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ON IMAGES OF CONTINUOUS FUNCTIONS FROM A COMPACT MANIFOLD TO EUCLIDEAN SPACE

R. MIRZAIE

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ABSTRACT. We show that typical elements of the set of continuous functions from a compact differentiable manifold M to R^n are nowhere differentiable. Then, we study the box dimensions of typical elements in the set of images of M in R^n.

1. **Introduction**

We recall that a subset Y of a metric space (X, d) is called a comeagre subset, if Y contains an intersection of a countable number of open dense subsets. Each element of a comeagre subset is called a *typical element*. By Baire's category theorem, if X is a complete metric space, then each comeagre subset of X is dense in X.

Let M be a differentiable compact submanifold of R^n and

$$C(M, R^n) = \{f : M \to R^n, f \text{ is continuous}\}.$$

We denote by $|a - b|$, the usual distance of points a, b in R^n. $C(M, R^n)$ (endowed with the max-metric d, defined by $d(f, g) = max_{x \in M}|f(x) - g(x)|$), is a complete metric space. If $I = [0, 1]$, then a well known theorem due to Banach states that *"Typical elements of* $C(I, R)$ *are nowhere differentiable."* Banach's theorem is a classic theorem, and there are similar results in more general cases. Here, we generalize Banach's theorem

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to $C(M, R^n)$.

In the second part of the paper, we consider the set $K_n = \{A \subset R^n : A \text{ is compact}\}$. K_n endowed with the Hausdorff metric is a complete metric space. It is interesting to characterize fractal elements in K_n or a given subset of K_n. For this purpose, we must study the Box or the Hausdorff dimension of sets. There are many interesting results concerning the Box or the Hausdorff dimension of typical elements in K_n or some subspaces of K_n (see [2], [4], [6]). We show here that typical elements of the following subspace of K_n,

$$Im(M) = \{f(M) : f \in C(M, R^n)\},$$

have integer box dimensions. We will use the max-metric on $C(M, R^n)$ and on $Im(M)$ (Max-metric on $Im(M)$ is defined by $d(f(M), g(M)) = d(f, g)$). But it is not hard to show that our conclusions on $Im(M)$ are also valid for the Hausdorff metric.

2. Results

The following notations will be used in the proofs:

(1) $D(M, R^n) = \{f \in C(M, R^n) : f \text{ is differentiable}\}$.
(2) $ND(M, R^n) = \{f \in C(M, R^n) : f \text{ is nowhere differentiable}\}$.
(3) $I^m = I \times I \times ... \times I$ (m times).

Remark 2.1. *By Banach's theorem, $ND(I, R)$ is a comeagre subset of $C(I, R)$.*

Remark 2.2. *If Y is a comeagre subset of a topological space X, and if $Y \subset Z \subset X$, then Z is also comeagre in X.*

Lemma 2.3. *$ND(I^m, R)$ is a comeagre subset of $C(I^m, R)$.*

Proof. By Banach's theorem, the lemma is true for $m = 1$. We suppose that $m \geq 2$. Let $ND_1(I^m, R) = \{f \in C(I^m, R) : \frac{\partial f}{\partial x_1} \text{ nowhere exist}\}$. We show that $ND_1(I^m, R)$ is a comeagre subset of $C(I^m, R)$. Let Q

be the rational numbers and $J = I \cap Q$. For each $f \in C(I^m, R)$, and $t = (t_1, ...t_{m-1}) \in J^{m-1}$, define the map $f_t : I \to R$ by

$$f_t(x) = f(x, t_1, ..., t_{m-1}).$$

J^{m-1} is countable, and so we can denote it by $J^{m-1} = \{a_1, a_2, ...\}$. For each $a_i \in J^{m-1}$, put $(C(I, R))_{a_i} = C(I, R)$ and consider the following set, with the product topology (see [5] for definition and details about the product topology),

$$C(I, R)_{a_1} \times C(I, R)_{a_2} \times \cdots .$$

Now, define the map $\phi : C(I^m, R) \to C(I, R)_{a_1} \times C(I, R)_{a_2} \times \cdots$ by $\phi(f) = (f_{a_1}, f_{a_2}, ...)$. ϕ is one to one and a continuous function. If we put $ND(I, R)_{a_i} = ND(I, R)$, then we have

$$\phi(ND_1(I^m, R)) \subset ND(I, R)_{a_1} \times ND(I, R)_{a_2} \times \cdots .$$

By Banach's theorem, $ND(I, R)$ is a comeagre subset of $C(I, R)$. So, there is a countable collection $\{U_k : k \in N\}$ of open and dense subsets of $C(I, R)$ such that

$$\bigcap_{k\in N} U_k \subset ND(I, R).$$

For each $l \in N$ and $a_i \in J^{m-1}$, put $(U_1 \cap ... \cap U_l)_{a_i} = (U_1 \cap ... \cap U_l)$ and let

$$\begin{aligned} W_l &= (U_1 \cap ... \cap U_l)_{a_1} \times (U_1 \cap ... \cap U_l)_{a_2} \times ... \times (U_1 \cap ... \cap U_l)_{a_l} \\ &\times C(I, R)_{a_{l+1}} \times C(I, R)_{a_{l+2}} \times \cdots . \end{aligned}$$

W_l is open and dense in $C(I, R)_{a_1} \times C(I, R)_{a_2} \times ....$ and we have

$$\bigcap_{l\in N} W_l \subset ND(I, R)_{a_1} \times ND(I, R)_{a_2} \times \cdots .$$

J^{m-1} is dense in I^{m-1}. Thus, $\frac{\partial f}{\partial x_1}(x, t)$ does not exist, for all $t \in I^{m-1}$, if and only if it does not exist for all $t \in J^{m-1}$. Thus, we can show that

$$\bigcap_{l\in N} \phi^{-1}(W_l) \subset ND_1(I^m, R). \tag{2.1}$$

Now, we show that for each $l \in N$, $\phi^{-1}(W_l)$ is a dense subset of $C(I^m, R)$. Consider a function $f \in C(I^m, R)$ and let $\epsilon > 0$. Since $U_1 \cap U_2 \cap ... \cap U_l$ is dense in $C(I, R)$, for each $i \in \{1, 2, ...l\}$ there is a $g_i \in U_1 \cap U_2 \cap ... \cap U_l$ such that

$$d(f_{a_i}, g_i) < \frac{\epsilon}{l}.$$

Let $\theta_i : I^{m-1} \to I$ be a continuous function such that

$$\theta_i(a_i) = 1 \quad and \quad \theta_i(a_j) = 0 \quad for \quad each \quad j \in \{1, 2, ..., l\} - \{i\}.$$

Now, define a map $h : I^m \to R$ by

$$h(x,t) = f(x,t) + \sum_{i=1}^{l} \theta_i(t)(g_i(x) - f_{a_i}(x)), \quad (x,t) \in I \times I^{m-1}.$$

We have

$$|h(x,t) - f(x,t)| \leq \sum_{i=1}^{l} |\theta_i(t)||g_i(x) - f_{a_i}(x)| < \sum_{i=1}^{l} \frac{\epsilon}{l} = \epsilon.$$

Thus, $d(h, f) < \epsilon$. If $\phi(h) = (h_{a_1}, ..., h_{a_l}, h_{a_{l+1}}, ...)$, then

$$h_{a_1} = g_1, h_{a_2} = g_2, ..., h_{a_l} = g_l \Rightarrow \phi(h) \in W_l \Rightarrow h \in \phi^{-1}(W_l).$$

Therefore, $\phi^{-1}(W_l)$ is dense in $C(I^m, R)$. Since $\phi^{-1}(W_l)$ is open in $C(I^m, R)$, then we get by (2.1) that $ND_1(I^m, R)$ is comeagre in $C(I^m, R)$. Since $ND_1(I^m, R) \subset ND(I^m, R)$, we get the result by Remark 2.2. □

Lemma 2.4. *$ND(M, R)$ is comeagre in $C(M, R)$.*

Proof. Let $m = dimM$ and for each point $p \in M$, consider a chart (U, ϕ) around p such that $I^m \subset \phi(U)$. Since M is compact, then there is a finite collection of this kind of charts, say $(U_1, \phi_1), ..., (U_l, \phi_l)$, such that $M \subset \phi_1^{-1}(I^m) \cup ... \cup \phi_l^{-1}(I^m)$. Put $W_i = \phi_i^{-1}(I^m)$, $1 \leq i \leq l$, and for each $f \in C(I^m, R)$, denote by f_i the restriction of f on W_i, and consider the following function:

$$\psi_i : C(M, R) \to C(W_i, R), \quad \psi_i(f) = f_i.$$

Since $\phi(W_i) = I^m$, then we get from Lemma 2.3 that $ND(W_i, R)$ is a comeagre subset of $C(W_i, R)$. So, there is a countable collection $\{V_k^i : k \in N\}$ of open and dense subsets of $C(W_i, R)$ such that

$$\bigcap_k V_k^i \subset ND(W_i, R).$$

We show that for each $i, k \in N$, $\psi_i^{-1}(V_k^i)$ is a dense subset of $C(M, R)$. Suppose $f \in C(M, R)$ and let $\epsilon > 0$. Since V_k^i is dense in $C(W_i, R)$, then there is a function $g \in V_k^i$ such that

$$d(f_i, g) < \frac{\epsilon}{2}. \tag{2.2}$$

Let $\hat{g} : M \to R$ be a continuous extension of g on M. Since f and $\hat{g}$ are continuous, then by (2.2), there is an open subset B of M such that $W_i \subset B$ and

$$x \in B \Rightarrow d(f(x), \hat{g}(x)) < \epsilon. \tag{2.3}$$

Now, let $\theta : M \to [0, 1]$ be a continuous function such that

$$\theta(x) = 1 \; for \; x \in W_i \; \text{ and } \; \theta(x) = 0, \; \text{ for } x \in M - B.$$

Consider the continuous function $h : M \to R$, defined by

$$h(x) = f(x) + \theta(x)(\hat{g}(x) - f(x)). \tag{2.4}$$

If $x \in W_i$, then $h(x) = g(x)$, and so $\psi_i(h) = g$. Thus, $h \in \psi_i^{-1}(V_k^i)$. Also, we have

$$|f(x) - h(x)| = |\theta(x)|\hat{g}(x) - f(x)| < \epsilon.$$

So, $\psi_i^{-1}(V_k^i)$ is dense in $C(M, R)$. It is easy to show that

$$\bigcap_{k \in N} \bigcap_{1 \leq i \leq l} \psi_i^{-1}(V_k^i) \subset ND(M, R).$$

Therefore, $ND(M, R)$ is comeagre in $C(M, R)$. □

Theorem 2.5. *Typical elements of $C(M, R^n)$ are nowhere differentiable.*

Proof. For each $f \in C(M, R^n)$, we have $f = (f_1, ..., f_n)$ such that $f_i \in C(M, R)$. Consider the map $\psi : C(M, R^n) \to C(M, R) \times ... \times C(M, R)$ (n times), and $\psi(f) = (f_1, ..., f_n)$. ψ is a homeomorphism and

$$\psi^{-1}[ND(M, R) \times ... \times ND(M, R)] \subset ND(M, R^n). \tag{2.5}$$

Since by Lemma 2.4, $ND(M, R)$ is comeagre in $C(M, R)$, then $ND(M, R) \times \cdots \times ND(M, R)$ is comeagre in $C(M, R) \times ... \times C(M, R)$. Thus, $\psi^{-1}[ND(M, R) \times ... \times ND(M, R)]$ must be comeagre in $C(M, R^n)$. Now, we get the result by (2.5) and Remark 2.2. □

3. **Box Dimension**

Let C be a bounded subset of R^n. We denote by $dim(C)$, the topological dimension of C. For each number $\epsilon > 0$, put

$$A_\epsilon(C) = sup\{card\{Z\} : Z \subset C \; \text{ and } \; \text{for } \; \text{each } \; x, y \in Z, \; \; |x - y| > \epsilon\}.$$

The upper and lower box dimensions of C are defined by:

$$\overline{dim}_B(C) = limsup_{\epsilon\to 0}\frac{logA_\epsilon(C)}{-log\epsilon},$$

$$\underline{dim}_B(C) = liminf_{\epsilon\to 0}\frac{logA_\epsilon(C)}{-log\epsilon}.$$

If $\overline{dim}_B(C) = \underline{dim}_B(C)$, then $dim_B(C) = lim_{\epsilon\to 0}\frac{logA_\epsilon(C)}{-log\epsilon}$ is the box dimension of C.

Remark 3.1. *Let M be a differentiable submanifold of R^n and $dim(M) = m$. Then,*
(1) $dim_B(M) = dim(M) = m$.
(2) If $g : M \to R^n$ is a differentiable map and $M_g = g(M)$, then

$$dim_B(M_g) = dim(M_g) \in \{0, 1, ..., m\}.$$

If $g : M \to R^n$ is nowhere differentiable, then M_g may be a fractal subset of R^n. Thus, we might expect that for typical M_g, $\overline{dim}_B(M_g) > dimM_g$. But, the following theorem shows that typical elements of $Im(M)$ have integer box dimensions. Therefore, the set of images of M, which have fractal behaviors, is a meagre subset of $Im(M)$.

Remark 3.2. *Since $D(M, R^n)$ is an algebra in $C(M, R^n)$, then by the Stone-Weierstrass theorem, it is dense in $C(M, R^n)$ (see [7], Chapter 7).*

Theorem 3.3. *If M is a compact differentiable submanifold of R^n, then typical elements of the set of images of M under continuous maps have integer box dimensions.*

Proof. For each $g \in C(M, R^n)$ put

$$M_g = g(M) \subset R^n.$$

Consider a real number $\epsilon > 0$. If $f, g \in C(M, R^n)$, $x, y \in M$, and $d(g, f) < \frac{\epsilon}{3}$, then

$$\begin{aligned} d(g(x), g(y)) &\leq d(g(x), f(x)) + d(f(x), f(y)) + d(f(y), g(y)) \\ &\leq \frac{2\epsilon}{3} + d(f(x), f(y)). \end{aligned}$$

Thus,

$$A_\epsilon(M_g) \leq A_{\frac{\epsilon}{3}}(M_f).$$

In a similar way, we can show that

$$A_{\frac{1}{3}\epsilon}(M_f) \leq A_{\frac{1}{9}\epsilon}(M_g).$$

So, if $0 < \epsilon < 1$, then

$$\frac{logA_\epsilon(M_g)}{-log\epsilon} \leq \frac{logA_{\frac{1}{3}\epsilon}(M_f)}{-log\frac{\epsilon}{3} - log3} \leq \frac{logA_{\frac{1}{9}\epsilon}(M_g)}{-log\frac{\epsilon}{9} - log9}.$$

If g is differentiable, then by Remark 3.1, we have

$$dim_B(M_g) = dimM_g.$$

Therefore,

$$lim_{\epsilon\to 0}\frac{logA_\epsilon(M_g)}{-log\epsilon} = lim_{\epsilon\to 0}\frac{logA_{\frac{\epsilon}{9}}(M_g)}{-log\frac{\epsilon}{9} - log9} = dimM_g.$$

Thus, for each $K \in N$, there is an open set $U_{K,g}$ in $C(M, R^n)$ containing M_g such that for each $f \in U_{K,g}$,

$$dimM_g - \frac{1}{K} \leq \frac{logA_{\frac{1}{3}\epsilon}M_f}{-log\frac{\epsilon}{3} - log3} \leq dimM_g + \frac{1}{K}.$$

Put

$$W_K = \bigcup_{g\in D(M,R^n)} U_{K,g}.$$

Since $D(M, R^n)$ is dense in $C(M, R^n)$, then W_K is an open and dense subset of $C(M, R^n)$. Now, put

$$W = \bigcap_{K\in N} W_K.$$

W is comeagre in $C(M, R^n)$. If $f \in W$, then there is a differentiable function g such that $\overline{dim}_B M_f = dimM_g$. So, $\overline{dim}_B M_f \in \{0, 1, 2, ..., m\}$. $\square$

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R. Mirzaie
Department of Mathematics, Imam Khomeini International University, Qazvin, Iran
Email: `R_mirzaioe@yahoo.com`