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φ-FACTORABLE OPERATORS AND WEYL-HEISENBERG FRAMES ON LCA GROUPS

R.A. KAMYABI GOL* AND R. RAISI TOUSI

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Abstract. We investigate φ-factorable operators and Weyl-Heisenberg frames with respect to a function-valued inner product, the so called φ-bracket product on $L^2(G)$, where G is a locally compact abelian group and φ is a topological isomorphism on G. We introduce φ-factorable operators on $L^2(G)$ and extend the Riesz Representation Theorems for these operators. Finally, as an application of the φ-bracket product, we show that several well known theorems for Weyl-Heisenberg frames in $L^2(\mathbb{R})$ remain valid in $L^2(G)$, and they are unified within of group theory, in connection with the φ-bracket product.

1. Introduction

In [13], we have defined the φ-bracket product as a function-valued inner product on $L^2(G)$, where G is a locally compact abelian (which will be abbreviated by "LCA") group and φ is a topological isomorphism on G. The φ-bracket product, as a new inner product on $L^2(G)$, is applicable to extend many ideas and constructions from the theory of shift invariant spaces, factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different

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*Corresponding author

way. Whereas our work in [13] was devoted to basic properties of the φ-bracket product and φ-orthonormal bases, here we deal with characterizing φ-factorable operators on $L^2(G)$ and establishing Riesz Representation Theorems for such operators. We continue our investigation following the line of approach worked by Casazza and Lammers [5], but in a more general setting, using various tools in abstract harmonic analysis. In fact, our results generalize some of the results developed in [5] on $\mathbb{R}^n$, in which the authors want to be able to scale the lattice, and so they introduce a positive parameter a and express their results relative to the lattice $a\mathbb{Z}$. Here, like in [13], we use a topological isomorphism which introduces an appropriate scale factor in the setting of LCA groups. φ-Factorable operators are useful and shed light to define and investigate φ-frames and φ-Riesz bases, which are worked out in a forthcoming paper. After investigating φ-Factorable operators, we then, as an application of the φ-bracket product, study Weyl-Heisenberg frames on LCA groups in connection with the φ-bracket product. Our results generalize some of the results appearing in the literature on the Weyl-Heisenberg frames. Such a unified approach is useful, since it determines the basic features of the Weyl-Heisenberg frames, and includes most of the special cases.

Here, we give some of the basics regarding LCA groups. For a comprehensive account of LCA groups, we refer to [8, 11]. Suppose G is an LCA group with the Haar measure dx. A subgroup L of G is called a *uniform lattice* if it is discrete and co-compact (i.e., G/L is compact). Let φ be a topological isomorphism on G. If L is a uniform lattice in G, then so is $\varphi(L)$. Indeed, obviously $\varphi(L)$ is discrete. Also, by [11, Theorem 5.34], $G/\varphi(L)$ is topologically isomorphic to G/L and so it is compact. Here, we always assume that $G/\varphi(L)$ is normalized, i.e., $|G/\varphi(L)| = 1$. Denote by $\varphi(L)^\perp$, the annihilator of $\varphi(L)$ in $\hat{G}$, i.e., $\varphi(L)^\perp = \{\gamma \in \hat{G};\ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [12-16]).

Let L be a uniform lattice in G. Choosing the counting measure on L, a relation between the Haar measures dx on G and $d\dot{x}$ on $G/\varphi(L)$ is given by the following special case of Weil's formula [8]:
For $f \in L^1(G)$, we have $\sum_{k\in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

$$\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1})\in\varphi(L)} f(x\varphi(k^{-1}))d\dot{x}, \tag{1.1}$$

where, $\dot{x} = x\varphi(L)$.

Let $f, g \in L^2(G)$. The φ-*bracket product* of f, g is defined by

$$[f,g]_\varphi(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1})), \tag{1.2}$$

for all $x \in G$. We define the φ-norm of f as $\|f\|_\varphi(\dot{x}) = ([f,f]_\varphi(\dot{x}))^{1/2}$. In the sequel, we recall some basic properties of the φ-bracket product, for the proofs of which and more details the reader is referred to [13]. Let $f,\ g \in L^2(G)$. Then, $|[f,g]_\varphi| \le \|f\|_\varphi \|g\|_\varphi$ (the Cauchy-Schwartz Inequality). Also, (1.1) implies $\int_{G/\varphi(L)} [f,g]_\varphi(\dot{x}) d\dot{x} =< f, g >_{L^2(G)}$. For $\gamma \in \hat{G}$, denote by M_γ, the modulation operator on $L^2(G)$, i.e., $M_\gamma f(x) = \gamma(x)f(x)$, for all $f \in L^2(G)$. Then, for $f, g \in L^2(G)$ and $\gamma \in \varphi(L)^\perp$, we have the following relation between the φ-bracket product and the usual inner product in $L^2(G)$:

$$\widehat{[f,g]_\varphi}(\gamma) =< f, M_\gamma g >_{L^2(G)} . \tag{1.3}$$

We say $g \in L^2(G)$ is φ-*bounded* if there exists $M > 0$ so that $\|g\|_\varphi \le M$ *a.e.*. For $f, g \in L^2(G)$, the function $[f,g]_\varphi g$ need not generally be in $L^2(G)$. But, we have the following result.

Proposition 1.1. *If $f, g, h \in L^2(G)$ and g, h are φ-bounded, then $[f,g]_\varphi h \in L^2(G)$.*

A sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is called φ-*orthonormal* if $[g_n, g_m]_\varphi = 0$, for all $n \neq m \in \mathbb{N}$ and $\|g_n\|_\varphi = 1$, for all $n \in \mathbb{N}$. Let $f \in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence in $L^2(G)$. An extension of [5, Theorem 4.13] from $\mathbb{R}$ to the setting of an LCA group gives Bessel's Inequality for φ-bracket products as follows:

$$\sum_{n\in\mathbb{N}} |[f,g_n]_\varphi(\dot{x})|^2 \le \|f\|_\varphi^2(\dot{x}), \quad \textit{for a.e. } \dot{x} \in G/\varphi(L). \tag{1.4}$$

A φ-orthonormal sequence $(g_n)_{n\in\mathbb{N}}$ is called a φ-*orthonormal basis* if $[f, g_n]_\varphi = 0$ a.e., for all $n \in \mathbb{N}$, implies $f = 0$ a.e.. Let $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence. It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain equivalent conditions for $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ to be a φ-orthonormal basis (see also [13]).

Proposition 1.2. *If $(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal sequence in $L^2(G)$, then the following are equivalent.*

(1) *$(g_n)_{n\in\mathbb{N}}$ is a maximal φ-orthonormal sequence, i.e., $(g_n)_{n\in\mathbb{N}}$ is not contained in any other φ-orthonormal set.*

(2) $(g_n)_{n\in\mathbb{N}}$ *is a* φ*-orthonormal basis.*
(3) *For each* $f \in L^2(G)$, $f = \sum_{n\in\mathbb{N}} [f, g_n]_\varphi g_n$ *a.e..*
(4) $\|f\|_\varphi^2 = \sum_{n\in\mathbb{N}} |[f, g_n]_\varphi|^2$ *a.e., for all* $f \in L^2(G)$ *(the Parseval Identity).*
(5) $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ *is an orthonormal basis for* $L^2(G)$.

Thanks to Zorn's Lemma and Proposition 1.2, $L^2(G)$ admits a φ-orthonormal basis.

The rest of this paper is organized as follows. In Section 2, we introduce a φ-factorable operator on $L^2(G)$, where G is an LCA group and establish the Riesz Representation Theorems for these operators.

Over the last ten years, there have been a lot of research on frame theory in general, and the Weyl-Heisenberg frame theory, in particular [2-4, 7, 18], most of which are on the Euclidean space. Our main goal in Section 3 is to represent the Weyl-Heisenberg frame identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product on an LCA group.

2. φ-factorable operators

Throughout this paper, we always assume that G is a second countable LCA group, φ is a topological isomorphism on G and the notation are as in Section 1.

A function $h \in L^\infty(G)$ is said to be φ-*periodic* if $h(x\varphi(k)) = h(x)$, for every $k \in L$, $x \in G$.

Definition 2.1. *We say an operator* $U : L^2(G) \to L^p(E)$, $1 \le p \le \infty$, *is* φ*-factorable if* $U(hf) = hU(f)$, *for all* $f \in L^2(G)$ *and all* φ*-periodic* $h \in L^\infty(G)$, *where* E *is a subgroup of* G *or* $G/\varphi(L)$.

A bounded operator U is φ-factorable if and only if it commutes with modulations. More precisely, we have the following result.

Lemma 2.2. *Let* U *be a bounded operator from* $L^2(G)$ *to* $L^2(E)$, *where* E *is a subgroup of* G *or* $G/\varphi(L)$. U *is* φ*-factorable if and only if*

$$U(M_\gamma g) = M_\gamma U(g), \text{ for all } g \in L^2(G),\ \gamma \in \varphi(L)^\perp. \tag{2.1}$$

Proof. If U is φ-factorable and $\gamma \in \varphi(L)^\perp (\subseteq \hat{G} \subseteq L^\infty(G))$, then since γ is φ-periodic, (2.1) obviously holds. Conversely, assume (2.1). Then, U is φ-factorable using the facts that $\varphi(L)^\perp (= \widehat{G/\varphi(L)})$ is an orthonormal basis for $L^2(G/\varphi(L))$ and $L^\infty(G/\varphi(L)) \subseteq L^2(G/\varphi(L))$. Note that there is a one-to-one correspondence between $L^\infty(G/\varphi(L))$ and the set of all φ-periodic $h \in L^\infty(G)$. $\square$

Our main goal in this section is to characterize φ-factorable operators $U : L^2(G) \to L^p(G/\varphi(L))$, for $p = 1$ and $p = 2$.

Clearly, the operator U, defined by $U(f) = [f, g]_\varphi$, for $f \in L^2(G)$, is φ-factorable. We will also show that every φ-factorable operator $U : L^2(G) \to L^1(G/\varphi(L))$ is of this form. First, we establish a lemma in which we show that two φ-factorable operators are equal on $L^2(G)$ if and only if their integrals over $G/\varphi(L)$ are the same.

Lemma 2.3. *Let $U_1, U_2 : L^2(G) \to L^1(G/\varphi(L))$ be two φ-factorable operators. Then, $U_1 = U_2$ if and only if*

$$\int_{G/\varphi(L)} U_1(f)(\dot{x})d\dot{x} = \int_{G/\varphi(L)} U_2(f)(\dot{x})d\dot{x},$$

for every $f \in L^2(G)$.

Proof. The necessity is obvious. For the converse, by [8, Theorem 4.33], it is enough to show that $\widehat{U_1(f)} = \widehat{U_2(f)}$, for all $f \in L^2(G)$. Let $\xi \in \varphi(L)^\perp$ and $f \in L^2(G)$. Since ξ as a function in $L^\infty(G)$ is φ-periodic, we obtain:

$$\begin{aligned} \widehat{U_1(f)}(\xi) &= \int_{G/\varphi(L)} U_1(f)(\dot{x})\bar{\xi}(\dot{x})d\dot{x} \\ &= \int_{G/\varphi(L)} U_1(\xi^{-1}.f)(\dot{x})d\dot{x} \\ &= \int_{G/\varphi(L)} U_2(\xi^{-1}.f)(\dot{x})d\dot{x} \\ &= \widehat{U_2(f)}(\xi). \end{aligned}$$

Hence, $U_1 = U_2$. $\square$

Now, we have the following Riesz Representation Theorem which generalizes [5, Theorem 4.5.5] and characterizes all φ-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

Theorem 2.4. *A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is φ-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover, $\|U\| = \|g\|$.*

Proof. Let $U : L^2(G) \to L^1(G/\varphi(L))$ be a bounded φ-factorable operator. Define the linear functional $\psi : L^2(G) \to \mathbb{C}$ by

$$\psi(f) = \int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x}.$$

By the standard Riesz Representation Theorem [9, Theorem 5.25], there exists $g \in L^2(G)$ such that $\psi(f) =< f, g >_{L^2(G)}$, for all $f \in L^2(G)$. Thus, $\int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x} = \psi(f) =< f, g >_{L^2(G)}= \int_{G/\varphi(L)}[f,g]_\varphi(\dot{x})d\dot{x}$. By Lemma 2.3, $U(f) = [f,g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover, for any $f \in L^2(G)$,

$$\begin{aligned} \|U(f)\|_1 &= \int_{G/\varphi(L)} |[f,g]_\varphi(\dot{x})|d\dot{x} \\ &\leq \int_{G/\varphi(L)} \|f\|_\varphi(\dot{x})\|g\|_\varphi(\dot{x})d\dot{x} \\ &\leq (\int_{G/\varphi(L)} \|f\|^2_\varphi(\dot{x})d\dot{x})^{1/2}(\int_{G/\varphi(L)} \|g\|^2_\varphi(\dot{x})d\dot{x})^{1/2} \\ &= \|f\|_2\|g\|_2. \end{aligned}$$

So, $\|U\| \leq \|g\|_2$. Also, $\|Ug\|_1 = \int_{G/\varphi(L)} |[g,g]_\varphi(\dot{x})|d\dot{x} = \|g\|_2^2$. Therefore, $\|U\| = \|g\|_2$. □

The following theorem, which generalizes [5, Theorem 4.5.8], characterizes φ-factorable operators from $L^2(G)$ to $L^2(G/\varphi(L))$.

Theorem 2.5. *A bounded operator $U : L^2(G) \to L^2(G/\varphi(L))$ is φ-factorable if and only if there exists a φ-bounded $g \in L^2(G)$ such that $U(f) = [f,g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover,*

$$\|U\|^2 = ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|^2_\varphi(\dot{x}).$$

Proof. Let $U(f) = [f,g]_\varphi$ *a.e.*, for some φ-bounded $g \in L^2(G)$. Then, obviously U is φ-factorable and by the Cauchy-Shwartz Inequality, we have

$$\begin{aligned} \|U(f)\|^2_{L^2(G/\varphi(L))} &= \int_{G/\varphi(L)} |U(f)(\dot{x})|^2d\dot{x} \\ &= \int_{G/\varphi(L)} |[f,g]_\varphi(\dot{x})|^2d\dot{x} \\ &\leq \int_{G/\varphi(L)} \|f\|^2_\varphi(\dot{x})\|g\|^2_\varphi(\dot{x})d\dot{x} \\ &\leq ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|^2_\varphi(\dot{x})\|f\|^2_{L^2(G)}. \end{aligned} \tag{2.2}$$

Letting $f = g$ above, we get $\|U\| = ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|_\varphi(\dot{x})$.

For the converse, let U be a φ-factorable operator from $L^2(G)$ to $L^2(G/\varphi(L))$. Since $G/\varphi(L)$ is compact, $L^2(G/\varphi(L)) \subseteq L^1(G/\varphi(L))$ and so by Theorem 2.4, there exists $g \in L^2(G)$ such that $U(f) = [f,g]_\varphi$ *a.e.*, for all $f \in L^2(G)$. But, also g is φ-bounded. To show this observe that $|U(g)(\dot{x})| \leq \|U\|\|g\|_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. In fact, for every φ-periodic $h \in L^\infty(G)$, we have

$$\begin{array}{rcl}\int_{G/\varphi(L)} |h(\dot{x})|^2 |U(g)(\dot{x})|^2 d\dot{x} & = & \int_{G/\varphi(L)} |U(hg)(\dot{x})|^2 d\dot{x} \\ & = & \|U(hg)\|^2_{L^2(G/\varphi(L))} \\ & \leq & \|U\|^2 \int_G |hg(x)|^2 dx \\ & = & \|U\|^2 \int_{G/\varphi(L)} \sum_{\varphi(k)\in\varphi(L)} |hg(x\varphi(k^{-1}))|^2 d\dot{x} \\ & = & \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \sum_{\varphi(k)\in\varphi(L)} |g(x\varphi(k^{-1}))|^2 d\dot{x} \\ & = & \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \|g\|^2_{\varphi}(\dot{x}) d\dot{x},\end{array}$$

that is, $|U(g)(\dot{x})| \leq \|U\|\|g\|_{\varphi}(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. So, we get $\|g\|^2_{\varphi}(\dot{x}) = |U(g)(\dot{x})| \leq \|U\|\|g\|_{\varphi}(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. Hence, $\|g\|_{\varphi}(\dot{x}) \leq \|U\|$ *a.e.* That is, g is φ-bounded. $\square$

Next, we show that every bounded φ-factorable operator on $L^2(G)$ is adjointable.

Proposition 2.6. *Let $U : L^2(G) \to L^2(G)$ be a bounded φ-factorable operator and U^* be its adjoint. Then, U^* is φ-factorable. Moreover,*

$$[U(f), g]_{\varphi} = [f, U^*(g)]_{\varphi}, \quad \textit{a.e., for all } f, g \in L^2(G). \tag{2.3}$$

Proof. Clearly U^* is φ-factorable. Indeed, for $f, g \in L^2(G)$ and φ-periodic $h \in L^{\infty}(G)$, we have

$$\begin{array}{rcl}< U^*(hf), g >_{L^2(G)} & = & < hf, U(g) >_{L^2(G)} \\ & = & < f, \bar{h}U(g) >_{L^2(G)} \\ & = & < f, U(\bar{h}g) >_{L^2(G)} \\ & = & < U^*(f), \bar{h}g >_{L^2(G)} \\ & = & < hU^*(f), g >_{L^2(G)} .\end{array}$$

Moreover, given $f, g \in L^2(G)$, we have

$$\begin{array}{rcl}\int_{G/\varphi(L)} [U(f), g]_{\varphi}(\dot{x}) d\dot{x} & = & < U(f), g >_{L^2(G)} \\ & = & < f, U^*(g) >_{L^2(G)} \\ & = & \int_{G/\varphi(L)} [f, U^*(g)]_{\varphi}(\dot{x}) d\dot{x},\end{array}$$

which implies (2.3). $\square$

Example 2.7. *Let $G = \mathbb{R}^n$, for $n \in \mathbb{N}$. Then, $L = \mathbb{Z}^n$ is a uniform lattice in G. Let A be an invertible $n \times n$ real matrix. Define $\varphi : G \to G$ by $\varphi(x) = Ax$, for $x \in \mathbb{R}^n$. Then, for $g \in L^2(G)$, the operator U given by $U(f) = [f, g]_{\varphi}$, where $[f, g]_{\varphi}(x) = \sum_{k\in\mathbb{Z}^n} f\bar{g}(x - Ak)$, is a φ-factorable operator from $L^2(G)$ to $L^1(G/\varphi(L))$ $(= L^1(\mathbb{T}^n))$.*

Example 2.8. *Fix a prime p. Let Δ_p denote the group of p-adic integers, as defined in* [11, Definition 10.2]. *Consider the LCA group $G = \mathbb{R} \times \Delta_p$ and let L be the subgroup $\{(n, n\boldsymbol{u})\}_{n\in\mathbb{Z}}$ of $\mathbb{R} \times \Delta_p$, where $\boldsymbol{u} = (1, 0, 0, ...)$. Then, L is a uniform lattice in $\mathbb{R} \times \Delta_p$ (obviously,*

L is discrete and by [11, Theorem 10.13], *$\mathbb{R} \times \Delta_p/L$ is compact). Let $\boldsymbol{a} := (1/p, 0, 0, ...) \in \Delta_p$. Then, the mapping $\varphi : \mathbb{R} \times \Delta_p \to \mathbb{R} \times \Delta_p$, defined for $(x, \boldsymbol{v}) \in \mathbb{R} \times \Delta_p$, by $\varphi(x, \boldsymbol{v}) = (2x, \boldsymbol{av})$, is a topological isomorphism on $\mathbb{R} \times \Delta_p$. For $g \in L^2(\mathbb{R} \times \Delta_p)$, the operator U, given by $U(f)(x, \boldsymbol{v}) = \sum_{k\in\mathbb{Z}} f\bar{g}(x - 2k, \boldsymbol{v} - k\boldsymbol{au})$, is a φ-factorable operator from $L^2(\mathbb{R} \times \Delta_p)$ to $L^1(\mathbb{R} \times \Delta_p/L)$.*

The next section is devoted to an application of the φ-bracket product to the Weyl-Heisenberg systems.

3. Applications to Weyl-Heisenberg frames

In this section, we investigate the Weyl-Heisenberg frames with regard to the φ-bracket product. For general references on the Weyl-Heisenberg frames on $\mathbb{R}$, we refer to the survey articles [2, 3].

Suppose L_1 and L_2 are two uniform lattices in G, $g \in L^2(G)$ and $T_{\varphi(k)}g$ is the translation of g by $\varphi(k)$. We call $(M_\gamma T_{\varphi(k)}g)_{\gamma\in\varphi(L_2)^\perp, k\in L_1}$, *a Weyl-Heisenberg system (Gabor's system).* If this system is a frame in $L^2(G)$, we call it a *Weyl-Heisenberg frame.* In this case, the frame operator associated with it is defined to be

$$S(f) = \sum_{\gamma\in\varphi(L_2)^\perp} \sum_{k\in L_1} < f, M_\gamma T_{\varphi(k)}g > M_\gamma T_{\varphi(k)}g.$$

We would like to consider the Weyl-Heisenberg frame Identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product. The following proposition is an extension of the Weyl-Heisenberg frame Identity ([5, Theorem 4.6.2]) with regards to the φ-bracket product; see also [6].

Proposition 3.1. *Let L_1 and L_2 be two uniform lattices in G. Let $g \in L^2(G)$ be φ-bounded. Then, for every $f \in L^2(G)$ which is bounded and compactly supported, we have*

$$\sum_{k\in L_1} \sum_{\gamma\in\varphi(L_2)^\perp} | < f, M_\gamma T_{\varphi(k)}g > |^2 = \tag{3.1}$$

$$\sum_{l\in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(l^{-1})}f, f]_{\varphi,L_1}(\dot{x})[g, T_{\varphi(l^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x},$$

where, $[f, g]_{\varphi,L_i}(\dot{x}) = \sum_{k\in L_i} f\overline{g}(x\varphi(k^{-1}))$, $i = 1, 2$.

Proof. For $k \in L_1$, using the Plancherel Theorem, we have

$\sum_{\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g >|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp} |\int_G f(x)\overline{M_\gamma T_{\varphi(k)} g(x)} dx|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp} |\int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))\overline{\gamma}(x) d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp} |\hat{F}_k(\gamma)|^2$
$= \|\hat{F}_k\|^2_{L^2(\widehat{G/\varphi(L_2)})}$
$= \|F_k\|^2_{L^2(G/\varphi(L_2))}$,

where, $F_k(x) = \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))$. So, we get

$\sum_{k\in L_1} \sum_{\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g >|^2$
$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} |\sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))|^2 d\dot{x}$
$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))$
$\sum_{\varphi(m)\in\varphi(L_2)} f(x\varphi(m))\overline{g}(x\varphi(mk^{-1})) d\dot{x}$ $\quad$ *(put $m = nl$)*
$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))$
$\sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(nl))\overline{g}(x\varphi(nlk^{-1})) d\dot{x}$
$= \sum_{k\in L_1} \int_G \bar{f}(x)g(x\varphi(k^{-1})) \sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(n))\overline{g}(x\varphi(nk^{-1})) dx$
$= \sum_{n\in L_2} \int_G \bar{f}(x)f(x\varphi(n)) \sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1})) dx$
$= \sum_{n\in L_2} \int_G \bar{f}(x)f(x\varphi(n))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(x) dx$
$= \sum_{n\in L_2} \int_{G/\varphi(L_1)} \sum_{\varphi(l)\in\varphi(L_1)} \overline{f}(x\varphi(l))T_{\varphi(n^{-1})}f(x\varphi(l))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x}) d\dot{x}$
$= \sum_{n\in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(n^{-1})}f, f]_{\varphi,L_1}(\dot{x})[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x}) d\dot{x}.$

□

As a consequence of Proposition 3.1, we have the following corollary.

Corollary 3.2. *Let L_1 and L_2 be two uniform lattices in G. Let $g \in L^2(G)$ such that*

(3.2)
$$B := sup_{\dot{x}\in G/\varphi(L_1)} \sum_{k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})| < \infty, \text{ and}$$
$$A := inf_{\dot{x}\in G/\varphi(L_1)} [\|g\|^2_{\varphi,L_1}(\dot{x}) - \sum_{1_G\neq k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})|] > 0.$$

Then, $(M_\gamma T_{\varphi(k)} g)_{k\in L_1,\gamma\in\varphi(L_2)^\perp}$ is a Weyl-Heisenberg frame with bounds A and B.

Proof. Put $H_n(x) = \sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))$. Then,

$$\sum_{0\neq k_2\in L_2} |T_{\varphi(k_2)} H_{k_2}(x)| = \sum_{0\neq k_2\in L_2} |H_{k_2}(x)|.$$

Using Proposition 3.1, we have

$\sum_{k\in L_1}\sum_{\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g >|^2$
$|\sum_{0\neq n\in L_2}\int_G \bar{f}(x) f(x\varphi(n))\sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))dx|$
$\leq \sum_{0\neq n\in L_2}\int_G |f(x)|\sqrt{|H_n(x)|}\ |T_{\varphi(n^{-1})}f(x)|\sqrt{|H_n(x)|}dx$
$\leq \sum_{0\neq n\in L_2}(\int_G |f(x)|^2|H_n(x)|dx)^{1/2}(\int_G |T_{\varphi(n^{-1})}f(x)|^2|H_n(x)|dx)^{1/2}$
$\leq (\sum_{0\neq n\in L_2}\int_G |f(x)|^2|H_n(x)|dx)^{1/2}(\sum_{0\neq n\in L_2}\int_G |T_{\varphi(n^{-1})}f(x)|^2|H_n(x)|dx)^{1/2}$
$\leq (\int_G |f(x)|^2\sum_{0\neq n\in L_2}|H_n(x)|dx)^{1/2}(\int_G |f(x)|^2\sum_{0\neq n\in L_2}|T_{\varphi(n)}H_n(x)|dx)^{1/2}$
$= \int_G |f(x)|^2\sum_{0\neq n\in L_2}|H_n(x)|dx.$

Thus, by (3.2) we, get the desired inequalities:

$$A\|f\|_2^2 \leq \sum_{k\in L_1 \gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g >|^2 \leq B\|f\|_2^2.$$

□

It is useful to note also that the Weyl-Heisenberg system has the following property.

Proposition 3.3. *Let L_1 and L_2 be two uniform lattices in G. If $f, g \in L^2(G)$ and g is φ-bounded, then*

(3.3)
$$\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |< f, M_\gamma T_{\varphi(k)} g >|^2 = \sum_{k\in L_1} \|[f, T_{\varphi(k)}g]_{\varphi,L_2}\|^2_{L^2(G/\varphi(L_2))}.$$

Proof. Using the Plancherel Theorem we have the following calculations which proves (3.3):

$\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |< f, M_\gamma T_{\varphi(k)} g >|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_G f(x)\overline{T_{\varphi(k)}g}(x)\overline{\gamma}(x)dx|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_{G/\varphi(L_2)}\sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{T_{\varphi(k)}g}(x\varphi(l))\overline{\gamma}(x)d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_{G/\varphi(L_2)}[f, T_{\varphi(k)}g]_{\varphi,L_2}(\dot{x})\overline{\gamma}(\dot{x})d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\widehat{[f, T_{\varphi(k)}g]}_{\varphi,L_2}(\gamma)|^2$
$= \sum_{k\in L_1} \|\widehat{[f, T_{\varphi(k)}g]}_{\varphi,L_2}\|^2_{L^2(\widehat{G/\varphi(L_2)})}$
$= \sum_{k\in L_1} \|[f, T_{\varphi(k)}g]_{\varphi,L_2}\|^2_{L^2(G/\varphi(L_2))}.$

□

In the sequel, we will identify the frame operator of a Weyl-Heisenberg frame. For this, we need a couple of lemmas.

Lemma 3.4. *Suppose $g \in L^2(G)$ is φ-bounded and φ-periodic. Let L be a uniform lattice in G. Then,*

$$\sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > M_\gamma g = [f,g]_\varphi g \quad a.e. \quad \text{for all } f \in L^2(G), \tag{3.4}$$

where the series converges in $L^2(G)$. In particular, if $\|g\|_\varphi = 1$ a.e., and P is the orthogonal projection onto $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$, then $Pf = [f,g]_\varphi g$ a.e..

Proof. Let $f \in L^2(G)$. By (1.3), we have
$\sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > \gamma(\dot{x}) = \sum_{\gamma\in\varphi(L)^\perp} \widehat{[f,g]_\varphi}(\gamma)\gamma(\dot{x}) = [f,g]_\varphi(\dot{x})$, for a.e. $\dot{x} \in G/\varphi(L)$. Hence, (3.4) holds, where the convergence of the series in $L^2(G)$ follows from Proposition 1.1. In particular, if $\|g\|_\varphi = 1$, then $(M_\gamma g)_{\gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$. So, $Pf = \sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > M_\gamma g = [f,g]_\varphi g$ a.e.. □

Lemma 3.5. *Let L_1 and L_2 be two uniform lattices in G, $g \in L^\infty(G/\varphi(L_1))$ and $(M_\gamma T_{\varphi(k)} g)_{\gamma\in\varphi(L_1)^\perp, k\in L_2}$ be a Bessel sequence with bound B in $L^2(G)$. Then, $\|g\|^2_{\varphi,L_2} \le B$.*

Proof. Let $f \in L^2(G)$ be φ-periodic and $k \in L_2$. Then, $f \cdot T_{\varphi(k)}\overline{g} \in L^2(G/\varphi(L_1))$. Since $\varphi(L_1)^\perp$ is an orthonormal basis for $L^2(G/\varphi(L_1))$, we have

$$\begin{aligned}\sum_{\gamma\in\varphi(L_1)^\perp} |< f\cdot T_{\varphi(k)}\overline{g}, M_\gamma >|^2 &= \|f\cdot T_{\varphi(k)}\overline{g}\|^2_{L^2(G/\varphi(L_1))} \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 |g(x\varphi(k^{-1}))|^2 d\dot{x}.\end{aligned}$$

So,

$$\begin{aligned}\sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} |< f, M_\gamma T_{\varphi(k)} g >|^2 &= \sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} |< f\cdot T_{\varphi(k)}\overline{g}, M_\gamma >|^2 \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 \sum_{k\in L_2} |g(x\varphi(k^{-1})|^2 d\dot{x} \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 \|g\|^2_{\varphi,L_2}(x) d\dot{x}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} |< f, M_\gamma T_{\varphi(k)} g >|^2 \le B\|f\|^2_{L^2(G/\varphi(L_1))}. \tag{3.6}$$

Hence, (3.5) and (3.6) imply that $\|g\|^2_{\varphi,L_2} \le B$, a.e.. □

The frame operator of a Weyl-Heisenberg frame is given by the following theorem, which is a generalization of [5, Theorem 4.6.8].

Theorem 3.6. *Let L_1 and L_2 be two uniform lattices in G and $g \in L^\infty(G/\varphi(L_1))$. Suppose $(M_\gamma T_{\varphi(k)} g)_{\gamma \in \varphi(L_1), k \in L_2}$ is a Weyl-Heisenberg frame with the frame operator S. Then, S has the form*

$$S(f) = \sum_{k \in L_2} [f, T_{\varphi(k)} g]_{\varphi, L_1} T_{\varphi(k)} g, \tag{3.7}$$

where the series converges unconditionally in $L^2(G)$.

Proof. By Lemma 3.5, $T_{\varphi(k)} g$ is φ-bounded, and so we can use Lemma 3.4 to obtain:

$$\begin{aligned} S(f) &= \sum_{\gamma \in \varphi(L_1)^\perp, k \in L_2} < f, M_\gamma T_{\varphi(k)} g > M_\gamma T_{\varphi(k)} g \\ &= \sum_{k \in L_2} [f, T_{\varphi(k)} g]_{\varphi, L_1} T_{\varphi(k)} g. \end{aligned}$$

□

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R. A. Kamyabi Gol
Department of Pure Mathematics, Ferdowsi University of Mashhad, and Center of Excellence in Analysis on Algebraic Structures (CEAAS) and a senior member of the center for research in applications of wavelets, P.O.Box 1159-91775, Mashhad, Iran
Email: `kamyabi@ferdowsi.um.ac.ir`

R. Raisi Tousi
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran
Email: `raisi@um.ac.ir`