*Bulletin of the Iranian Mathematical Society Vol.* 37 *No. 1* (*2011*), *pp 143-159.*

# A COMPOSITE EXPLICIT ITERATIVE PROCESS WITH A VISCOSITY METHOD FOR LIPSCHITZIAN SEMIGROUP IN A SMOOTH BANACH SPACE

P. KATCHANG AND P. KUMAM*

Communicated by Tony Lau

Abstract. We introduce a new explicit composite iteration scheme with a viscosity iteration method for approximating a common fixed point of Lipschitzian semigroup on a compact convex subset of a smooth Banach space. We show that the iterative sequence converges strongly to a common fixed point under some parameter controlling conditions. Our results extend and improve the recent results by Saeidi [S. Saeidi, *Fixed Point Theory Appl.* (2008) Art. ID 363257 17pp.], Zhang et al. [S.-S. Zhang, L. Yang and J.-A. Liu, *Appl. Math. Mech. (English Ed.)* **28** (2007) 1287–1297.] and several others.

## 1. Introduction

Let $E$ be a Banach space and let $E^*$ be the topological dual of $E$. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\}.$$

Using the Hahn-Banach theorem, it immediately follows that $J(x) \neq \emptyset$, for each $x \in E$. A Banach space $E$ is said to be smooth if the duality

MSC(2010): Primary: 47H10; Secondary: 47H09, 43A07, 47H20, 47J20.
Keywords: Nonexpansive mapping, fixed point, amenable semigroup, iteration, left reversible.
Received: 28 May 2009, Accepted: 20 December 2009.
*Corresponding author

mapping $J$ of $E$ is single valued. We know that if $E$ is smooth, then $J$ is norm to weak-star continuous; see [7, 20].

Let $C$ be a nonempty closed convex subset of $E$. A mapping $T : C \to C$ is said to be
(i) *Lipschitzian* with Lipschitz constant $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C;$$

(ii) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in C.$$

Recall that a self mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|.$$

Clearly, every nonexpansive mapping $T$ is asymptotically nonexpansive with sequence $\{1\}$. Also, every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian with $L = \sup_{n\in\mathbb{N}} k_n$.

In 1953, Mann [9] introduced an iterative process as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.1}$$

where, the initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [10]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [2]. Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Saeidi [16] considered an implicit iteration process of Mann's type for (asymptotically) quasi-nonexpansive affine mappings in normed and Banach spaces and prove the weak and strong convergence of the process to a fixed point of the mappings.

On the other hand, let $C$ be a nonempty closed convex subset of a Banach space $E$. Then, $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of Lipschitzian mappings* from $C$ into itself if it satisfies the following conditions:

(i) for each $s > 0$, there exists a function $k(\cdot) : (0, \infty) \to (0, \infty)$ such that

$$\|T(s)x - T(s)y\| \le k(s)\|x - y\|, \quad \forall x, y \in C;$$

(ii) $T(0)x = x$ for each $x \in C$;
(iii) $T(s_1 + s_2)x = T(s_1)T(s_2)x$ for any $s_1, s_2 \in \mathbb{R}^+$ and $x \in C$;
(iv) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into $C$ is continuous.

If $k(s) = L$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of uniformly L-Lipschitzian mappings*. If $k(s) = 1$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of nonexpansive mappings* (see [12]). For a semigroup $S$, we can define a partial preordering $\prec$ on $S$ by $a \prec b$ if and only if $aS \supset bS$. If $S$ is a *left reversible semigroup* (i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$), then it is a directed set. (Indeed, for every $a, b \in S$, applying $aS \cap bS \neq \emptyset$, there exist $a', b' \in S$ with $aa' = bb'$; by taking $c = aa' = bb'$, we have $cS \subseteq aS \cap bS$, and then $a \prec c$ and $b \prec c$.) If a semigroup $S$ is left amenable, then $S$ is left reversible [5].

Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup $S$ as Lipschitzian mappings on $C$ with Lipschitz constants $\{k(s) : s \in S\}$. We shall say that $\mathcal{S}$ is an *asymptotically nonexpansive semigroup* on $C$, if there holds the uniform Lipschitzian condition $\lim_s k(s) \le 1$ on the Lipschitz constants. (Note that a left reversible semigroup is a directed set.) It is worth mentioning that there is a notion of asymptotically nonexpansive defined depending on left ideals in a semigroup in [4] and [6].

In 2007, Lau et al. [8] introduced the following Mann's explicit iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad \forall n \ge 1 \tag{1.2}$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset $C$ of a smooth and strictly convex Banach space.

Extending the above results to the nonexpansive semigroup case, Zhang et al. [17] introduce the following composite iteration scheme:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.3}$$

where, $\{T(t) : t \ge 0\}$ is a nonexpansive semigroup from $C$ to $C$, $u$ is an arbitrary (but fixed) element in $C$, $\{\alpha_n\} \subset (0, 1)$ *and* $\{\beta_n\} \subset [0, 1], \{t_n\} \subset \mathbb{R}^+$, and proved some strong convergence theorems of explicit composite iteration scheme for nonexpansive semigroups in the framework of a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly smooth Banach space and uniformly convex Banach space with a weakly continuous normalized duality mapping.

Saeidi [13] introduced the following viscosity iterative scheme,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \forall n \geq 1 \tag{1.4}$$

for a representation of $S$ as Lipschitzian mappings on a compact convex subset $C$ of a smooth Banach space $E$ with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$; for some related results, we refer the readers to [7, 20].

Here, motivated and inspired by the idea of Zhang et al. [17] and Saeidi [13], we introduce the composite explicit viscosity iterative schemes as follows:

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)T(\mu_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \geq 1 \end{cases} \tag{1.5}$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ on a compact convex subset $C$ of a smooth Banach space $E$ with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$. Then, we prove that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$, which is the unique solution of the variational inequality,

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

Equivalently, we have $z = Pfz$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$. Our results improve and extend the recent results of Saeidi [13] and Zhang Shi-Sheng et al. [17] to Lipschitzian semigroup mapping.

## 2. Preliminaries

Let $E$ be a Banach space and let $C$ be a closed convex subset of $E$. Then,

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in E$ and $\lambda \in [0,1]$.

Let $S$ be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on $S$ with the supremum norm. For each $s \in S$, we define $l_s$ and $r_s$ on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$

containing 1 and let $X^*$ be its topological dual. An element $\mu$ of $X^*$ is said to be a mean on $X$ if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$, instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let $X$ be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$), for each $s \in S$. A mean $\mu$ on $X$ is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. $X$ is said to be left (resp. right) *amenable* if $X$ has a left (resp. right) invariant mean. $X$ is amenable if $X$ is both left and right amenable. A net $\{\mu_\alpha\}$ of means on $X$ is said to be *strongly left regular* if

$$\lim_{\alpha} \|l_s^* \mu_\alpha - \mu_\alpha\| = 0$$

for each $s \in S$, where $l_s^*$ is the adjoint operator of $l_s$. Let $C$ be a nonempty closed and convex subset of $E$. Throughout this paper, $S$ will always denote a semigroup with an identity $e$. $S$ is called left reversible if any two right ideals in $S$ have nonvoid intersection, i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, we can define a partial ordering $\prec$ on $S$ by $a \prec b$ if and only if $aS \supset bS$. It is easy to see $t \prec ts, (\forall t, s \in S)$. Furthermore, if $t \prec s$, then $pt \prec ps$ for all $p \in S$. If a semigroup $S$ is left amenable, then $S$ is left reversible. But the converse is not true.

$\mathcal{S} = \{T(s) : s \in S\}$ is called a representation of $S$ as Lipschitzian mappings on $C$ if for each $s \in S$, the mapping $T(s)$ is Lipschitzian mapping on $C$ with Lipschitz constant $k(s)$, and $T(st) = T(s)T(t)$ for $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, and by $C_a$ the set of almost periodic elements in $C$, i.e., all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of $E$. We will call a subspace $X$ of $l^\infty(S)$, $\mathcal{S} -$ *stable* if the functions $s \mapsto \langle T(s)x, x^* \rangle$ and $s \mapsto \|T(s)x - y\|$ on $S$ are in $X$ for all $x, y \in C$ and $x^* \in E^*$. We know that if $\mu$ is a mean on $X$ and if for each $x^* \in E^*$, the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in $X$ and $C$ is weakly compact, then there exists a unique point $x_0$ of $E$ such that

$$\mu_s \langle T(s)x, x^* \rangle = \langle x_0, x^* \rangle$$

for each $x^* \in E^*$. We denote such a point $x_0$ by $T(\mu)x$. Note that $T(\mu)z = z$ for each $z \in F(\mathcal{S})$; see [3, 14, 19]. Let $D$ be a subset of $B$, where $B$ is a subset of a Banach space $E$ and let $P$ be a retraction of $B$ onto $D$. Then, $P$ is said to be sunny [11] if for each $x \in B$ and $t \geq 0$, with $Px + t(x - Px) \in B$,

$$P(Px + t(x - Px)) = Px.$$

A subset $D$ of $B$ is said to be a sunny nonexpansive retract of $B$ if there exists a sunny nonexpansive retraction $P$ of $B$ onto $D$. We know that if $E$ is smooth and $P$ is a retraction of $B$ onto $D$, then $P$ is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \leq 0. \tag{2.3}$$

For more details see [7, 20].

We need the following lemmas to prove our main results.

**Lemma 2.1.** ([15]) *Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from a nonempty weakly compact convex subset $C$ of a Banach space $E$ into $C$, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants of the mappings. Let $X$ be a left invariant $\mathcal{S} -$stable subspace of $l^\infty(S)$ containing 1, and $\mu$ be a left invariant mean on $X$. Then, $F(\mathcal{S}) = F(T(\mu)) \cap C_a$.*

**Corollary 2.2.** ([13]) *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on $X$. If $z \in C_a$ and $\liminf_{n\to\infty} \|T(\mu_n)z - z\| = 0$, then $z$ is a common fixed point for $\mathcal{S}$.*

**Lemma 2.3.** ([13]) *Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from a nonempty weakly compact convex subset $C$ of a Banach space $E$ into $C$, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants of the mappings. Let $X$ be a left invariant subspace of $l^\infty(S)$ containing 1 such that the mappings $s \mapsto \langle T(s)x, x^*\rangle$ be in $X$ for all $x \in X$ and $x^* \in E^*$, and $\{\mu_n\}$ be a strongly left regular sequence of means on $X$. Then,*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|) \leq 0.$$

**Remark 2.1.** Taking in Lemma 2.3,

$$c_n = \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|), \forall n, \tag{2.4}$$

we obtain $\limsup_{n\to\infty} c_n \leq 0$. Moreover,

$$\|T(\mu_n)x - T(\mu_n)y\| \leq \|x - y\| + c_n, \forall x, y \in C. \tag{2.5}$$

**Corollary 2.4.** ([13]) *Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from a nonempty compact convex subset $C$ of a Banach space $E$ into $C$, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$. Let $X$ be a left invariant*

$\mathcal{S}-$*stable subspace of* $l^\infty(S)$ *containing 1, and* $\mu$ *be a left invariant mean on* $X$*. Then,* $T(\mu)$ *is nonexpansive and* $F(\mathcal{S}) \neq \emptyset$*. Moreover, if* $E$ *is smooth, then* $F(\mathcal{S})$ *is a sunny nonexpansive retract of* $C$ *and the sunny nonexpansive retraction of* $C$ *onto* $F(\mathcal{S})$ *is unique.*

**Lemma 2.5.** ([7, 20]) *Let* $X$ *be a real Banach space and let* $J$ *be the duality mapping. Then, for any given* $x, y \in X$ *and* $j(x+y) \in J(x+y)$*, there holds the inequality,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

**Lemma 2.6.** ([18]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* $X$ *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*. Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$*, for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$*. Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Lemma 2.7.** ([21]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0,$$

*where,* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then,* $\lim_{n\to\infty} a_n = 0$.

## 3. Main Results

In this section, we prove a strong convergence theorem for Lipschitzian semigroup in a smooth Banach space.

**Theorem 3.1.** *Let* $S$ *be a left reversible semigroup and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of* $S$ *as Lipschitzian mappings from a nonempty compact convex subset* $C$ *of a smooth Banach space* $E$ *into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$*, and* $f$ *be a contraction of* $C$ *into itself with coefficient* $\alpha \in (0,1)$*. Let* $X$ *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* $X$ *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by* (2.4)*. Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$*,* $n \geq 1$*. The following conditions are satisfied:*

*(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

*(ii)* $\lim_{n\to\infty} \delta_n = 0$*;*

*(iii)* $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$*; (note that, by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*);*
*(iv)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
*If for arbitrary given* $x_1 \in C$*, the sequence* $\{x_n\}$ *is generated by* (1.5)*, then* $\{x_n\}$ *converges strongly to* $z \in F(\mathcal{S})$*, which is the unique solution of the variational inequality*

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

*Equivalently, we have* $z = Pfz$*, where* $P$ *is the unique sunny nonexpansive retraction of* $C$ *onto* $F(\mathcal{S})$*.*

*Proof.* First, we prove that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{S})$. Then, by the nonexpansiveness of $T(\mu_n)$ and (2.5), we have

$$\begin{aligned}
\|y_n - p\| &= \|\delta_n x_n + (1-\delta_n)T(\mu_n)x_n - p\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\|T(\mu_n)x_n - p\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)(\|x_n - p\| + c_n) \\
&= \delta_n\|x_n - p\| + (1-\delta_n)\|x_n - p\| + (1-\delta_n)c_n \\
&\leq \|x_n - p\| + c_n,
\end{aligned}$$

and so,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\
&= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(y_n - p)\| \\
&\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\
&\quad +\gamma_n\|x_n - p\| + \gamma_n c_n \\
&= (1 - \alpha_n + \alpha\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| + \gamma_n c_n \\
&= (1 - \alpha_n(1-\alpha))\|x_n - p\| + \gamma_n c_n \\
&\quad +\alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha}.
\end{aligned}$$

By induction and (2.4), we get

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}.$$

This implies that $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{y_n\}$. In fact, letting $M = \|p\| + \max\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}$, for any $n \geq 1$, we have

$$\|T(\mu_n)x_n\| \leq \|T(\mu_n)x_n - p\| + \|p\| \leq \|x_n - p\| + \|p\| \leq M,$$

and then we also have $\|T(\mu_n)x_n\|$ is bounded.

Let $\{\omega_n\}$ be a sequence in $C$. By Saeidi ([13], Theorem 3.1, STEP 1, p. 7), we can show that

$$\lim_{n\to\infty} \|T(\mu_{n+1})\omega_n - T(\mu_n)\omega_n\| = 0. \tag{3.1}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and by Lemma 2.3, we observe that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})T(\mu_{n+1})x_{n+1} \\
&\quad - (\delta_n x_n + (1-\delta_n)T(\mu_n)x_n)\| \\
&= \|\delta_{n+1}x_{n+1} - \delta_{n+1}x_n + \delta_{n+1}x_n \\
&\quad + (1-\delta_{n+1})T(\mu_{n+1})x_{n+1} - (1-\delta_{n+1})T(\mu_n)x_n \\
&\quad + (1-\delta_{n+1})T(\mu_n)x_n - \delta_n x_n - (1-\delta_n)T(\mu_n)x_n\| \\
&= \|\delta_{n+1}(x_{n+1} - x_n) + (\delta_{n+1} - \delta_n)x_n \\
&\quad + (1-\delta_{n+1})(T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n) \\
&\quad + (\delta_n - \delta_{n+1})T(\mu_n)x_n\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| \\
&\quad + \|T(\mu_n)x_{n+1} - T(\mu_n)x_n\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad + \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|x_{n+1} - x_n\| + c_n.
\end{aligned}$$

Setting $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, we see that $z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. Then, we compute

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} \\
&\quad + \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} - \frac{\gamma_{n+1}y_n}{1-\beta_{n+1}} + \frac{\gamma_{n+1}y_n}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n}\|
\end{aligned}$$

$$
\begin{aligned}
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n)) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(y_{n+1}-y_n) \\
&\quad +(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})f(x_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})y_n\| \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\
&\quad +|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} - \frac{1-\beta_n-\alpha_n}{1-\beta_n}|\|y_n\| \\
&= \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +\|y_{n+1}-y_n\| \\
&\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +\delta_{n+1}\|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad +\|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|x_{n+1}-x_n\| + c_n.
\end{aligned}
$$

Therefore, we observe that

$$
\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\leq (\frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} + \delta_{n+1})\|x_{n+1}-x_n\| \\
&\quad +|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|y_n\|) \\
&\quad +|\delta_{n+1}-\delta_n|(\|x_n\| + \|T(\mu_n)x_n\|) \\
&\quad +\|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + c_n.
\end{aligned}
$$

It follow from $(i)$, $(ii)$, $(iv)$, (3.1) and Lemma 2.3, that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Applying Lemma 2.6, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$, and also

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|z_n - x_n\| \to 0,$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.2}$$

Next, we show that the set of all limit points of $\{x_n\}$ is a subset of $F(\mathcal{S})$. Let $p$ be a limit point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging strongly to $p$. Note that

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= \;& \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n(\delta_n x_n + (1-\delta_n)T(\mu_n)x_n) - x_n\| \\
= \;& \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n(\delta_n x_n + (1-\delta_n)T(\mu_n)x_n)\| \\
= \;& \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n\delta_n x_n + \gamma_n T(\mu_n)x_n - \gamma_n\delta_n T(\mu_n)x_n\| \\
= \;& \|\alpha_n f(x_n) - (1-\beta_n)x_n + \gamma_n\delta_n x_n + (1-\alpha_n-\beta_n)T(\mu_n)x_n - \gamma_n\delta_n T(\mu_n)x_n\| \\
= \;& \|\alpha_n(f(x_n) - T(\mu_n)x_n) + (1-\beta_n)(T(\mu_n)x_n - x_n) + \gamma_n\delta_n(x_n - T(\mu_n)x_n)\| \\
= \;& \|\alpha_n(f(x_n) - T(\mu_n)x_n) + (-1+\beta_n+\gamma_n\delta_n)(x_n - T(\mu_n)x_n)\| \\
\leq \;& \alpha_n\|f(x_n) - T(\mu_n)x_n\| + (-1+\beta_n+\gamma_n\delta_n)\|x_n - T(\mu_n)x_n\|.
\end{aligned}$$

So,

$$\|x_n - T(\mu_n)x_n\| \leq \frac{1}{1-\beta_n-\gamma_n\delta_n}(\alpha_n\|f(x_n) - T(\mu_n)x_n\| - \|x_{n+1} - x_n\|).$$

Hence, by $(i)$, $(ii)$, $(iv)$ and (3.2), we have

$$\lim_{n\to\infty}\|x_n - T(\mu_n)x_n\| = 0. \tag{3.3}$$

From this and Lemma 2.3, we obtain:

$$\begin{aligned}
\limsup_{k\to\infty}\|p - T(\mu_{n_k})p\| \;\leq\; & \limsup_{k\to\infty}(\|p - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| \\
& +\|T(\mu_{n_k})x_{n_k} - T(\mu_{n_k})p\|) \\
\leq\; & \limsup_{k\to\infty}(2\|p - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| + c_{n_k}) \\
\leq\; & 0.
\end{aligned}$$

Therefore, applying Corollary 2.2, we get $p \in F(\mathcal{S})$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle \leq 0$, where, $z = Pfz$. We know from Corollary 2.4 and the proof of Corollary 2.2 [13], that there exists a unique sunny nonexpansive retraction $P$ of $C$ onto $F(\mathcal{S})$. The Banach Contraction Mapping Principle guarantees that $Pf$ has a unique fixed point $z$, which by (2.3) is the unique solution of

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \quad \forall p \in F(\mathcal{S}). \tag{3.4}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle (f-I)z, J(x_{n_k} - z)\rangle = \limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle. \tag{3.5}$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ converges to some $p \in C$ such that $p \in F(\mathcal{S})$. Smoothness of $E$ and a combination of (3.4) and (3.5) give

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_n-z)\rangle = \langle (f-I)z, J(p-z)\rangle \le 0, \tag{3.6}$$

as required.

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = Pfz$. Now, we have

$$\begin{aligned}
\|y_n - z\| &= \|\delta_n x_n + (1-\delta_n)T(\mu_n)x_n - z\| \\
&= \|(1-\delta_n)(T(\mu_n)x_n - z) + \delta_n(x_n - z)\| \\
&\le (1-\delta_n)\|T(\mu_n)x_n - z\| + \delta_n\|x_n - z\| \\
&\le (1-\delta_n)\|x_n - z\| + c_n + \delta_n\|x_n - z\| \\
&= \|x_n - z\| + c_n. \qquad (3.7)
\end{aligned}$$

By using Lemma 2.5, (3.7) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - z\|^2 \\
&= \|(\gamma_n(y_n - z) + \beta_n(x_n - z)) + \alpha_n(f(x_n) - z)\|^2 \\
&\le \|\gamma_n(y_n - z) + \beta_n(x_n - z)\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - z, J(x_{n+1} - z)\rangle \\
&= \|(1-\beta_n)\frac{\gamma_n}{1-\beta_n}(y_n - z) + \beta_n(\frac{1-\beta_n}{1-\beta_n})(x_n - z)\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(z), J(x_{n+1} - z)\rangle \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le (1-\beta_n)\|\frac{\gamma_n}{1-\beta_n}(y_n - z)\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le \frac{\gamma_n^2}{1-\beta_n}\|y_n - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \alpha\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\le \frac{\gamma_n^2}{1-\beta_n}\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} + \beta_n\|x_n - z\|^2 \\
&\quad + \alpha\alpha_n\|x_n - z\|^2 + \alpha\alpha_n\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle
\end{aligned}$$

$$\begin{aligned}
&= (\frac{\gamma_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad +\alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (\frac{((1-\beta_n) - \alpha_n)^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (\frac{(1-\beta_n)^2 - 2(1-\beta_n)\alpha_n + \alpha_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 \\
&\quad + \frac{\gamma_n^2 c_n}{1-\beta_n} + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= (1 - \beta_n - 2\alpha_n + \frac{\alpha_n^2}{1-\beta_n} + \beta_n + \alpha\alpha_n)\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= ((1 - \alpha\alpha_n) + (2\alpha\alpha_n - 2\alpha_n) + \frac{\alpha_n^2}{1-\beta_n})\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{1-\beta_n} \\
&\quad + \alpha\alpha_n\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} \\
&\quad + \frac{\alpha_n^2}{(1-\alpha\alpha_n)(1-\beta_n)})\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{(1-\alpha\alpha_n)(1-\beta_n)} \\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\leq (1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n})\|x_n - z\|^2 + \frac{\alpha_n}{1-\alpha\alpha_n}(\frac{\alpha_n}{1-\beta_n}\|x_n - z\|^2 \\
&\quad + \frac{\gamma_n^2 c_n}{\alpha_n(1-\beta_n)} + 2\langle f(z) - z, J(x_{n+1} - z)\rangle) \\
&:= (1 - \sigma_n)\|x_n - z\|^2 + \rho_n,
\end{aligned}$$

where, $\sigma_n := \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}$ and $\rho_n := \frac{\alpha_n}{1-\alpha\alpha_n}(\frac{\alpha_n}{1-\beta_n}\|x_n - z\|^2 + \frac{\gamma_n^2 c_n}{\alpha_n(1-\beta_n)} + 2\langle f(z) - z, J(x_{n+1} - z)\rangle)$. Now, from $(i)$, $(iii)$, $(iv)$, (3.6) and Lemma 2.7, we get $\|x_n - z\| \to 0$, as $n \to \infty$. This completes the proof. □

**Corollary 3.2.** *Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from a nonempty compact convex subset $C$ of a smooth Banach space $E$ into itself, with the*

*uniform Lipschitzian condition* $\lim_s k(s) \leq 1$. *Let* $X$ *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* $X$ *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by* (2.4). *Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. *The following conditions are satisfied:*
*(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(ii)* $\lim_{n\to\infty} \delta_n = 0$;
*(iii)* $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$; *(by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*)*;
*(iv)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
*If arbitrary given* $x_1 \in C$, *the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)T(\mu_n)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \forall n \geq 1, \end{cases} \tag{3.8}$$

*then* $\{x_n\}$ *converges strongly to* $z \in F(\mathcal{S})$, *which is the unique solution of the variational inequality,*

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

*Equivalently, we have* $z = Pfz$, *where* $P$ *is the unique sunny nonexpansive retraction of* $C$ *onto* $F(\mathcal{S})$.

*Proof.* Taking $f(x) = u$ for all $x \in C$ in (1.5), we get (3.8), and we can conclude the desired conclusion easily. This completes the proof. □

**Corollary 3.3.** [13, Theorem 3.1] *Let* $S$ *be a left reversible semigroup and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of* $S$ *as Lipschitzian mappings from a nonempty compact convex subset* $C$ *of a smooth Banach space* $E$ *into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *and* $f$ *be a contraction of* $C$ *into itself with coefficient* $\alpha \in (0,1)$. *Let* $X$ *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* $X$ *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by*

$$c_n = \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x-y\|), \forall n.$$

*Suppose the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. *The following conditions are satisfied:*
*(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(ii)* $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$; *(by Remark 2.1,* $\limsup_{n\to\infty} c_n \leq 0$*)*;
*(iii)* $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
*If arbitrary given* $x_1 \in C$, *the sequence* $\{x_n\}$ *is generated by* (1.4), *then*

*$\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$, which is the unique solution of the variational inequality,*

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

*Equivalently, we have $z = Pfz$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.*

*Proof.* Taking $\delta_n = 0$ for all $n \in \mathbb{N}$ in (1.5), we get (1.4), and we can conclude the desired conclusion easily. This completes the proof. $\square$

## 4. **Application**

**Corollary 4.1.** *Let $C$ be a nonempty compact convex subset of a smooth Banach space $E$ and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of Lipschitzian mappings from $C$ into itself, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ and $\{t_n\}$ be an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. Let $f$ be a contraction of $C$ into itself with coefficient $\alpha \in (0,1)$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*
*(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(ii)* $\lim_{n\to\infty} \delta_n = 0$;
*(iii)* $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$,
*where,* $c_n = \sup_{x,y\in C}\{\|\frac{1}{t_n}\int_0^{t_n} T(s)xds - \frac{1}{t_n}\int_0^{t_n} T(s)yds\| - \|x-y\|\}$;
*(iv)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
*If for arbitrary given $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \geq 1, \end{cases} \tag{4.1}$$

*then $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$, which is the unique solution of the variational inequality,*

$$\langle (f-I)z, J(p-z)\rangle \leq 0, \forall p \in F(\mathcal{S}).$$

*Equivalently, we have $z = Pfz$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.*

*Proof.* For $n \geq 1$, define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)dt$ for each $f \in C(\mathbb{R}^+)$, where, $C(\mathbb{R}^+)$ is the space of all real valued bounded continuous functions on $\mathbb{R}^+$ with the supremum norm. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ [1]. Furthermore, for each $x \in C$, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(s)xds$. Therefore, we apply Theorem 3.1 to conclude the result. $\square$

**Acknowledgements**

The first author was partially supported by the King Mongkut's Dimond scholarship for Ph.D. student at KMUTT and King Mongkuts University of Technology Thonburi during the preparation of this manuscript. Moreover, we also would like to thanks the National Research University Project of Thailand's Office of the Higher Education Commission for financial support (under the project NRU-CSEC no. 54000267). Finally, the authors are grateful to the referees for their valuable comments and suggestions.

## References

1. S. Atsushiba and W. Takahashi, Strong convergence of Mann's-type iterations for nonexpansive semigroups in general Banach spaces, *Nonlinear Anal.* **61** (2005) 881–899.
2. A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel J. Math.* **22** (1975) 81–86.
3. N. Hirano, K. Kido and W. Takahashi, Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces, *Nonlinear Anal.* **12** (1988) 1269–1281.
4. R. D. Holmes and A. T. Lau, Asymptotically non-expansive actions of topological semigroups and fixed points, *Bull. London Math. Soc.* **3** (1971) 343–347.
5. R. D. Holmes and A. T. Lau, Non-expansive actions of topological semigroups and fixed points, *J. London Math. Soc. (2)* **5** (1972) 330–336.
6. R. D. Holmes and P. P. Narayanaswamy, On asymptotically nonexpansive semigroups of mappings, *Canad. Math. Bull.* **13** (1970) 209–214.
7. W. A. Kirk and B. Sims (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
8. A. T. Lau, H. Miyake and W. Takahashi, Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **67** (2007) 1211–1225.
9. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953) 506–510.
10. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **67** (1979) 274–276.
11. S. Reich, Asymptotic behavior of contractions in Banach spaces, *J. Math. Anal. Appl.* **44** (1973) 57-70.
12. C. Sahu and D. O'Regan, Convergence theorems for semigroup-type families of non-self mappings, *Rend. Circ. Mat. Palermo (2)* **57** (2008) 305–329.
13. S. Saeidi, Approximating common fixed points of Lipschitzian semigroup in smooth Banach spaces, *Fixed Point Theory Appl.* (2008) Art. ID 363257 17pp.
14. S. Saeidi, Existence of ergodic retractions for semigroups in Banach spaces, *Nonlinear Anal.* **69** (2008) 3417–3422.

15. S. Saeidi, Strong convergence of Browder's type iterations for left amenable semigroups of Lipschitzian mappings in Banach spaces, *J. Fixed Point Theory Appl.* **5** (2009) 93–103.
16. S. Saeidi, Convergence of an implicit iteration for affine mappings in normed and Banach spaces, *Bull. Iranian Math. Soc.* **33** (2007) 27–35.
17. S.-S. Zhang, L. Yang and J.-A. Liu, Strong convergence theorems for nonexpansive semi-groups in Banach spaces, *Appl. Math. Mech. (English Ed.)* **28** (2007) 1287–1297.
18. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005) 227–239.
19. W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, *Proc. Amer. Math. Soc.* **81** (1981) 253–256.
20. W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*. Yokohama Publishers, Yokohama, 2000.
21. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* **298** (2004) 279–291.

**P. Katchang and P. Kumam**
Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Thrungkru, 10140, Bangkok, Thailand
Email: p.katchang@hotmail.com
Email: poom.kum@kmutt.ac.th