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ON THE DENSENESS OF THE INVERTIBLE GROUP IN UNIFORM FRÉCHET ALGEBRAS

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Abstract. We first extend the Arens-Royden theorem to unital, commutative Fréchet algebras under certain conditions. Then, we show that if A is a uniform Fréchet algebra on $X = M_A$, where M_A is the continuous character space of A, then A does not have dense invertible group, if we impose some conditions on X. On the other hand, if A has dense invertible group, then it is shown that $A = C(X)$, with certain conditions on X. Thus, the results of Dawson and Feinstein on denseness of the invertible group in Banach algebras are extended to uniform Fréchet algebras.

1. Introduction

Here, we assume that all algebras are unital and commutative.

Let B be a unital commutative Banach algebra (B-algebra) with the character space (maximal ideal space) $M_B = X$, and let $G(B)$ denote the group of invertible elements in B and $exp(B) = \{e^x : x \in B\}$. The multiplicative group $G(B)/\exp(B)$ is denoted by $H^1(B)$. The Arens-Royden theorem asserts that $H^1(B)$ and $H^1(C(X))$ are isomorphic [8, p. 413], or in other words, for every f in $C(X)$, which does not vanish

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on X, there exists $g \in G(B)$ such that $f/\hat{g}$ has a continuous logarithm on X [5, III. Theorem 7.2].

Here, we obtain a result similar to the Arens-Royden theorem, for certain unital commutative Fréchet algebras, and then we extend some results of Dawson and Feinstein on denseness of the invertible group [3] to uniform Fréchet algebras.

Definition 1.1. *Let A be a topological algebra. Then, A is a locally multiplicatively convex algebra or an LMC algebra if there is a base of neighbourhoods (V_α) of the origin consisting of sets which are absolutely convex and multiplicative, i.e., $V_\alpha \cdot V_\alpha \subseteq V_\alpha$. Equivalently, an LMC algebra is a topological algebra whose topology is defined by a separating family (p_α) of submultiplicative seminorms, i.e., $p_\alpha(fg) \leq p_\alpha(f)p_\alpha(g)$, for all $f, g \in A$.*

A Fréchet algebra (F-algebra) A is an LMC-algebra which is also a complete metrizable space. Its topology can be defined by an increasing sequence (p_n) of submultiplicative seminorms. Without loss of generality, we may assume that $p_n(1) = 1$, for all $n \in \mathbb{N}$, if A has a unit. A uniform Fréchet algebra (uF-algebra) is a Fréchet algebra A with the defining sequence (p_n) of seminorms such that, for all $f \in A$ and $n \in \mathbb{N}$, $p_n(f^2) = (p_n(f))^2$ [6, Definition 4.1.2]. *For those terms concerning topological algebras or Fréchet algebras, which are not defined here, one may refer, for example, to* [2], [6] *and* [7].

Definition 1.2. *The weak* topology on the dual space A^* is denoted by $\sigma = \sigma(A^*, A)$, so that $\varphi_\nu \to \varphi$ in (A^*, σ) if and only if $\varphi_\nu(f) \to \varphi(f)$, for all $f \in A$. The continuous character space of an F-algebra $(A, (p_n))$, denoted by M_A, is the set of all non-zero continuous complex-valued homomorphisms on A. The space M_A is taken to have the related weak* topology from A^*, which is called the Gelfand topology. We always endow M_A with the Gelfand topology.*

The Gelfand transform of an element $f \in A$ is defined by $\hat{f}(\varphi) = \varphi(f)$, for all $\varphi \in A^*$. The definition of the weak topology shows immediately that $\hat{f}$ is continuous on (A^*, σ). We also take $\hat{A}$ to be the set of all Gelfand transforms $\hat{f}$ of elements f in A, see, for example,* [2, Section 4.10].

Definition 1.3. *A Hausdorff space X is hemicompact if there exists a sequence (K_n) of compact subsets of X such that, for all $n \in \mathbb{N}$,*

$K_n \subseteq K_{n+1}$, and each compact subset K of X is contained in some K_n. The sequence (K_n) is called an admissible exhaustion of X.

A Hausdorff space is a k-space if every subset intersecting each compact subset in a closed set is itself closed.

Examples of k-spaces are locally compact spaces and first countable spaces [4, p. 248]. It is also known that if X is a σ-compact and locally compact space then it is hemicompact. Moreover, if X is a hemicompact k-space, then $C(X)$ is a Fréchet algebra with respect to the compact open topology [6, Remark 3.1.10]. Note that a complex-valued function f on a k-space X is continuous if and only if it is continuous on each compact subset of X. Hence, whenever X is a hemicompact k-space with the admissible exhaustion (X_n), a necessary and sufficient condition for the continuity of a complex-valued function f on X is that f is continuous on each X_n.

For an F-algebra $(A, (p_n))$, let A_n be the completion of $A/\ker p_n$ with respect to the norm $p'_n(f + \ker p_n) = p_n(f)$ $(f \in A)$. It is known that $A = \varprojlim A_n$, the projective limit of the sequence (A_n) of B-algebras. Furthermore, $M_A = \bigcup_{n=1}^{\infty} M_{A_n}$, as sets, in a natural way. In fact, M_A is a hemicompact space with the admissible exhaustion (M_{A_n}). For further information, see, for example, [6, pp. 77-80] or [2, pp. 581-582].

We now give examples of hemicompact k-spaces and uniform algebras on them.

Example 1.4. (*i*) *Let U be an open subset of the complex plane. Then, U is the union of a sequence K_n of compact subsets of U such that for all $n \in \mathbb{N}$, $K_n \subseteq K_{n+1}$, and each compact subset K of U is contained in some K_n* [10, Theorem 13.3]. *Hence, U is a hemicompact space. Since U is locally compact, it is also a k-space. Now, let $C(U)$ denote the algebra of continuous functions on U. If we define $p_n(f) = \|f\|_{K_n}$ $(f \in C(U))$, then (p_n) is an increasing sequence of submultiplicative seminorms on $C(U)$, which defines the compact-open topology on $C(U)$ and with respect to this topology, $C(U)$ is a uniform Fréchet algebra. Also, $C(U) = \varprojlim C(K_n)$, the projective limit of the sequence $(C(K_n))$ of B-algebras.*

(*ii*) *Let $A(U)$ denote the algebra of analytic functions on U. Then, $A(U)$ is a closed subalgebra of $C(U)$, and so it is a uniform Fréchet algebra.*

Definition 1.5. *Let X be a non-empty topological space, and let A be an algebra of complex functions on X. Then, A is a function algebra on*

X if A separates the points of X and contains the constants and if the A-topology on X is the given topology. The algebra A is a Fréchet function algebra (Ff-algebra) or a Banach function algebra (Bf-algebra) on X if A is a function algebra which is also an F-algebra or a B-algebra, respectively, with respect to some topology [2, Definition 4.1.1].

It is interesting to note that if there is a function algebra on a topological space X, then X must be Hausdorff and completely regular.

Definition 1.6. *Let A be an Ff-algebra (Bf-algebra) on X such that the evaluation homomorphisms $\varphi_x : A \longrightarrow \mathbb{C}$ are all continuous, where $\varphi_x(f) = f(x)$, for $f \in A$ and $x \in X$. Obviously, the map $x \longrightarrow \varphi_x$ from X into M_A is continuous and injective. If this map is also surjective, then it is in fact a homeomorphism from X onto M_A, since the topology on X is the A-topology. In this case, we say that A is a natural Ff-algebra (Bf-algebra) on X, and we identify X with M_A.*

Note that evaluation homomorphisms are always continuous in Bf-algebras.

2. Continuous Functions of Exponential Type

In this section, we impose some conditions on the elements of $G(C(X))$ to guarantee that they are in $\exp(C(X))$, when X is a hemicompact space.

First, we need the following elementary lemma, which can be proved by a classical argument.

Lemma 2.1. *Let X be a connected, Hausdorff space. If $f, g \in C(X)$ such that $\exp f = \exp g$ on X, then there exists an integer k such that $f = g + 2k\pi i$ on X.*

Theorem 2.2. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n is connected. If $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$.*

Proof. By the hypothesis, for each $n \in \mathbb{N}$, there exists $g_n \in C(X_n)$ such that $f = \exp g_n$ on X_n. Since $\exp g_{n+1} = \exp g_n$ on X_n, by Lemma 2.1, there exists an integer $k_n \in \mathbb{Z}$ such that $g_{n+1} = g_n + 2k_n\pi i$ on X_n. Taking $k_0 = 0$ and defining g on each X_n by $g = g_n - 2(\sum_{j=0}^{n-1} k_j)\pi i$, we observe that g is a well-defined continuous function on X. It is then clear that $f = \exp g$ on X. □

Theorem 2.3. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. If $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$.*

Proof. Let $x \in X$ and $n \in \mathbb{N}$. If $x \in X_n$, take $C_n(x)$ to be the component of X_n containing x; otherwise, take $C_n(x) = \phi$. Clearly, each $C_n(x)$ is compact and $C_n(x) \subseteq C_{n+1}(x)$. For each $x \in X$, take $C(x) = \bigcup_{n=1}^{\infty} C_n(x)$. We will show that $C(x)$ is a hemicompact k-space with the admissible exhaustion $(C_n(x))$.

Obviously, any closed subset of a hemicompact k-space is again a hemicompact k-space. So, it is enough to show that $C(x)$ is closed. Let K be an arbitrary compact subset of X having nonempty intersection with $C(x)$. Since X is hemicompact, $K \subseteq X_n$, for some $n \in \mathbb{N}$. Let $Y_1, Y_2, \ldots, Y_m$ be components of X_n which have nonempty intersection with $C(x)$. For each i, $1 \le i \le m$, there exists a large enough $k_i \in \mathbb{N}$ such that $k_i \ge n$ and $Y_i \cap C_{k_i}(x) \neq \phi$. Since Y_i is connected and $C_{k_i}(x)$ is a component of X_{k_i}, it follows that $Y_i \subseteq C_{k_i}(x)$ $(1 \le i \le m)$. Hence, $\bigcup_{i=1}^{m} Y_i \subseteq C_k(x)$, where $k = \max\{k_1, k_2, \ldots, k_m\}$, and so

$K \cap C(x) \subseteq C(x) \cap X_n = C(x) \cap (\cup_{i=1}^{m} Y_i) \subseteq C(x) \cap C_k(x) = C_k(x)$.

Therefore, $K \cap C(x) = K \cap C_k(x)$, which implies that $K \cap C(x)$ is a closed subset of X. Since X is a k-space, we conclude that $C(x)$ is a closed subset of X.

Now, let K be a compact subset of $C(x)$. By following a similar argument as in the above paragraph, we have $K = K \cap C(x) = K \cap C_k(x)$, for some $k \in \mathbb{N}$, and hence $(C_n(x))$ is an admissible exhaustion of $C(x)$.

It is easy to see that for every $x, y \in X$, either $C(x) = C(y)$ or $C(x) \cap C(y) = \phi$. Therefore, the family $\{C(x) : x \in X\}$ is, in fact, a partition of X. We may choose Y to be a subset of X such that X is disjoint union of the $C(x)$, when x runs over Y.

Now, let $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for each $n \in \mathbb{N}$. Since $f|_{C_n(x)} \in \exp(C(C_n(x)))$, for each $x \in Y$ and $n \in \mathbb{N}$ when $C_n(x) \neq \phi$, then by Theorem 2.2, for every $x \in Y$, there exists a continuous function g_x on $C(x)$ such that $f = \exp g_x$ on $C(x)$. Let g be a function defined on X by $g = g_x$ on $C(x)$, for each $x \in Y$. Clearly, g is well-defined. Since for every $x \in Y$, each component of X_n having nonempty intersection with $C(x)$, is contained in some $C_k(x)$, we conclude that g is continuous on each component of X_n, and so it is continuous on X_n. Therefore, g is continuous on X. Clearly, $f = \exp g$, which completes the proof of the theorem. $\square$

In general, when X_n has infinitely many components, we do not know whether the above theorem is still valid. However, the following example shows that the result may be true even if X_n has infinitely many components.

Example 2.4. *Let $M = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, and $X = \mathbb{R} \times M$. For each $n \in \mathbb{N}$, take $X_n = [-n, n] \times M$. Then, (X_n) is an admissible exhaustion of X, i.e., X is a hemicompact space. Moreover, X is a k-space* [4, p. 249]. *Since for each $n \in \mathbb{N}$, $\mathbb{C} \setminus X_n$ is connected, $H^1(C(X_n))$ is trivial, for all $n \in \mathbb{N}$ ([1] and [3]). Now, it is clear from Theorem 2.2 that the group $H^1(C(X)) = G(C(X))/\exp(C(X))$ is also trivial.*

Clearly, if $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for each $n \in \mathbb{N}$, then $f(x) \neq 0$, for every $x \in X$, and hence $f \in G(C(X)) = \exp(C(X))$. So, Theorem 2.3 is valid, although, for each $n \in \mathbb{N}$, X_n has infinitely many components.

To present the main result of this section, we need the following preliminaries.

If A is a uF-algebra, then $\hat{A}$ is a natural Ff-algebra on $X = M_A$. Since the Gelfand map is a topological and algebraical isomorphism on a uF-algebra, we can consider each uF-algebra A, as a pointseparating complete unital subalgebra of $C(X)$, endowed with the compact-open topology, where $X = M_A$ is a hemicompact space [6, Theorem 4.1.3]. On the other hand, if we take $X_n = M_{A_n}$ then $\hat{A} = \varprojlim \hat{A}_{X_n}$, the projective limit of the dense projective system $\ldots \longrightarrow \hat{A}_{X_{n-1}} \xrightarrow{r_n} \hat{A}_{X_n} \longrightarrow \ldots$, where (X_n) is the admissible exhaustion of $X = M_A$, and $\hat{A}_{X_n}$ is the completion of $\hat{A}|_{X_n}$ with respect to the supremum norm $\|\cdot\|_{X_n}$, and the r_n are the restriction mappings. Since $\hat{A}_{X_n}$ is algebraically and topologically isomorphic with A_n ($n \in \mathbb{N}$), it is then a natural uniform (Banach) algebra on X_n. Hence, identifying A with $\hat{A}$, each uF-algebra A is the projective limit of a sequence of natural uniform Banach algebras. In this case, we say that A is a uF-algebra on $X = M_A$.

Theorem 2.5. *Let A be a uF-algebra on $X = M_A$, where X is a k-space, and for each $n \in \mathbb{N}$, $X_n = M_{A_n}$ has finitely many components. If $f \in A$ and $f|_{X_n} \in \exp(A_{X_n})$, for all $n \in \mathbb{N}$, then $f \in \exp(A)$.*

Proof. By the hypothesis, for each $n \in \mathbb{N}$, there exists $g_n \in A_{X_n}$ such that $f|_{X_n} = \exp g_n$. By Theorem 2.3, there exists an element $g \in C(X)$ such that $f = \exp g$. Since $f|_{X_n} \in A_{X_n}$, and A_{X_n} is a natural Bf-algebra on X_n, we can apply the Implicit Function Theorem [8, Theorem 3.5.12] to the function $F(w,z) = e^w - z$, to obtain $h_n \in A_{X_n}$ such that $f|_{X_n} = \exp h_n$ and moreover, $h_n = g|_{X_n}$, for each $n \in \mathbb{N}$. Since $A = \varprojlim A_{X_n}$ and $g|_{X_n} \in A_{X_n}$, it follows that $g \in A$, and this completes the proof of the theorem. □

3. Extension of the Arens-Royden Theorem to Fréchet Algebras

Let X be a topological space, and let T be the unit circle in the plane. The group of homotopy classes of maps $X \longrightarrow T$, which is denoted by $\pi^1(X)$, is called the first cohomotopy group of X. If X is a compact Hausdorff space, then it is known that $H^1(C(X)) = G(C(X))/\exp(C(X))$ is isomorphic with $\pi^1(X)$. See, for example, [8, p. 411]. Following the same technique as in [8, p. 411], we apply Theorem 2.3 to extend the above result.

Theorem 3.1. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. Then, $\pi^1(X)$ is isomorphic with $H^1(C(X))$.*

Proof. Let L denote the set of all continuous functions from X into the unit circle T. For an element $g \in L$, the homotopy class of g is denoted by $[g]$. Let $\theta : G(C(X)) \longrightarrow \pi^1(X)$ be the group homomorphism, defined by $\theta(f) = [\frac{f}{|f|}]$. Obviously, θ is surjective. Now, we show that $\ker\theta = \exp(C(X))$.

If $g \in C(X)$, then $\exp g/|\exp g| = \exp(iImg)$. Hence, the function $G(x,t) = \exp(i(1-t)Img(x))$, for $x \in X$ and $t \in [0,1]$, defines a homotopy between 1 and $\exp(iImg)$ so that $\theta(\exp g)$ is the identity element in $\pi^1(X)$. Hence, $\exp(C(X)) \subseteq \ker\theta$.

For the reverse inclusion, let $g \in G(C(X))$ and $\theta(g) = 1$. Then, there exists a homotopy $F : X \times [0,1] \longrightarrow T$ with $F(x,0) = \frac{g(x)}{|g(x)|}$ and $F(x,1) = 1$. For each $n \in \mathbb{N}$, take $F_n = F|_{X_n \times [0,1]}$ and $g_n = g|_{X_n}$. Clearly, $t \longrightarrow F_n(\cdot,t)$ is a continuous path from 1 to $\frac{g_n}{|g_n|}$ in $G(C(X_n))$, and $\exp(C(X_n))$ is the principal component of $G(C(X_n))$. Therefore,

$\frac{g_n}{|g_n|} \in \exp(C(X_n))$. On the other hand, the function $I(x,t) = \frac{g(x)}{1-t+t|g(x)|}$ defines a homotopy between g and $\frac{g}{|g|}$. If we take $I_n = I|_{X_n \times [0,1]}$, then $t \longrightarrow I_n(.,t)$ is a continuous path connecting g_n to $\frac{g_n}{|g_n|}$. Thus, $g_n \in \exp(C(X_n))$, for each $n \in \mathbb{N}$. Now, Theorem 2.3 implies that $g \in \exp(C(X))$. Therefore, $\ker\theta = \exp(C(X))$, that is, θ induces a group isomorphism between $H^1(C(X))$ and $\pi^1(X)$. □

Before presenting next results, let us recall some elementary notions, which can be found in [6]. Let $(A,(p_n))$ be a unital, commutative F-algebra, and let A_n denote the completion of $A/\ker p_n$ with respect to the norm $p_n'(f+\ker p_n) = p_n(f)$. For each $n \in \mathbb{N}$, let $\pi_{n+1,n} : A_{n+1} \longrightarrow A_n$ be the extension of the norm decreasing homomorphism $A/\ker p_{n+1} \longrightarrow A/\ker p_n$, $x + \ker p_{n+1} \mapsto x + \ker p_n$. Then, $\pi_{n+1,n}$ is a continuous, dense range homomorphism. Therefore, $\pi^*_{n+1,n}$ is a homeomorphism between M_{A_n} and $\pi^*_{n+1,n}(M_{A_n}) \subseteq M_{A_{n+1}}$. Identifying M_{A_n} with $\pi^*_{n+1,n}(M_{A_n})$, for each $x \in A_{n+1}$, we have $\hat{x}|_{M_{A_n}} = \widehat{\pi_{n+1,n}}(x) \in \hat{A}_n$. Now, by considering the function $\theta_n : H^1(A_{n+1}) \longrightarrow H^1(A_n)$, $x \cdot \exp(A_{n+1}) \mapsto \pi_{n+1,n}(x) \cdot \exp(A_n)$ $(x \in G(A_{n+1}))$, one can easily show that $\{H^1(A_n), \theta_n\}$ is a projective system of groups so that $\varprojlim H^1(A_n)$ is a subgroup of $\prod_{n=1}^{\infty} H^1(A_n)$.

In particular, whenever A is a uF-algebra on X, we can define θ_n by $f \cdot \exp(A_{n+1}) \mapsto r_n(f) \cdot \exp(A_n)$ $(f \in G(A_{n+1}))$, where r_n is the restriction map $A_{n+1} \longrightarrow A_n$, $f \mapsto f|_{X_n}$ $(f \in A_{n+1})$.

Theorem 3.2. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. Then, $H^1(C(X))$ is isomorphic with $\varprojlim H^1(C(X_n))$.*

Proof. Define the function $\varphi : H^1(C(X)) \longrightarrow \varprojlim H^1(C(X_n))$, by

$$\varphi(f \cdot \exp(C(X)) = (f|_{X_n} \cdot \exp(C(X_n)))_n \quad ; \quad (f \in G(C(X))).$$

Clearly, φ is a group homomorphism. If $f \in G(C(X))$ such that $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$, by Theorem 2.3. Hence, φ is injective. For the surjectivity of φ, let $(f_n \cdot \exp(C(X_n)))_n \in \varprojlim H^1(C(X_n))$. For each $n \in \mathbb{N}$, there exists $g_n \in C(X_n)$ such that $f_{n+1}|_{X_n} = f_n \cdot \exp g_n$. Since X is a normal space [6, Remarks 3.1.10], each g_n has a continuous extension to X, which is again denoted by g_n. Adopting a similar method as in the proof of Theorem 2.2, set $g_0 = 0$

and $h_n = f_n \cdot \exp(-\sum_{i=0}^{n-1} g_i)$ on X_n. Since $f_{n+1}|_{X_n} = f_n \cdot \exp g_n$, we conclude that $h_{n+1}|_{X_n} = h_n$. This shows that the function h, defined by h_n on each X_n, is well-defined, and since X is a k-space, $h \in C(X)$. In fact, $h \in G(C(X))$. Moreover, $\varphi(h \cdot \exp C(X)) = (f_n \cdot \exp(C(X_n)))_n$, and hence φ is surjective. $\square$

Theorem 3.3. *Let A be a unital commutative F-algebra. Then $\varprojlim H^1 A_n)$ is isomorphic with $\varprojlim H^1(C(M_{A_n}))$.*

Proof. Let $(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n)$. For each $n \in \mathbb{N}$, there exists an element y_n in A_n such that $\pi_{n+1,n}(x_{n+1}) = x_n \cdot \exp y_n$, and hence $\hat{x}_{n+1}|_{M_{A_n}} = \widehat{\pi_{n+1,n}(x_{n+1})} = \hat{x}_n \cdot \exp \hat{y}_n$. Therefore, $(\hat{x}_n \cdot \exp(C(M_{A_n})))_n \in \varprojlim H^1(C(M_{A_n}))$. This shows that the function $\psi : \varprojlim H^1(A_n) \longrightarrow \varprojlim H^1(C(M_{A_n}))$, defined by $(x_n \cdot \exp(A_n))_n \longrightarrow (\hat{x}_n \cdot \exp(C(M_{A_n})))_n$, is well-defined. Clearly, ψ is a group homomorphism.

Let $(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n)$ be such that $\hat{x}_n \in \exp(C(M_{A_n}))$, for all $n \in \mathbb{N}$. By the Implicit Function Theorem, for each $n \in \mathbb{N}$, there exists $c_n \in A_n$ such that $x_n = \exp c_n$. Thus, ψ is injective. For the surjectivity of ψ, let $(f_n \cdot \exp(C(M_{A_n})))_n \in \varprojlim H^1(C(M_{A_n}))$. For each $n \in \mathbb{N}$, there exists $h_n \in C(M_{A_n})$ such that $f_{n+1}|_{M_{A_n}} = f_n \cdot \exp h_n$. On the other hand, by the Arens-Royden Theorem, for each $n \in \mathbb{N}$, there exist $x_n \in G(A_n)$ and $g_n \in C(M_{A_n})$ such that $\hat{x}_n = f_n \cdot \exp g_n$. Therefore, for each $n \in \mathbb{N}$,

$$\widehat{\pi_{n+1,n}(x_{n+1})} = \hat{x}_{n+1}|_{M_{A_n}} = \hat{x}_n \cdot \exp(-g_n + h_n + g_{n+1}).$$

By applying the Implicit Function Theorem, $\pi_{n+1,n}(x_{n+1}) \cdot x_n^{-1} \in \exp(A_n)$, for all $n \in \mathbb{N}$, i.e.,

$$(x_n \cdot \exp(A_n))_n \in \varprojlim H^1(A_n).$$

Obviously, $\psi((x_n \cdot \exp(A_n))_n) = (f_n \cdot \exp(C(M_{A_n})))_n$.

$\square$

As a consequence of the above two theorems we obtain the following result, which is an extension of the Arens-Royden theorem.

Corollary 3.4. *Let A be a unital commutative F-algebra such that M_A is a k-space, and M_{A_n} has finitely many components, for each $n \in \mathbb{N}$. Then $H^1(C(M_A))$ is isomorphic with $\varprojlim H^1(A_n)$.*

Corollary 3.5. *Let A be a uF-algebra on $X = M_A$ such that X is a k-space, and each $X_n = M_{A_n}$ has finitely many components. If $\pi^1(X)$ is trivial, or equivalently, if $G(C(X)) = \exp(C(X))$, then $G(A) = \exp(A)$.*

Proof. By Corollary 3.4 and Theorem 3.1, $\pi^1(X)$ is isomorphic with $\varprojlim H^1(A_{X_n})$. Obviously, if $f \in G(A)$, then $f_n = f|_{X_n} \in G(A_{X_n})$, for all $n \in \mathbb{N}$. By considering the continuous functions θ_n, which were defined before Theorem 3.2, it follows that $(f_n \cdot \exp(A_{X_n}))_n \in \varprojlim H^1(A_{X_n})$. Since $\pi^1(X)$ is trivial, $f_n \in \exp(A_{X_n})$, for all $n \in \mathbb{N}$. Now, the result follows by Theorem 2.5. □

Remark 3.6. *(i) Recall that if X is a compact plane set, then $\pi^1(X)$ is trivial, i.e., X has trivial fundamental group if and only if $\mathbb{C}\backslash X$ is connected* [1].

(ii) If X is a hemicompact k-space with the admissible exhaustion (X_n) such that $\mathbb{C}\backslash X_n$ is connected for each $n \in \mathbb{N}$, then each $\pi^1(X_n)$ is trivial by (i). If, moreover, each X_n has finitely many components, then by Theorem 2.3 $\pi^1(X)$ is trivial.

Remark 3.7. *In Corollary 3.5, the fact that $\pi^1(X)$ is trivial does not pass to compact subsets of X, i.e., $\pi^1(X_n)$ may not be trivial, in general, as the following example shows.*

Example 3.8. *Consider the admissible exhaustion of the open unit disc X consisting of an expanding sequence (X_n) of compact subsets, each of which consists of the union of a closed disc centred on the origin together with a disjoint closed annulus with two slightly larger radii. Each X_n has then 2 components, and since $\mathbb{C}\backslash X_n$ is not connected, none of the X_n have trivial fundamental group, i.e., $G(C(X_n) \neq exp(C(X_n))$, by Remark 3.6(i), although X does, i.e., $G(C(X)) = exp(C(X))$, by Remark 3.6(ii) (it is enough to take the exhaustion (K_n) for X as the increasing closed discs with centres at the origin). Nevertheless, the projective limit of the fundamental groups $\varprojlim H^1(C(X_n))$ is trivial by theorems 3.1 and 3.2.*

It is interesting to note that $H^1(C(X))$ may be trivial even if X is disconnected (consider, for example, a union of two disjoint closed intervals or discs).

Remark 3.9. *Let A be a uF-algebra on $X = M_A$, which is a k-space, with each X_n having finitely many components. If $A|_{X_n}$ is closed in $C(X_n)$, that is, $A_{X_n} = A|_{X_n}$, for each $n \in \mathbb{N}$, then, by following the same technique as in Theorem 3.2, we can obtain an isomorphism between $H^1(A)$ and $\varprojlim H^1(A_{X_n})$. Hence, in this case, $H^1(C(M_A))$ is isomorphic with $H^1(A)$. However, in general, we do not yet know whether there exists an isomorphism between $H^1(C(M_A))$ and $H^1(A)$.*

4. On the Denseness of Invertible Elements in a UF-Algebra

In 2003, Dawson and Feinstein obtained some results on the denseness of the invertible group in Banach function algebras [3]. Following the same technique and making use of the theorems proved in Section 3, we extend some of these results to *uF*-algebras. For the definition of topological dimension, refer to [9].

Theorem 4.1. *Let A be a uF-algebra on $X = M_A$ such that X is a k-space, and each $X_n = M_{A_n}$ is locally connected. Let $\pi^1(X)$ be trivial, or equivalently, let $G(C(X)) = \exp(C(X))$. Then,*

(i) *If X contains a closed subspace E whose topological dimension is at least 2, and if $E_n = E \cap X_n$ is locally connected, for all $n \in \mathbb{N}$, then A does not have a dense invertible group.*

(ii) *If A has a dense invertible group, then $A = C(X)$.*

Proof. We first note that each X_n has finitely many components.

(i) By Corollary 3.5, $G(A) = \exp(A)$. Suppose on the contrary that A has a dense invertible group. Let $B = A|_E$, and $I = \{f \in A : f|_E = 0\}$. It is easy to see that B is algebraically isomorphic with A/I. Since I is a closed ideal of A, A/I is a Fréchet algebra with respect to the sequence of seminorms $(q_k)_k$, defined by

$$q_k(f + I) = \inf\{\|f + g\|_{X_k} : g \in I\}, \quad (f \in A).$$

If we endow B with this sequence of seminorms, then B is an Ff-algebra on E. Note that E is a hemicompact k-space with the admissible exhaustion $(E_n)_n$. Hence, $C(E)$ is an F-algebra.

Let $F = \exp(A)|_E$, and let C be the closure of B in $C(E)$ with respect to the compact-open topology. Therefore, C is a *uF*-algebra on E. By our assumption, A has a dense invertible group, and so F is dense in C.

Clearly, we may write $C = \varprojlim C_{E_n}$, where C_{E_n} is the completion of $C|_{E_n}$ with respect to the norm $\|\cdot\|_{E_n}$, for each $n \in \mathbb{N}$. Obviously, $F|_{E_n}$ is dense in C_{E_n} and each element in $F|_{E_n}$ has a square root in C_{E_n}. Since each E_n is locally connected, it follows from Čirka's Theorem [11, pp. 131-134] that $C_{E_n} = C(E_n)$, and hence $C = C(E)$. Since E is normal and the invertible elements are dense in $C(E)$, by [9, Theorem 3.2, p. 128], we have $\dim E \leq 1$, which is a contradiction. Therefore, A does not have a dense invertible group.

(ii) Since $\pi^1(X)$ is trivial, Corollary 3.5 implies that $G(A) = exp(A)$. By the hypothesis, $G(A)$ is dense in A. Hence, $exp(A_{X_n})$ is dense in A_{X_n}, for all $n \in \mathbb{N}$. Since each X_n is locally connected, again by Čirka's Theorem [11, pp. 131-134], we conclude that $A_{X_n} = C(X_n)$, for all $n \in \mathbb{N}$. Therefore, $A = \varprojlim A_{X_n} = \varprojlim C(X_n) = C(X)$. □

The following result for Fréchet algebras is also similar to Corollary 1.8 in [3].

Corollary 4.2. *Let A be a natural uF-algebra on $\mathbb{R}$. If A has a dense invertible group, then $A = C(\mathbb{R})$.*

Proof. Set $I_n = [-n, n]$, then $C(\mathbb{R}) = \varprojlim C(I_n)$. Since $\mathbb{C}\backslash I_n$ is connected, $\pi^1(I_n)$ is trivial. Hence, $H^1(C(I_n))$ is trivial. By theorems 3.1 and 3.2, $\pi^1(\mathbb{R})$ is isomorphic with $\varprojlim H^1(C(I_n))$. Therefore, $\pi^1(\mathbb{R})$ is trivial, and hence $A = C(\mathbb{R})$, by Theorem 4.1 (ii). □

Note that we can also deduce the triviality of $\pi^1(\mathbb{R})$ and $\pi^1(\mathbb{C})$ from Remark 3.6(ii).

Remark 4.3. *One may think that the above corollary is also true if we replace $\mathbb{R}$ by $\mathbb{C}$. Actually this case cannot arise, since $C(\mathbb{C})$ does not have a dense invertible group. To see this, it is interesting to note that for a hemicompact k-space X, it is known that the uF-algebra $C(X)$ has a dense invertible group if and only if $dim(X) \leq 1$, where $dim(X)$ is the topological dimension of X. Since $dim(\mathbb{R}) = 1$ and $dim(\mathbb{C}) = dim(\mathbb{R}^2) =$* 2 [9, Theorem 3.2.7]*, it follows that $C(\mathbb{R})$ has a dense invertible group, whereas $C(\mathbb{C})$ does not have a dense invertible group.*

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