + {r_{_F}}^2(T).$$

Hence, $|c' - c| = 0$. So, $c' = c$. This proves the uniqueness assertion.
Let $x = q_{_T}(c) \in F \cap Q_{_T}(c)$. We have $|q_{_T}(c) - q_{_T}(q_{_T}(c))|^2 \geq |q_{_T}(c) - c|^2 + {r_{_F}}^2(T)$, and so $|q_{_T}(c) - q_{_T}(q_{_T}(c))|^2 \geq 2{r_{_F}}^2(T)$. Hence, $d(T)^2 \geq |q_{_T}(c) - q_{_T}(q_{_T}(c))|^2 \geq 2{r_{_F}}^2(T)$.
(iii) By (i) with $x = x_0$, we have $|c - q_{_T}(c)|^2 + |x_0 - c|^2 \leq |x_0 - q_{_T}(x_0)|^2$.
Hence,

$$|c - q_{_T}(c)|^2 \leq |x_0 - q_{_T}(x_0)|^2 - |x_0 - c|^2. \tag{2.4}$$

But,

$$\langle c - x_0 - (c - q_T(x_0)), c - x_0 - (c - q_T(x_0))\rangle = |q_T(x_0) - x_0|^2.$$

Therefore,

$$\langle c - x_0, c - x_0\rangle + \langle c - q_T(x_0), c - q_T(x_0)\rangle + \langle c - x_0, c - q_T(x_0)\rangle$$
$$+\langle c - q_T(x_0), c - x_0\rangle = |x_0 - q_T(x_0)|^2.$$

Using our assumption on x_0, we obtain:

$$\langle c - x_0, c - x_0\rangle + \langle c - q_T(x_0), c - q_T(x_0)\rangle = |x_0 - q_T(x_0)|^2.$$

Therefore, $|c-q_T(x_0)|^2+|x_0-c|^2 = |x_0-q_T(x_0)|^2$. It follows from (2.4) that

$$|c - q_T(c)|^2 \leq |x_0 - q_T(x_0)|^2 - |x_0 - c|^2 = |c - q_T(x_0)|^2 \leq |c - q_T(c)|^2.$$

Hence, $|c - q_T(c)| = |c - q_T(x_0)|$. Due to the fact that T is uniquely remotal, $q_T(c) = q_T(x_0)$.
If $q_T(c) \in F$ and $\langle c-q_T(c), c-q_T(q_T(c))\rangle + \langle c-q_T(q_T(c)), c-q_T(c)\rangle = 0$, then by the first part of (iii) with $x_0 = q_T(c)$, we have $q_T(c) = q_T(q_T(c))$. Hence, $T = \{x_0\}$. Conversely, if T is a singleton set, then $q_T(c) = q_T(q_T(c))$ and $|c - q_T(c)| \leq |q_T(c) - q_T(q_T(c))| = 0$. So
$c - q_T(c) = 0$, i.e., $T = \{c\}$. Therefore, $c = q_T(q_T(c))$ and we conclude that $\langle c - q_T(c), c - q_T(q_T(c))\rangle = 0$.

Corollary 2.4. *Let T be a uniquely remotal subset M such that $d(T)$ exists, and let c be a Chebyshev center of T. Then, the following assertions are satisfied:*
(i) $|x - q_T(x)|^2 \geq |x - c|^2 + r^2(T)$.
(ii) If T is not a singleton, then $d(T) \geq \sqrt{2}r(T)$.

Proof. (i) This part follows immediately from assertion (i) of Theorem 2.4 with $F = M$.
(ii) We know that $d(T)^2 \geq |q_T(c) - q_T(q_T(c))|^2$. We infer therefore that $|q_T(c)-q_T(q_T(c))|^2 \geq 2{r_F}^2(T) \geq 2r^2(T)$, by part (i) of Theorem 2.4. □

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