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EXTENSIONS OF BAER AND QUASI-BAER MODULES

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ABSTRACT. We study the relationships between the Baer, quasi-Baer and p.q.-Baer property of an R-module M and the polynomial extensions of module M. As a consequence of our results, we obtain some results of [C.Y. Hong, N.K. Kim and T.K. Kwak, *J. Pure Appl. Algebra* **151** (2000) 215-226.] and [E. Hashemi and A. Moussavi, *Acta Math. Hungar.* **107** (2005) 207-224.].

1. Introduction

Throughout the paper, R will always denote an associative ring with identity and M_R will stand for a right R-module. Recall from [15] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [15], Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. The class of Baer rings includes the von Neumann algebras. In [9], Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with identity over an algebraically closed field is isomorphic to

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a twisted matrix units semigroup algebra. Every prime ring is a quasi-Baer ring. Another generalization of Baer rings is the p.p.-rings. A ring R is called *right (resp. left) p.p.* if right *(resp. left)* annihilator of an element of R is generated by an idempotent. Birkenmeier, et al. in [6] introduced the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal of R is generated by an idempotent.

In 1974, Armendariz considered the behavior of a polynomial ring over a Baer ring by obtaining the following result: Let R be a *reduced* ring (i.e., R has no nonzero nilpotent elements). Then, $R[x]$ is a Baer ring if and only if R is a Baer ring ([4], Theorem B). Armendariz provided an example to show that the reduced condition is not superfluous. In [6], Birkenmeier, et al. showed that the quasi-Baer condition is preserved by many polynomial extensions. Also, Birkenmeier, et al. [6] showed that a ring R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer.

From now on, we always denote the Ore extension ring (or Ore polynomial ring) by $S := R[x;\alpha,\delta]$, where $\alpha : R \to R$ is an endomorphism and $\delta : R \to R$ is an α-derivation. Recall that an α-derivation δ is an additive operator on R with the property that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. The Ore extension S is then the ring consisting of all (left) polynomials of the form $\sum a_i x^i$ ($a_i \in R$), which are multiplied using the distributive law and the Ore commutation rule $xa = \alpha(a)x + \delta(a)$, for all $a \in R$. From this rule, an inductive argument can be made to calculate an expression for $x^j a$, for all $j \in \mathbb{N}$ and $a \in R$.

Notation [19]. Let δ be an α-derivation of R. For integers $j \geq i \geq 0$, write f_i^j for the sum of all "words" in α and δ in which there are i factors of α and $j - i$ factors of δ. For instance, $f_j^j = \alpha^j$, $f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$.

Using recursive formulas for the f_i^j and induction, as in [19], one can show with a routine computation that

$$x^j a = \sum_{i=0}^{j} f_i^j(a) x^i. \tag{1.1}$$

This formula uniquely determines a general product of (left) polynomials in S and will be used freely in what follows.

Given a right R-module M_R, we can make $M[x]$ into a right S-module by allowing polynomials from S to act on polynomials in $M[x]$ in the obvious way, and applying the above "twist" whenever necessary. The verification that this defines a valid S-module structure on $M[x]$ is almost identical to the verification that S is a ring, and it is straightforward.

For a nonempty subset X of M, put $ann_R(X) = \{a \in R \mid Xa = 0\}$. In [21], Lee and Zhou introduced the notions of Baer, quasi-Baer and p.p.-modules as follows: (1) M_R is called *Baer* if for any subset X of M, $ann_R(X) = eR$, where $e^2 = e \in R$. (2) M_R is called *quasi-Baer* if, for any submodule $X \subseteq M$, $ann_R(X) = eR$, where $e^2 = e \in R$. (3) M_R is called *p.p.* if for any element $m \in M$, $ann_R(m) = eR$, where $e^2 = e \in R$. Clearly, a ring R is Baer (resp. p.p. or quasi-Baer) if and only if R_R is Baer (resp. p.p. or quasi-Baer) module. If R is a Baer (resp. p.p. or quasi-Baer) ring, then for any right ideal I of R, I_R is Baer (resp. p.p. or quasi-Baer) module.

The module M_R is called *principally quasi-Baer* (or simply p.q.-Baer) if for any $m \in M$, $ann_R(mR) = eR$, where $e^2 = e \in R$. It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

Here, we impose (α, δ)-compatibility assumption on the module M_R and prove the following results, extending many results on rings to modules:

(1) The module M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer), where $S = R[x; \alpha, \delta]$.

(2) If M_R is (α, δ)-Armendariz, then M_R is Baer (resp. p.p.) if and only if $M[x]_S$ is Baer (resp. p.p.).

Also, we give examples to show that (α, δ)-compatibility assumption on M_R in the preceding results is not superfluous. Among applications, we obtain some results of [12] and [10] as corollaries of our results.

2. Polynomials over Baer and Quasi-Baer Modules

Definition 2.1. (*Annin* [3]) *Given a module M_R, an endomorphism $\alpha : R \to R$, and an α-derivation $\delta : R \to R$, we say that M_R is α-compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, we say that M_R is δ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Rightarrow m\delta(r) = 0$. If M_R is both α-compatible and δ-compatible, we say that M_R is (α, δ)-compatible.*

Recall that an R-module N_R is called *prime* if $N \neq 0$ and $ann_R(N) = ann_R(N')$, for every nonzero submodule $N' \subseteq N$.

The following example shows that there exists an (α, δ)-compatible module M_R such that M_R and $M[x]_{R[x;\alpha,\delta]}$ are quasi-Baer.

Example 2.2. [3, Example 4.6] *Let R_0 be a domain of characteristic zero, and $R := R_0[t]$. Define $\alpha|_{R_0} = Id$ and $\alpha(t) = -t$. Now, for $a \in R_0$, set*

$$\delta(at^l) := \begin{cases} at^{l-1} & \text{if l is odd} \\ 0 & \text{if l is even.} \end{cases}$$

It is shown in [19] *that δ is an α-derivation on R. Let $M_R := R_0 \oplus R_0 \oplus R_0 \oplus \cdots$, where $t \in R$ acts on M_R as follows: for $(m_0, m_1, m_2, \cdots) \in M$, we set $(m_0, m_1, m_2, \cdots)t := (0, m_0k_0, m_1k_1, m_2k_2, \cdots)$, where the k_i ($i \in \mathbb{N}$) are fixed nonzero integers. We show that M_R is (α, δ)-compatible. For this, it suffices to show that $ann_R(m) = 0$, whenever $0 \neq m \in M$. Suppose that $(a_0, a_1, a_2, \cdots)(b_rt^r + b_{r+1}t^{r+1} +$ "higher terms"$) = 0$, where $a_i, b_i \in R_0$, for every $i \in \mathbb{N}$ and $b_r \neq 0$. First, applying t^r to $(a_0, a_1, a_2, \cdots)$ gives:*

$$(0, 0, \cdots, 0, a_0k_0k_1 \cdots k_{r-1}, a_1k_1k_2 \cdots k_r, \cdots)(b_r + b_{r+1}t + \text{“higher terms”}) = 0.$$

Upon computing this expression, we deduce that $a_0k_0k_1 \cdots k_{r-1}b_r = 0$. Since the characteristic is zero, R is a domain, and $k_0k_1 \cdots k_{r-1}b_r \neq 0$, we deduce that $a_0 = 0$. Now, we may proceed inductively to show that $a_i = 0$, for all i. From this calculation, we deduce at once that M_R is (α, δ)-compatible. Moreover, the calculation implies that M_R is prime, and $ann_R(N) = \{0\}$, for each nonzero submodule N of M. Therefore, M_R is quasi-Baer. Hence, $M[x]_{R[x;\alpha,\delta]}$ is quasi-Baer, by Theorem 2.11.

Remark 2.3. (*a*) *If M_R is α-compatible (resp. δ-compatible), then so is any submodule of M_R.*
(*b*) *If M_R is α-compatible (resp. δ-compatible), then M_R is α^i-compatible (resp. δ^i-compatible), for all $i \geq 1$.*

Lemma 2.4. *Let M_R be an (α, δ)-compatible R-module. Let $m \in M$, and $a, b \in R$. Then, we have the followings:*

(1) *If $ma = 0$, then $m\alpha^i(\delta^j(a)) = 0 = m\delta^j(\alpha^i(a))$, for any positive integers i, j.*
(2) *If $mab = 0$, then $m\alpha^i(a)\delta^j(b) = 0 = m\delta^j(a)\alpha^i(b)$, for any positive integers i, j.*
(3) *$ann_R(ma) = ann_R(m\alpha(a)) \subseteq ann_R(m\delta(a))$.*

Proof. (1) It follows from Remark 2.3.

(2) It is enough to show that $m\alpha(a)\delta(b) = 0 = m\delta(a)\alpha(a)$. Since M_R is δ-compatible, $mab = 0$ implies that $ma\delta(b) = 0$ and $m\delta(ab) = m\delta(a)b + m\alpha(a)\delta(b) = 0$. Since M_R is α-compatible, $mab = 0$ implies that $m\alpha(ab) = m\alpha(a)\alpha(b) = 0$, and so $m\alpha(a)b = 0$. Thus, $m\alpha(a)\delta(b) = 0$. Hence, $m\delta(a)b = 0$ and $m\delta(a)\alpha(a) = 0$.

(3) Observe that the α-compatibility of M_R yields $m\alpha(a)b = 0 \Leftrightarrow m\alpha(a)\alpha(b) = 0 \Leftrightarrow m\alpha(ab) = 0 \Leftrightarrow mab = 0$, for each $b \in R$. It is remains only to show that $ann_R(ma) \subseteq ann_R(m\delta(a))$. Let $mab = 0$, for some $b \in R$. Using δ-compatibility, we get $0 = m\delta(ab) = m\alpha(a)\delta(b) + m\delta(a)b = 0$ and hence $m\delta(a)b = 0$, as desired. □

Lemma 2.5. *Let M_R be an (α, δ)-compatible module, $m(x) = m_0 + \cdots + m_k x^k \in M[x]$ and $r \in R$. If $m(x)r = 0$, then $m_i r = 0$, for each i.*

Proof. An easy calculation using Eq. (1.1) shows that $0 = m(x)r = \sum_{i=0}^{k} \sum_{j=i}^{k} m_j f_i^j(r) x^i$ and so

$$\sum_{j=i}^{k} m_j f_i^j(r) = 0 \text{ for each } i \leq k. \tag{2.1}$$

Starting with $i = k$, Eq. (2.1) yields $m_k \alpha^k(r) = 0$, and so α-compatibility of M_R yields $m_k r = 0$. Now, assume inductively that $m_j r = 0$, for each $j > i$. By (α, δ)-compatibility of M_R, for $j > i$ we have $m_j f_i^j(r) = 0$. Using Eq. (2.1) again, we deduce that $m_i \alpha^i(r) = 0$, and so $m_i r = 0$ as needed. □

Following Anderson and Camillo [1], a module M_R is called *Armendariz* if whenever $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, we have $m_i a_j = 0$, for all i, j.

Definition 2.6. *Given a module M_R, an endomorphism $\alpha : R \to R$, and an α-derivation $\delta : R \to R$, we say M_R is (α, δ)*-quasi Armendariz (*resp. (α, δ)-Armendariz*), *if whenever $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ satisfy $m(x)R[x; \alpha, \delta]f(x) = 0$ (resp. $m(x)f(x) = 0$), we have $m_i x^i R b_j x^j = 0$ (resp. $m_i x^i a_j x^j = 0$), for all i, j.*

For a module M_R, put

$$\mathrm{Ann}_R(\mathrm{sub}(M)) = \{\mathrm{ann}_R(N) \mid N \text{ is a submodule of } M\}.$$

Clearly, $A = \mathrm{ann}_R(N)$ is an ideal of R for each submodule N of M.

Proposition 2.7. *Let M_R be an (α, δ)-compatible module and S be the skew polynomials ring $R[x; \alpha, \delta]$. Then, the following statements are equivalent:*

(1) *M_R is (α, δ)-quasi Armendariz.*
(2) *$\psi : Ann_R(sub(M)) \to Ann_S(sub(M[x])); A \to AS$ is bijective.*

Proof. (2) $\Rightarrow$ (1). Let $m(x) = m_0 + m_1x + ... + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + ... + b_mx^m \in S$ satisfy $m(x)Sf(x) = 0$. Then, $f(x) \in ann_S(m(x)S) = AS$, where A is an ideal of R. Hence, $b_0, \cdots, b_m \in A$, and so $m(x)Rb_j = 0$, for $j = 0, \cdots, m$. By lemmas 2.4 and 2.5, $m_ix^iRb_jx^j = 0$, for all i, j. Therefore, M_R is (α, δ)-quasi Armendariz.

(1) $\Rightarrow$ (2). Let $A \in Ann_R(sub(M))$. Then, there exists a submodule N of M such that $A = ann_R(N)$, and hence $ann_S(N[x]) = AS$, by Lemmas 2.4 and 2.5. Thus, ψ is a well defined map. Assume that $B \in Ann_S(sub(M[x]))$. Then, there exists a submodule N of $M[x]$ such that $B = ann_S(N)$. Let B_1 denote the set of all coefficients of elements of B in R and N_1 denote the set of all coefficients of elements of N in M. We claim that $ann_R(N_1R) = B_1R$. Let $m(x) = m_0 + m_1x + ... + m_kx^k \in N$ and $f(x) = b_0 + b_1x + ... + b_mx^m \in B$. Then, $m(x)Sg(x) = 0$. Since M_R is (α, δ)-quasi Armendariz and (α, δ)-compatible, $m_iRb_j = 0$, for all i, j. Thus, $(N_1R)(B_1R) = 0$, and so $B_1R \subseteq ann_R(N_1R)$. Since M_R is (α, δ)-compatible, $ann_R(N_1R) \subseteq B_1R$. Thus, $ann_R(N_1R) = B_1R$, and so $ann_S(N) = (B_1R)S$. □

Following Tominaga [25], an ideal I of R is said to be *left s-unital* if for each $a \in I$ there is an $x \in I$ such that $xa = a$. If an ideal I of R is left s-unital, then, for any finite subset F of I, there exists an element $e \in I$ such that $ex = x$, for each $x \in F$. A submodule N of a right R-module M is called a *pure submodule* if $N \otimes_R L \longrightarrow M \otimes_R L$ is a monomorphism for every left R-module L. By [25, Proposition 11.3.13], an ideal I is left s-unital if and only if R/I is flat as a right R-module if and only if I is pure as a right ideal of R.

Proposition 2.8. *Let M_R be an (α, δ)-compatible module and $S = R[x; \alpha, \delta]$. Then, the followings are equivalent:*

(1) *$ann_R(mR)$ is left s-unital for any element $m \in M$.*
(2) *$ann_S(m(x)S)$ is left s-unital for any element $m(x) \in M[x]$. In this case, M_R is (α, δ)-quasi Armendariz.*

Proof. (1) $\Rightarrow$ (2). First, we prove that M_R is (α,δ)-quasi Armendariz. Suppose that $(m_0+m_1x+...+m_kx^k)S(b_0+b_1x+...+b_nx^n)=0$, with $m_i\in M$ and $b_j\in R$. Then,

$$(m_0+m_1x+...+m_kx^k)R(b_0+b_1x+...+b_nx^n)=0. \tag{2.2}$$

Since M_R is α-compatible, $m_kRb_n=0$. Then, $b_n\in ann_R(m_kR)$, and so $m_kx^kRb_nx^n=0$, by Lemma 2.4. Since $ann_R(m_kR)$ is left s-unital, there exists $e_k\in ann_R(m_kR)$ such that $e_kb_n=b_n$. Replacing R by Re_k in Eq. (2.2), and using Lemma 2.4, we obtain $(m_0+m_1x+...+m_{k-1}x^{k-1})Re_k(b_0+b_1x+...+b_nx^n)=0$. Hence, $m_{k-1}Rb_n=0$, since M_R is α-compatible. Then, $b_n\in ann_R(m_{k-1}R)$, and so $m_{k-1}x^{k-1}Rb_nx^n=0$, by Lemma 2.4. Hence, $b_n\in ann_R(m_kR)\cap ann_R(m_{k-1}R)$. Since $ann_R(m_{k-1}R)$ is left s-unital, there exists $f\in ann_R(m_{k-1}R)$ such that $fb_n=b_n$. If we put $e_{k-1}=e_mf$, then $e_{k-1}b_n=b_n$ and $e_{k-1}\in ann_R(m_kR)\cap ann_R(m_{k-1}R)$. Next, replacing R by Re_{k-1} in Eq. (2.2), and using Lemma 2.4, we obtain $(m_0+m_1x+...+m_{k-2}x^{k-2})Re_{k-1}(b_0+b_1x+...+b_nx^n)=0$. Hence, we have $b_n\in ann_R(m_{k-2}R)$, and so $m_{k-2}x^{k-2}Rb_nx^n=0$, by Lemma 2.4. Continuing this process, we get $m_ix^iRb_nx^n=0$, for $i=0,\cdots,k$. Using induction on $k+n$, we obtain $m_ix^iRb_jx^j=0$, for all i,j. Therefore, M_R is (α,δ)-quasi Armendariz. Let $m(x)=m_0+m_1x+...+m_kx^k\in M[x]$ and $f(x)=b_0+b_1x+...+b_mx^m\in ann_S(m(x)S)$. Then, $m_iRb_j=0$, for all i,j. Since $ann_R(m_iR)$ is left s-unital, there exists $e_i\in ann_R(m_iR)$ such that $b_j=e_ib_j$, for $j=0,1,\cdots,m$. Put $e=e_0e_1\cdots e_k$. Then, $b_j=eb_j$, for $j=0,1,\cdots,m$, and so $ef(x)=f(x)$. Clearly, $e\in ann_S(m(x)S)$. Therefore, $ann_S(m(x)S)$ is left s-unital.

(2) $\Rightarrow$ (1). Let $m\in M$. By using Lemma 2.4, $ann_R(mR)\subseteq ann_S(mS)$. Hence, for any $b\in ann_R(mR)$, there exists a polynomial $f(x)\in S$ such that $f(x)b=b$. Let a_0 be the constant term of $f(x)$. Then, $a_0b=b$, by (α,δ)-compatibility of M_R. Clearly, $a_0\in ann_R(mR)$. Therefore, $ann_R(mR)$ is left s-unital. □

By Proposition 2.8, if $ann_R(mR)$ is left s-unital for any element $m\in M$, then M_R is α-quasi Armendariz. But the converse is not true, in general. The following example shows that there exists an α-compatible ring R such that R_R is α-quasi Armendariz, but $ann_R(mR)$ is not left s-unital for some $m\in R$.

Example 2.9. [26, Example 2.4] *For a given field F, let*

$$S = \{(a_n)_{n=1}^{\infty} \in \prod F \mid a_n \text{ is eventually constant}\},$$

which is a subring of the countably infinite direct product $\prod F$. Then, S is a commutative ring. Let $R = S[[x]]$. Clearly S is a reduced ring. Suppose that $f(x) = a_0 + a_1x + \cdots$ and $g(x) = b_0 + b_1x + \cdots \in S[[x]]$ are such that $f(x)g(x) = 0$. Then, from [1, p. 2269]*, it follows that $a_ib_j = 0$, for all i, j. Thus, R is a reduced ring. Let α be the S-automorphism of R such that $\alpha(x) = -x$. Clearly, R_R is α-compatible. Hence R is α-quasi Armendariz, by* [12, Proposition 6]*, and* [10, Lemma 2.2]*. We show that there exists $m \in R$ such that $ann_R(mR)$ is not left s-unital. Let $m = m_0 + m_1x + \cdots$, where $m_0 = (0, 1, 0, 0, \cdots), m_1 = (0, 1, 0, 1, 0, 0, \cdots), m_2 = (0, 1, 0, 1, 0, 1, 0, 0, \cdots), \cdots$. We show that $ann_R(mR)$ is not left s-unital. Suppose that $ann_R(mR)$ is left s-unital. Let $f = f_0 + f_1x + \cdots \in R$, where*

$$f_0 = (1, 0, 0, 0, \cdots), f_1 = (1, 0, 1, 0, 0, 0, \cdots), f_2 = (1, 0, 1, 0, 1, 0, 0, 0, \cdots), \cdots.$$

Then, $mf = 0$, and so $mRf = 0$, since R is reduced. Hence, $f \in ann_R(mR)$. Thus, there exists $h \in ann_R(mR)$ such that $hf = f$. Suppose that $h = h_0 + h_1x + \cdots$. Now, $mh = 0$ and from [1, p. 2269]*, it follows that $m_ih_j = 0$, for all i, j, and so there exists $n_j \in \mathbb{N}$ such that h_j has the form $(b_1^j, 0, b_3^j, 0, \cdots, b_{2n_j+1}^j, 0, 0, 0, \cdots)$, where $b_k^j \in F$, $j = 0, 1, 2, \cdots$. From $(h-1)f = 0$, it follows that $(h_0 - 1)f_i = 0$ and $h_jf_i = 0$, for all i and $j \geq 1$, and so there exists $m_j \in \mathbb{N}$ such that h_j has the form $(0, b_2^j, 0, b_4^j, 0, \cdots, b_{2m_j}^j, 0, 0, 0, \cdots)$, where $b_k^j \in F$, $j = 1, 2, \cdots$. Thus, $h_1 = h_2 = \cdots = 0$, and so $h = h_0$. This contradicts with $h_0f_i = f_i$, $i = 0, 1, \cdots$. Thus, $ann_R(mR)$ is not left s-unital.*

Clearly, if M_R is quasi-Baer, then $ann_R(mR)$ is left s-unital for each $m \in M$. But the converse is not true, in general. The following example shows that there exists a ring R such that $ann_R(mR)$ is left s-unital for each $m \in R$, but R is not quasi-Baer. Recall that a ring R is called a *right Bezout* ring if every finitely generated right ideal of R is principal. Recall that the weak global dimension of a ring R is defined as $sup\{fd(A) \mid A \text{ is a right } R\text{-module}\}$. Note that the weak global dimension ≤ 1 if and only if every right ideal of R is flat.

Example 2.10. [26, Example 2.5] *Let $\mathbb{Z}$ be the ring of integers and let*

$$S = (\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z})/(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}).$$

Then, S is clearly a Boolean ring and by [8, p. 64]*, the weak global dimension of $S[[x]]$ is one and $S[[x]]$ is not semihereditary. Let $R = S[[x]]$. Then, every principal ideal of R is flat, and so $R/ann_R(aR) = R/ann_R(a) \cong aR$ is flat. Thus, $ann_R(aR)$ is pure as a right ideal of R, for every $a \in R$. In* [8, Theorem 43]*, it was shown that the power series ring $A[[x]]$ over a von Neumann regular ring A is semihereditary if and only if $A[[x]]$ is a Bezout ring, in which all principal ideals are projective. On the other hand, by* [8, Theorem 42]*, $S[[x]]$ is a Bezout ring since the weak global dimension of $S[[x]]$ is one. Thus, R is not p.q.-Baer.*

Since quasi-Baer (p.q.-Baer) modules satisfy the hypotheses of Proposition 2.8, by using Proposition 2.7 we have the following results.

Theorem 2.11. *Let M_R be an (α, δ)-compatible module. Then, M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer); in this case, M_R is an (α, δ)-quasi Armendariz module.*

The following examples show that the α-compatibility condition on M_R in Theorem 2.11 is not superfluous.

Example 2.12. [3, Example 2.7] *Let F be any field of characteristic zero, and set $R := F[t]$. Let α be the F-automorphism of R such that $\alpha(t) = t + 1$, and set $S := R[x; \alpha]$. Consider the right R-module $M_R := \frac{F[t]}{(t^2)}$ and the right S-module $P_S := M[x]_S$. Using "$-$" to mean "modulo (t^2)", note that since $\bar{t}.t = \overline{0}$ but $\bar{t}.(t + 1) \neq \overline{0}$, the α-compatibility condition fails here. We show that P_S is prime. It suffices to show that, for any nonzero submodule $P'_S \subseteq P_S$, we have $ann_S(P') = 0$. Choose any $0 \neq p' \in P'$. We may write*

$$p' = \overline{g_k(t)}x^k + \overline{g_{k+1}(t)}x^{k+1} + \cdots \in P,$$

where $\overline{g_k(t)} \neq \overline{0}$ in M_R. It suffices to show that $ann(p'S_S) = 0$. Suppose there exists $s \in S$ with $(p'S)s = 0$. Write $s = f_0(t) + f_1(t)x + \cdots \in S$ with $f_j(t) \in R$, for each j. Now, for each $i \geq 0$, we have

$$\begin{aligned}\overline{0} &= (\overline{g_k(t)}x^{k+i} + \text{"higher terms"})(f_0(t) + \text{"higher terms"}) \\ &= \overline{g_k(t)}f_0(t + k + i)x^{k+i} + \text{"higher terms."}\end{aligned}$$

Hence, we have $\overline{g_k(t)}f_0(t + k + i) = \overline{0}$ in M_R. So, for each $i \geq 0$, we have $g_k(t)f_0(t + k + i) \in (t^2)$ in R. But $\overline{g_k(t)} \neq \overline{0}$ implies that $g_k(t) \notin (t^2)$. From this, we conclude that t divides $f_0(t + k + i)$, for *each $i \geq 0$. Putting $t = 0$, we have that $f_0(k+i) = 0$, for each $i \geq 0$. Since F has characteristic zero, we conclude that $f_0(t) = 0$. Now, we may go back and repeat this argument for $f_1, f_2, \cdots$, in turn, eventually concluding that $s = 0$. Thus, as desired, we have $ann_S(p'S) = 0$. Hence, P_S is prime with $ann_S(P) = 0$. Thus, $M[x]_S$ is quasi-Baer. Since $ann_R(M) = (t^2)$ and (t^2) does not have any idempotents, M_R is not quasi-Baer.*

Example 2.13. *Let R_0 denote any domain and let $R := R_0[t]$. Let $\alpha : R \to R$ be defined by $\alpha(t) = 0$ and $\alpha|_{R_0} = Id$. Next, let $M := R$ and $S = R[x;\alpha]$. Observe that α-compatibility evidently fails in this case. Since R is a domain, it is quasi-Baer. Now, consider the S-submodule $Q = xS$. Then, $ann_S(Q) = tS$ and tS does not have any idempotents. Hence, $M[x]_S$ is not quasi-Baer.*

The following example shows that δ-compatibility condition on R_R in Theorem 2.11 is not superfluous.

Example 2.14. [4, Example 11] *There is a ring R and a derivation δ of R such that $R[x;\delta]$ is a Baer (hence a quasi-Baer) ring, but R is not quasi-Baer. In fact, let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$, where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $R[x;\delta]$. If we set $e_{11} = \bar{t}x, e_{12} = \bar{t}, e_{21} = \bar{t}x^2 + x$, and $e_{22} = 1 + \bar{t}x$ in $R[x;\delta]$, then they form a system of matrix units in $R[x;\delta]$. Now, the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore, $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So, $R[x;\delta]$ is a Baer ring, but R is not quasi-Baer.*

Corollary 2.15. [7, Corollary 2.8] *Let R be a ring. Then, R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x]$ is quasi-Baer (resp. right p.q.-Baer).*

Corollary 2.16. [10, Corollary 2.8] *Let R be an (α, δ)-compatible ring. Then, R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x;\alpha,\delta]$ is quasi-Baer (resp. right p.q.-Baer).*

According to Lee-Zhou [21], a module M_R is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that R is a reduced ring if an only if R_R is reduced. If M_R is reduced, then M_R is p.p. if and only if M_R is p.q.-Baer.

Lemma 2.17. *The followings are equivalent for a module M_R.*

(1) *M_R is reduced and (α,δ)-compatible.*
(2) *The following conditions hold: for any $m \in M$ and $a \in R$,*
  (a) *$ma = 0$ implies $mRa = 0 = mR\alpha(a)$.*
  (b) *$m\alpha(a) = 0$ implies $ma = 0$.*
  (c) *$ma = 0$ implies $m\delta(a) = 0$.*
  (d) *$ma^2 = 0$ implies $ma = 0$.*

Proof. The proof is straightforward. □

Lemma 2.18. *Let M_R be a reduced (α,δ)-compatible module. Then, M_R is (α,δ)-Armendariz.*

Proof. Let $m(x) = m_0 + \cdots + m_k x^k \in M[x]$, and $f(x) = a_0 + \cdots + a_n x^n \in R[x;\alpha,\delta]$ such that $m(x)f(x) = 0$. Hence, $m_k R a_n = 0$, by Lemmas 2.4 and 2.17. Thus, the coefficient of x^{k+n-1} in equation $m(x)f(x) = 0$ is $m_k\alpha^k(a_{n-1}) + m_{k-1}\alpha^{k-1}(a_n) = 0$. Multiplying this equation by a_n from the right-hand side, we obtain $m_{k-1}\alpha^{k-1}(a_n)a_n = 0$. Hence, $m_{k-1}a_n^2 = 0$, and so $m_{k-1}a_n = 0$, by Lemma 2.17. Therefore, $m_k a_{n-1} = 0$, and so $m_k x^k a_{n-1} x^{n-1} = m_{k-1} x^{k-1} a_n x^n = 0$, by Lemma 2.4. Continuing this process, we can prove $m_i x^i a_j x^j = 0$, for each i, j. □

For a module M_R, put $\mathrm{Ann}_R(2^M) = \{\mathrm{ann}_R(N) \mid N \text{ is a subset of } M\}$. In a similar way as in the proof of Proposition 2.7, we can prove the following.

Proposition 2.19. *Let M_R be an (α,δ)-compatible module and S be the skew polynomial ring $R[x;\alpha,\delta]$. Then, the following statements are equivalent.*

(1) *M_R is (α,δ)-Armendariz.*
(2) *$\psi : Ann_R(2^M) \to Ann_S(2^{M[x]}); A \to AS$ is bijective.*

Theorem 2.20. *Let M_R be an (α,δ)-compatible module and $S = R[x;\alpha,\delta]$. If M_R is (α,δ)-Armendariz, then M_R is Baer (resp. p.p.) if and only if $M[x]_S$ is Baer (resp. p.p.).*

Proof. It follows from Lemma 2.18 and Proposition 2.19. □

According to Krempa [18], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R.

Corollary 2.21. [12, Theorem 14] *Let R be an α-rigid ring. Then, R is Baer (resp. p.p.) if and only if $R[x; \alpha, \delta]$ is Baer (resp. p.p.).*

Proof. Since α-rigid rings are reduced and (α, δ)-compatible, the proof follows from Lemma 2.18 and Theorem 2.20. □

Corollary 2.22. [4, Theorem B] *Let R be a reduced ring. Then, R is Baer (resp. p.p.) if and only if $R[x]$ is Baer (resp. p.p.).*

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