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LOWER BOUNDS OF COPSON TYPE FOR THE TRANSPOSE OF MATRICES ON WEIGHTED SEQUENCE SPACES

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ABSTRACT. Let $A = (a_{n,k})_{n,k\geq 0}$ be a non-negative matrix. Denote by $L_{w,p,q}(A)$, the supremum of those L, satisfying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} a_{n,k} x_k\right)^q\right)^{\frac{1}{q}} \ge L \left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}}$$

where, $x \ge 0$ and $x \in l_p(w)$ and also $w = (w_n)$ is a decreasing, nonnegative sequence of real numbers. If p = q, then we use $L_{w,p}(A)$ inested of $L_{w,p,p}(A)$. Here, we focus on the evaluation of $L_{w,p}(A^t)$ for a lower triangular matrix A, where, 0 . In particular,we apply our results to summability matrices, weighted mean matrices, Nörlund matrices. Our results also generalize some resultsin Chen and Wang [C.-P. Chen and K.-Z. Wang, J. Math. Anal.Appl.**341**(2008) 1284-1294.], Foroutannia and Lashkaripour [D.Foroutannia and R. Lashkaripour, Lobachevskii J. Math.**27**(2007)15-29.], and Lashkaripour and Foroutannia [R. Lashkaripour andD. Foroutannia, J. Sci. Islam. Repub. Iran**18**(2007) 49-56.].

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1. Introduction

Let $p \in \mathbb{R} \setminus \{0\}$ and let $l_p(w)$ denote the space of all real sequences x = $\{x_k\}_{k=0}^{\infty}$ such that $\|x\|_{w,p} := (\sum_{k=0}^{\infty} w_k x_k^p)^{1/p} < \infty$, where, $w = (w_n)_{n=0}^{\infty}$ is a decreasing, non-negative sequence of real numbers with $\sum_{n=0}^{\infty} \frac{w_n}{n+1} =$ ∞ with $w_0 = 1$.

We write $x \ge 0$ if $x_k \ge 0$, for all k. We also write $x \uparrow$ for the case that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$. The symbol $x \downarrow$ is defined in a similar way. For $p, q \in \mathbb{R} \setminus \{0\}$, the lower bound involved here is the number $L_{w,p,q}(A)$, which is defined as the supremum of those L obeying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} a_{n,k} x_k\right)^q\right)^{\frac{1}{q}} \ge L \left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}}, \qquad (x \ge 0, x \in l_p(w)),$$

where, $A \ge 0$, that is, $A = (a_{n,k})_{n,k\ge 0}$ is a non-negative matrix. We have $L_{w,p,q}(A) \le ||A||_{w,p,q}.$

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In [3], the author obtained $L_{w,p}(C(1)^t) = p, (0 , where, <math>(.)^t$ denotes the transpose of (.) and $C(1) = (a_{n,k})_{n,k \ge 0}$ is the Cesaro matrix defined by

$$a_{n,k} = \begin{cases} \frac{1}{n+1} & 0 \le k \le n\\ 0 & \text{otherwise.} \end{cases}$$

This is an analogue of Copson's result [2, Eq. (1.1)] (see also [4], Theorem 344) for weighted sequence space $l_p(w)$ and has been generalized by Foroutannia [3]. He extended it in [3, Theorem 2.7.17 and Theorem 2.7.19 to those summability matrices A, whose rows are increasing or decreasing. Also, he gave upper bounds or lower bounds for $L_{w,p}(A)$, for such A. For the case of Hausdorff matrices, the related result with 0 has been established in [3, Theorem 4.3.2], giving a Hardytype formula for $L_{w,p}(H^t_{\mu})$.

Obviously, the lower bound problems of Copson type for the weighted mean matrices, $(A_W^{WM}) = (a_{n,k})_{n,k\geq 0}$, and the Nörlund matrices, (A_W^{NM}) $=(b_{n,k})_{n,k\geq 0}$, or more generally for the summability matrices on weighted sequence spaces are still less satisfactory (cf. [1, problem 4.20]), where

the weighted mean matrices and the Nörlund matrices are defined as:

$$a_{n,k} = \begin{cases} \frac{w'_n}{W'_n} & 0 \le k \le n\\ 0 & \text{otherwise,} \end{cases}$$

and

Then,

$$b_{n,k} = \left\{ \begin{array}{ll} \frac{w_{n-k}'}{W_n'} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{array} \right.$$

Here, $W'_n = \sum_{k=0}^n w'_k$, and $w' = (w'_n)$ is a non negative sequence with $w'_0 > 0$.

Here we are concerned with the problem of finding $L_{w,p}(A^t)$ and $L_{w,p^*}(A)$ (see Theorem 2.3), where, $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$ and A is a non-negative lower triangular matrix. Our result gives a lower estimate for these two values in terms of the constant M, defined by:

(1.1)
$$a_{n,k} \le M a_{n,j}, \ (0 \le k \le j \le n).$$

Here, $M \ge 1$. We shall assume that M is the smallest value appearing in (1.1). If (1.1) is not satisfied, then we set $M = \infty$. As a consequence, we prove that Theorem 2.3 generalizes some works of Lashkaripour and Foroutannia ([3], pp.53-54). Also, we obtain lower estimate and upper estimate for the weighted mean matrix and the Nörlund matrix in some cases.

2. Main Result

The purpose of this section is to establish general lower bounds for $L_{w,p}(A^t)$ and $L_{w,p^*}(A)$, where, $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$ and A is a non-negative lower triangular matrix. First, we generalize Lemma 5.2 of [2] to the weighted sequence space $l_p(w)$.

Lemma 2.1. Suppose that $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$ and $N \in \mathbb{N}$. Let $C_N^1 = (c_{n,k}(N))_{n,k\geq 0}$ be the matrix with entries

$$c_{n,k}(N) = \begin{cases} \frac{1}{n+N} & 0 \le k < n+N \\ 0 & k \ge n+N. \end{cases}$$
$$L_{w,p}\left((C_N^1)^t\right) = L_{w,p^*}\left(C_N^1\right) = p.$$

Moreover, for $r \in \mathbb{N}$ and $r > \max\{N-2, \frac{1}{p}\}$, there exists a sequence $\{x_N^m\}_{m=0}^{\infty}$ such that $x_N^m = (0, ..., 0, x_{r-N+1}^m, ...) \ge 0, \ x_{r-N+1}^m \ge x_{r-N+2}^m \ge 0$

 $\dots, \|x_N^m\|_{w,p} = 1$, for all m, and also

$$\lim_{m \to \infty} \|x_N^m\|_{w,1} = 0 \ , \lim_{m \to \infty} \left\| \left(C_N^1 \right)^t x_N^m \right\|_{w,p} = p.$$

Proof. Applying Proposition 2.5 of [6], it suffices to prove the case $L_{w,p}\left((C_N^1)^t\right) = p$. For $x \ge 0$, we have

$$\left\| \left(C_{N}^{1} \right)^{t} x \right\|_{w,p} = \left\| C(1)^{t} x' \right\|_{w,p},$$

where, $x' = \{x'_k\}_{k=0}^{\infty}$ is defined by

(2.1)
$$x'_{k} = \begin{cases} 0 & 0 \le k < N-1, \\ x_{k-N+1} & k \ge N-1. \end{cases}$$

This implies that $L_{w,p}\left((C_N^1)^t\right) \geq L_{w,p}\left(C(1)^t\right) = p$. For the rest of the proof, it suffices to prove the existence of $\{x_N^m\}_{m=0}^{\infty}$, for $r \in \mathbb{N}$, with $r > max\{N-2, \frac{1}{p}\}$. Choose a sequence, say $\{\rho_m\}_{m=0}^{\infty}$, such that $\rho_0 \leq r$ and $\rho_m \downarrow \frac{1}{p}$. Define $x_N^m = \{x_k^m\}_{k=0}^{\infty}$ by

$$x_k^m = \begin{cases} 0 & 0 \le k < r - N + 1, \\ (\phi(\rho_m))^{-1} & \binom{k+N-1-\rho_m}{k+N-1-r} / \binom{k+N-1}{r} & k \ge r - N + 1, \end{cases}$$

where,

$$\phi(t) = \left(\sum_{k=r-N+1}^{\infty} w_k \left\{ \binom{k+N-1-t}{k+N-1-r} \middle/ \binom{k+N-1}{r} \right\}^p \right)^{\frac{1}{p}}.$$

We have $x_N^m = (0, ..., 0, x_{r-N+1}^m, ...) \ge 0$, $x_k^m \downarrow$, for all $k \ge r - N + 1$, and $||x_N^m||_{w,p} = 1$, for all m. Applying ([7, Vol.I], p.77, Eq. (1.15)), we have

$$\binom{k+N-1-\rho_m}{k+N-1-r} / \binom{k+N-1}{r} \sim \frac{\Gamma\left(r+1\right)}{\Gamma\left(r-\rho_m+1\right)} (k+N-1-r)^{-\rho_m}, \quad as \ k \to \infty$$

Since $\rho_m \downarrow, \frac{1}{p}$ and $\frac{1}{p} > 1$, it follows from the monotone convergence theorem that $\lim_{m\to\infty} \phi(\rho_m) = \infty$. Moreover, there exists a constant C such that

$$\limsup_{m \to \infty} \sum_{k=r-N+1}^{\infty} w_k \left\{ \binom{k+N-1-\rho_m}{k+N-1-r} \right\} \left| \binom{k+N-1}{r} \right| \le C \sum_{n=1}^{\infty} w_n n^{-1/p} < \infty.$$

So, $\lim_{m\to\infty} \|x_m^N\|_{w,1} = 0$. We know that C(1) is the same as the Hausdorff matrix H_{μ} with $d\mu(\theta) = d\theta$. By modifying the argument given in ([3], pp. 80-81), we can prove that

$$||(C_N^1)^t x_N^m||_{w,p} = ||C(1)^t (x_N^m)'||_{w,p} \to p, \quad as \quad m \to \infty,$$

where, $(x_N^m)'$ is obtained from x_N^m by means of (2-1). This completes the proof of the lemma

In the following lemma, we extend Lemma 2.1 from matrix C_N^1 to general matrix C_N^l , with $l \in \mathbb{N}$.

Lemma 2.2. Suppose that $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$ and $l, N \in \mathbb{N}$. Let $C_N^l = (c_{n,k}^l)_{n,k\geq 0}$ be the matrix with

$$c_{n,k}^{l} = \begin{cases} \frac{1}{n+N} & 0 \le k < n+N-l + \\ 0 & k \ge n+N-l+1. \end{cases}$$

Then,

$$L_{w,p}\left((C_N^l)^t\right) = L_{w,p^*}\left(C_N^l\right) \le p.$$

Moreover, the following two assertions hold: (i) For $l \leq N$ and $x \geq 0$ with $x \downarrow$, we have

(2.2)
$$\left\| (C_N^l)^t x \right\|_{w,p}^p \leq \left\| C(1)^t x' \right\|_{w,p}^p \leq \left\| (C_N^l)^t x \right\|_{w,p}^p + \frac{l^p (l+1)}{N^p} \left\| x \right\|_{w,p}^p,$$

where, $x' = \{x'_k\}_{k=0}^{\infty}$ is defined by (2.1). (ii) There exists a sequence $\{x_N\}_{N=0}^{\infty}$ such that $x_N \ge 0, x_N \downarrow$, $\|x_N\|_{w,p} = 1$, and also

$$\lim_{N \to \infty} \|x_N\|_{w,1} = 0, \lim_{N \to \infty} \left\| (C_N^l)^t x_N \right\|_{w,p} = p.$$

Proof. For $x \ge 0$, $\|(C_N^l)^t x\|_{w,p}^p \le \|(C_N^1) x\|_{w,p}^p$. Applying Lemma 2.1, we have we have

$$L_{w,p}((C_N^l)^t) \le L_{w,p^*}((C_N^1)^t) = p.$$

The left side in (2.2) follows from the observation,

$$\|(C_N^l)^t x\|_{w,p}^p \le \|(C_N^1)^t) x\|_{w,p}^p = \|C(1)^t x'\|_{w,p}^p \quad (x \ge 0).$$

Hence, to prove (i) it is suffices to show the right side of (2-2). Assume that $l \leq N, x \geq 0$ and $x \downarrow$. Applying definition of x'_k , we get

(2.3)
$$\|C(1)^{t}x'\|_{w,p}^{p} = \sum_{k=0}^{N-1} w_{k} \left(\sum_{n=N-1}^{\infty} \frac{x'_{n}}{n+1}\right)^{p} + \sum_{k=N}^{\infty} w_{k} \left(\sum_{n=k}^{\infty} \frac{x'_{n}}{n+1}\right)^{p}$$
$$\leq \sum_{k=0}^{N} w_{k} \left(\sum_{n=0}^{\infty} \frac{x_{n}}{n+N}\right)^{p} + \sum_{k=N+1}^{\infty} w_{k} \left(\sum_{n=k-N+1}^{\infty} \frac{x_{n}}{n+N}\right)^{p}$$
$$= \Sigma_{1} + \Sigma_{2}.$$

We know that $a^p + b^p \ge (a+b)^p$, for all $a, b \ge 0$. Hence,

(2.4)
$$\Sigma_{1} \leq \sum_{k=0}^{N-l} w_{k} \left(\sum_{n=0}^{\infty} c_{n,k}^{l} x_{n} \right)^{p} + \sum_{k=N-l+1}^{N} w_{k} \left\{ \left(\sum_{n=0}^{k-N+l-1} \frac{x_{n}}{n+N} \right)^{p} + \left(\sum_{n=k-N+l}^{\infty} c_{n,k}^{l} x_{n} \right)^{p} \right\}.$$

The monotonicity of x_n implies that $\sum_{n=0}^{k-N+l-1} \frac{x_n}{n+N} \leq \binom{l}{N} x_0$, for all $N-l < k \leq N$. Inserting this into (2.4), yields:

(2.5)
$$\Sigma_1 \leq \sum_{k=0}^{N-l} w_k \left(\sum_{n=0}^{\infty} c_{n,k}^l x_n \right)^p + \frac{l^{p+1} x_0^p}{N^p} + \sum_{k=N-l+1}^N w_k \left(\sum_{n=0}^{\infty} c_{n,k}^l x_n \right)^p.$$

In the same way as in (2.4), one can show

(2.6)

$$\Sigma_{2} \leq \sum_{k=N+1}^{\infty} w_{k} \left\{ \left(\sum_{n=k-N+1}^{k-N+l-1} \frac{x_{n}}{n+N} \right)^{p} + \left(\sum_{n=k-N+l}^{\infty} c_{n,k}^{l} x_{n} \right)^{p} \right\}$$

$$\leq \frac{l^{p}}{N^{p}} \sum_{k=N+1}^{\infty} w_{k} x_{k-N+1}^{p} + \sum_{k=N+1}^{\infty} w_{k} \left(\sum_{n=0}^{\infty} c_{n,k}^{l} x_{n} \right)^{p}.$$

Putting (2.3), (2.5) and (2.6) together, yields:
$$\|C(1)^{t}x'\|_{w,p}^{p} \leq \left\| \left(C_{N}^{l}\right)^{t}x \right\|_{w,p}^{p} + \frac{l^{p}(l+1)}{N^{p}} \|x\|_{w,p}^{p}$$

This completes the proof of (i). (ii). Let $x_0 = x_1 = \dots = x_{\left\lfloor \frac{1}{p} \right\rfloor + 1} = e_0$, where, $e_0 = (1, 0, 0, \dots)$. For each $N > \frac{1}{p} + 1$, it follows from the case r = N - 1 of Lemma 2.1

that there exist x_N with the properties: $x_N \ge 0$, $x_N \downarrow$, $||x_N||_{w,p} = 1$, $||x_N||_{w,1} \le \frac{1}{N}$ and

$$p - \frac{1}{N} \le \left\| \left(C_N^l \right)^t x_N \right\|_{w,p} \le p + \frac{1}{N}.$$

Obviously,

$$\lim_{N \to \infty} \|x_N\|_{w,1} = 0, \lim_{N \to \infty} \left\| \left(C_N^l \right)^t x_N \right\|_{w,p} = p.$$

Applying (2.2), we get

$$\left\| \left(C_{N}^{l} \right)^{t} x_{N} \right\|_{w,p}^{p} \leq \left\| C(1)^{t} x_{N}^{\prime} \right\|_{w,p}^{p} = \left\| \left(C_{N}^{1} \right)^{t} x_{N} \right\|_{w,p}^{p}$$
$$\leq \left\| \left(C_{N}^{l} \right)^{t} x_{N} \right\|_{w,p}^{p} + \frac{l^{p}(l+1)}{N^{p}}. \quad (N \geq l)$$

Making $N \to \infty$, it follows that

$$\lim_{N \to \infty} \left\| \left(C_N^l \right)^t x_N \right\|_{w,p} = \lim_{N \to \infty} \left\| \left(C_N^1 \right)^t x_N \right\|_{w,p}^p = p.$$

This completes the proof.

Note that, in general, $L_{w,p}((C_N^l)^t) \neq p$. In fact, we have $L_{w,p}((C_N^N)^t) \leq \frac{1}{N} < p$, if $N > \frac{1}{p}$. One can see this by considering the definition of C_N^N .

Theorem 2.3. Let $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$ and $A = (a_{n,k})_{n,k\geq 0}$ be a lower triangular matrix with $A \geq 0$. Then,

(2.7)
$$pM^{p-1}(\inf_{n\geq 0}\sum_{k=0}a_{n,k})\leq L_{w,p}(A^t)$$

Also, the same inequality holds, if $L_{w,p}(A^t)$ is replaced by $L_{w,p^*}(A)$. Here, M is defined by (1.1).

Proof. Applying Proposition 4.3.6 of [3], we have $L_{w,p}(A^t) = L_{w,p^*}(A)$, and so it suffices to prove (2.7). Let $x \ge 0$ with $||x||_{w,p} = 1$. Since

p-1<0, from Lemma 2.7.18 of [3] with (1.1) and Fubini's theorem, it follows that:

$$\|A^{t}x\|_{w,p}^{p} = \sum_{k=0}^{\infty} w_{k} \left(\sum_{n=k}^{\infty} a_{n,k}x_{n}\right)^{p}$$

$$\geq p\left\{\sum_{k=0}^{\infty} w_{k}\sum_{j=k}^{\infty} a_{j,k}x_{j} \left(\sum_{n=j}^{\infty} a_{n,k}x_{n}\right)^{p-1}\right\}$$

$$\geq pM^{p-1}\sum_{k=0}^{\infty} w_{k}\sum_{j=k}^{\infty} a_{j,k}x_{j} \left(\sum_{n=j}^{\infty} a_{n,j}x_{n}\right)^{p-1}$$

$$\geq pM^{p-1}\sum_{j=0}^{\infty} w_{j}x_{j} \left(\sum_{n=j}^{\infty} a_{n,j}x_{n}\right)^{p-1} \left(\sum_{k=0}^{j} a_{j,k}\right)$$

$$\geq pM^{p-1} \left(\inf_{j\geq 0}\sum_{k=0}^{j} a_{j,k}\right) \left\{\sum_{j=0}^{\infty} w_{j}x_{j} \left(\sum_{n=j}^{\infty} a_{n,j}x_{n}\right)^{p-1}\right\}.$$

Applying Hölder's inequality, we deduce that

$$\begin{split} \sum_{j=0}^{\infty} w_j x_j \left(\sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} &= \sum_{j=0}^{\infty} w_j^{\frac{1}{p}} x_j \left(w_j^{\frac{1}{p^*(p-1)}} \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{p-1} \\ &\geq \left(\sum_{j=0}^{\infty} w_j x_j^p \right)^{\frac{1}{p}} \left(\sum_{k=0}^{\infty} \left(w_k^{\frac{1}{p}} \sum_{j=k}^{\infty} a_{j,k} x_j \right)^p \right)^{\frac{1}{p^*}} \\ &= \| x \|_{w,p} \| A^t x \|_{w,p}^{p-1} . \end{split}$$

Inserting this estimate into the corresponding term in (2.8), gives

$$||A^t x||_{w,p} \ge p M^{p-1} \left(\inf_{j \ge 0} \sum_{k=0}^j a_{j,k} \right) ||x||_{w,p}.$$

This leads us to the lower estimate in (2.7).

Theorem 2.3 has some applications. For example, consider the weighted mean matrix, say (A_{Wl}^{WM}) , associated with the sequence $W' = (w'_n)_{n=0}^{\infty}$, where, $l = 0, 1, 2, \dots, w'_0 = w'_1 = \dots = w'_l = 1$ and $w'_n = \frac{1}{2}$, for n > l. Applying inequality (2.7) for M = 2, we have

$$L_{w,p}((A_{W l}^{WM})^t) \ge p2^{p-1}.$$

Next, consider the Nörlund matrix (A_W^{NM}) , where, $w' = (w'_n)_{n=0}^{\infty}$ is a non-negative sequence with $w'_0 > 0$ and $W'_n = \sum_{k=0}^n w'_k$. If $w'_n \downarrow$, then

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M = 1. Applying (2.7), we deduce that

$$L_{w,p}((A_W^{NW})^t) \ge p.$$

In general, for the summability matrix A (see [1]), with increasing rows M = 1, we observe that (2.7) has the following form:

(2.9)
$$p \le L_{w,p}(A^t) = L_{w,p^*}(A).$$

Inequality (2.9) is an analogue of ([4], Theorem 4.2), obtained by a different way.

Theorem 2.4. Let $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_0 > 0$ and $w'_n \ge 0$, for all $n \ge 1$ and $also \lim_{n \to \infty} W'_n = \infty$. Then, the following assertions are true:

$$(i) \ L_{w,p}\left(\left(A_{W}^{NM}\right)^{t}\right) = L_{w,p^{*}}\left(A_{W}^{NM}\right) \leq p\left(\lim_{l \to \infty} K(l)\right),$$

where, $K(l) := \sup_{n \geq 0, \ N \geq l, \ l \leq k \leq n+N} \frac{(n+N+1)w'_{k}}{W'_{n+N}}.$
$$(ii) \ L_{w,p}\left(\left(A_{W}^{WM}\right)^{t}\right) = L_{w,p^{*}}\left(A_{W}^{WM}\right) \leq p\left(\lim_{l \to \infty} k(l)\right),$$

where, $k(l) := \sup_{n \geq 0, \ l \leq k \leq n} \frac{(n+1)w'_{k}}{W'_{n}}.$

Obviously, $k(l) \leq K(l)$, for all $l \geq 1$. Since $k(l) \downarrow$ and $K(l) \downarrow$, then the limits in (i) and (ii) can be replaced by $\inf_{l \in \mathbb{N}}$. We have

$$K(l) \leq \left(\sup_{n \geq l} w'_n\right) / \left(\inf_{n \geq l} \frac{W'_n}{n+1}\right).$$

Proof. Let x_N and x'_N be defined as in Lemma 2.2. Since $a^p + b^p \ge (a+b)^p$, for all $a, b \ge 0$, we deduce that

(2.10)
$$\left\| \left(A_W^{NM} \right)^t x'_N \right\|_{w,p}^p \le \left\| \left(A_1^l \right)^t x'_N \right\|_{w,p}^p + \left\| \left(A_2^l \right)^t x'_N \right\|_{w,p}^p \quad (N \ge 0),$$

where, $A_2^l = A_W^{NM} - A_1^l$ and $A_1^l = (a_{n,k})_{n,k \ge 0}$ is the matrix obtained from A_W^{NW} by replacing the (n, k)th entry of A_W^{NW} with 0, for all n, k, with $n-l < k \le n$. Consider $N \ge l+1$. Obviously, $a_{n+N-1,k} \le K(l)/n + N$, for $0 \le k < n+N-l$, and $a_{n+N-1,k} = 0$, for $k \ge n+N-l$. This implies

that

(2.11)
$$\left\| \left(A_1^l \right)^t x_N' \right\|_{w,p}^p \le K(l)^p \left\| \left(C_N^l \right)^t x_N \right\|_{w,p}^p$$

On the other hand, it follows from the definition of A_2^l that

(2.12)
$$\left\| \left(A_2^l \right)^t x_N' \right\|_{w,p}^p \le l \left(\frac{\max\{w_0', w_1', \dots, w_{l-1}'\}}{W_{N-1}'} \right)^p \|x_N\|_{w,p}^p.$$

Putting (2.10) and (2.11) together with (2.12), yields:

$$\left\| \left(A_W^{NM} \right)^t x'_N \right\|_{w,p}^p \le \left(K(l) \right)^p \left\| \left(C_N^l \right)^t x_N \right\|_{w,p}^p$$
$$+ l \left(\frac{\max\{w'_0, w'_1, \dots, w'_{l-1}\}}{W'_{N-1}} \right)^p \|x_N\|_{w,p}^p.$$

We have $||x_N||_{w,p} = 1$ and $W'_N \to \infty$, as $N \to \infty$, and applying Lemma 2.2(*ii*), we get $L_{w,p}((A_W^{NM})^t) \leq pK(l)$. Hence,

$$L_{w,p}((A_W^{NM})^t) \le p(\inf_{l \in \mathbb{N}} K(l)) = p \lim_{l \to \infty} K(l).$$

This proves (i).

Now, consider (*ii*). Let $\{x_N^m\}_{m=0}^{\infty}$ be the corresponding sequence given in Lemma 2.2. Similar to A_W^{NM} , write $A_W^{WM} = A_1^l + A_2^l$, where, A_1^l is the matrix obtained from A_W^{WM} by replacing the (n, k)th entry of A_W^{WM} with 0, for all $n \ge 0$ and $0 \le k < l$. As seen above, one can easily derive:

$$\begin{split} \left\| \left(A_{W}^{WM} \right)^{t} (x_{N}^{m})' \right\|_{w,p}^{p} &\leq \\ \left\| (A_{1}^{l})^{t} (x_{N}^{m})' \right\|_{w,p}^{p} + \\ \left\| (A_{2}^{l})^{t} (x_{N}^{m})' \right\|_{w,p}^{p} \\ &\leq \\ (k(l))^{p} \left\| \left(C_{N}^{1} \right) x_{N}^{m} \right\|_{w,p}^{p} \\ &+ \\ l (\frac{\max\{w_{0}', w_{1}', \dots, w_{l-1}'\}}{W'})^{p} \left\| x_{N}^{m} \right\|_{w,p}^{p}, \end{split}$$

which gives $L_{w,p}(A_W^{WM})^t) \leq pk(l)$, for all $l \in \mathbb{N}$. Therefore,

$$L_{w,p}((A_W^{WM})^t) \le p(\inf_{l \in \mathbb{N}} k(l)) = p \lim_{l \to \infty} k(l).$$

This completes the proof of the (ii).

Applying (2.9) for the summability matrix A, with increasing rows, we have

$$p \le L_{w,p}(A^t) = L_{w,p^*}(A).$$

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Also, applying Theorem 2.4(i), we deduce the following corollaries.

Corollary 2.5. Let $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \downarrow \alpha$ and $\alpha > 0$. Then, $L_{w,p}((A_W^{NM})^t) = L_{w,p^*}((A_W^{NM})) = p$.

Remark 2.6. The case $\alpha = 0$ in Corollary 2.5 is false. In general, a counterexample is the Nörlund matrix (A_W^{NM}) , where, $w'_0 = 1$, $w'_n \downarrow 0$, $\inf_{k\geq 0} \frac{w'_0}{w'_0+\ldots+w'_k} > p$. For this matrix, $\alpha = 0$, but



In ([5], Theorem 4.1), the upper bound of $L_{w,p}(A^t)$ is established for those summability matrices A, whose rows are decreasing, where, such matrices, $L_{w,p}(A^t) \leq p$. For this of type matrix, applying (2.7), we have

$$pM^{p-1} \le L_{w,p}(A^t) \le p.$$

Also, we have the following results for particular cases of such matrices.

Corollary 2.7. Let $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \downarrow \alpha$ and $\alpha \ge 0$. Then,

$$p\left(\frac{w_0}{\alpha}\right)^{p-1} \le L_{w,p}\left(\left(A_W^{WM}\right)^t\right) = L_{w,p^*}\left(A_W^{WM}\right) \le p.$$

Corollary 2.8. Let $0 , <math>\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \uparrow \alpha$ and $w'_0 > 0$. Then,

$$p\left(\frac{\alpha}{w'_0}\right)^{p-1} \le L_{w,p}\left(\left(A_W^{NM}\right)^t\right) = L_{w,p^*}\left(A_W^{NM}\right) \le p.$$

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