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DUAL MULTIWAVELET FRAMES WITH SYMMETRY FROM TWO-DIRECTION REFINABLE FUNCTIONS

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Abstract. Motivated by [B. Han and Q. Mo, *Adv. Comp. Math.* **18** (2003) 211-245] and [B. Han and Z. Shen, *Constr. Approx.* **29** (2009) 369-406], we propose dual two-direction frames in dual Sobolev spaces $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, with $s > 0$. Based on the dual two-direction frames from a pair of two-direction refinable functions, dual multiwavelet frames with symmetry $\{\Psi^\ell(x) := (\psi_1^\ell(x), \psi_2^\ell(x))^T\}_{\ell=1}^d$ and $\{\widetilde{\Psi}^\ell(x) := (\widetilde{\psi}_1^\ell(x), \widetilde{\psi}_2^\ell(x))^T\}_{\ell=1}^d$ can be constructed very easily. The vanishing moment of the constructed multiwavelet frames is discussed. An example is given to illustrate our results.

1. Introduction

Recently, wavelet and multiwavelet frames in $L^2(\mathbb{R})$ have been studied extensively in the literature. To mention only a few here, see [1-5] and the references therein. In [1], Daubechies and Han gave an algorithm for constructing pairs of dual wavelet frames from any two refinable functions. In [3], Han and Mo constructed multiwavelet frames in $L^2(\mathbb{R})$ from any pair of refinable function vectors. In order to construct dual frames with symmetry, Han and Mo in [3] proposed a new factorization algorithm of mask symbols of multiwavelets. In [6], we used the

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factorization algorithm to construct symmetric multiwavelets with high approximation orders and regularity. However, the process of factorization is complex, as can be seen in [6]. Recently, we proposed a two-direction refinable function and a two-direction wavelet with dilation factor d, and got some nice results [7,8]. On the other hand, Han and Shen in [9] constructed dual wavelet frames in the dual Sobolev spaces $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, which are of interest in numerical algorithm and characterization of function spaces. Based on [9], we constructed pairs of dual multiwavelet frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, from which we obtained sampling theorem in $H^s(\mathbb{R})$, with $s > 1/2$ (see [11]). Motivated by the results in [3, 7-9], we shall propose dual two-direction frames in Sobolev spaces. Based on dual two-direction wavelet frames, dual symmetric multiwavelet frames can be obtained very easily through a linear combination. The vanishing moments of the dual symmetric multiwavelet frames are discussed amply.

Firstly, we introduce some conceptions about two-direction refinable functions. More details can be seen in [7,8].

If a function $\phi(x) : \mathbb{R} \mapsto \mathbb{C}$ satisfies the following refinement equation,

$$\phi(x) = d\sum_k p_k^+ \phi(dx - k) + d\sum_k p_k^- \phi(k - dx) \tag{1.1}$$

with $d \geq 2$ being an integer, we say that $\phi(x)$ is a two-direction refinable function. Moreover, $P^+(w) := \sum_k p_k^+ e^{-ikw}$ and $P^-(w) := \sum_k p_k^- e^{-ikw}$ are called the positive direction mask symbol and negative direction mask symbol of $\phi(x)$, respectively.

In [7, 8], we gave the conditions for the existense of the solution (1.1), and constructed two-direction refinable functions with nice properties including orthogonality, high approximation orders and regularity.

In order to investigate the existence of solution of (1.1), consider

$$\phi(-x) = d\sum_k p_k^+ \phi(-dx - k) + d\sum_k p_k^- \phi(dx + k). \tag{1.2}$$

Taking the Fourier transform of both sides of (1.2), we have

$$\overline{\widehat{\phi}(w)} = \overline{P^+(\frac{w}{d})\widehat{\phi}(\frac{w}{d})} + \overline{P^-(\frac{w}{d})\widehat{\phi}(\frac{w}{d})}. \tag{1.3}$$

From (1.1) and (1.3), we get

$$\left[\begin{array}{c}\widehat{\phi}(w)\\ \hline \widehat{\phi}(w)\end{array}\right]=\left[\begin{array}{cc}\frac{P^+(\frac{w}{d})}{P^-(\frac{w}{d})} & \frac{P^-(\frac{w}{d})}{P^+(\frac{w}{d})}\end{array}\right]\left[\begin{array}{c}\widehat{\phi}(\frac{w}{d})\\ \hline \widehat{\phi}(\frac{w}{d})\end{array}\right]. \tag{1.4}$$

It is easy to see that (1.1) has a solution if and only if (1.4) has a solution. That is, (1.1) is equivalent to (1.4). In [7], we gave conditions that equation (1.1) has distribution solutions or L^2- stable solutions. Throughout this article, assume that $\widehat{\phi}(0)=1$ which, together with (1.4), leads to $P^+(0)+P^-(0)=1$.

Let

$$\Phi(x)=\left[\begin{array}{c}\phi(x)\\ \phi(-x)\end{array}\right]=d\sum_{k\in Z}\left[\begin{array}{cc}p_k^+ & p_k^-\\ p_{-k}^- & p_{-k}^+\end{array}\right]\Phi(dx-k).$$

Then, (1.4) is the d-refinement equation in frequency field of $\Phi(x)$. Its refinement mask symbol is:

$$P(w):=\left[\begin{array}{cc}\frac{P^+(w)}{P^-(w)} & \frac{P^-(w)}{P^+(w)}\end{array}\right].$$

Therefore, (1.1) has an unique compactly supported distributional solution if and only if $P(w)$ satisfies condition E.

Now, let us introduce some conceptions about multiwavelet frames and propose dual two-direction frames in dual Sobolev space ($H^s(\mathbb{R})$, $H^{-s}(\mathbb{R})$) with $H^s(\mathbb{R})$ defined to be

$$H^s(\mathbb{R})=\{f:\frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{f}(w)|^2(1+|w|^2)^s dw<\infty\}.$$

Denote:

$$\nu_2(f)=\sup\{s:f\in H^s(\mathbb{R})\}.$$

Note that $H^s(\mathbb{R})$ is a Hilbert space with inner product defined to be

$$\langle f,g\rangle_{H^s(\mathbb{R})}=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi)\overline{\widehat{g}}(\xi)(1+|\xi|^2)^s d\xi,\ \ f,g\in H^s(\mathbb{R}).$$

For $f\in H^s(\mathbb{R})$ and $g\in H^{-s}(\mathbb{R})$, define

$$<f,g>=\int_{\mathbb{R}}f(x)\overline{g(x)}dx.$$

Suppose $\Phi(x)=(\phi_1(x),\cdots,\phi_r(x))^T\in(H^s(\mathbb{R}))^r$ and $\widetilde{\Phi}(x)=(\widetilde{\phi}_1(x),\cdots,\widetilde{\phi}_r(x))^T\in(H^{-s}(\mathbb{R}))^r$ are two refinable function vectors. They satisfy

the following refinement equations:

$$\Phi(x) = d\sum_k P_k\Phi(dx-k), \tag{1.5}$$

$$\widetilde{\Phi}(x) = d\sum_k \widetilde{P}_k\widetilde{\Phi}(dx-k), \tag{1.6}$$

where $\{P_k\}_k$ and $\{\widetilde{P}_k\}_k$ are some $r\times r$ matrices of numbers with the two mask symbols $P(w) := \sum_k P_k e^{-ikw}$ and $\widetilde{P}(w) := \sum_k \widetilde{P}_k e^{-ikw}$ being called the mask symbols of $\Phi(x)$ and $\widetilde{\Phi}(x)$, respectively. Let $\{\Psi^1,\cdots,\Psi^L\}$ be a finite set of $r\times 1$ function vectors in $H^s(\mathbb{R})$. Denote

$$\begin{aligned} X^s(\Phi;\Psi_1,\cdots,\Psi_L) \quad &:= \{\Phi_{n;0,k} : n=1,\cdots,r; k\in\mathbb{Z}\} \\ &\cup\{\Psi^{\ell,s}_{n;j,k}(\cdot) : n=1,\cdots,r; k\in\mathbb{Z}; j\in\mathbb{N}_0; \\ &\quad \ell=1,\cdots,L\} \end{aligned}$$

with $\Phi_{n;0,k}(\cdot) := \Phi_n(\cdot-k)$ and $\Psi^{\ell,s}_{n;j,k}(\cdot) := d^{j(1/2-s)}\Psi^{\ell}_n(d^j\cdot-k)$. If there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 \parallel f \parallel^2 \le & \textstyle\sum_{k\in Z} | < f,\Phi(x-k) > |^2 \\ & + \textstyle\sum_{\ell=1}^L \sum_{j\in Z}\sum_{k\in Z} | < f,\Psi^{\ell,s}_{j,k} > |^2 \\ & \le C_2 \parallel f \parallel^2, \qquad \forall f\in H^{-s}(\mathbb{R}), \end{aligned}$$

then we say that the set $X^s(\Phi;\Psi_1,\cdots,\Psi_L)$ generates a d-multiwavelet frame in $H^s(\mathbb{R})$, where, $| < f,\Phi(\cdot-k) > |^2 =< f,\Phi(\cdot-k) >< \Phi(\cdot-k), f >$ is the square of the ℓ_2 Euclidean norm of the column vector $< f,\Phi(\cdot-k) >$ in $\mathbb{R}^{r\times 1}$. If there exists another set of function vectors $\{\widetilde{\Psi}^1(x),\cdots,\widetilde{\Psi}^L(x)\}$ in $H^{-s}(\mathbb{R})$ such that for all $f\in H^{-s}(\mathbb{R}), g\in H^s(\mathbb{R})$,

$$\begin{aligned} < f,g > \quad &= \textstyle\sum_{k\in Z} < f(x),\Phi(x-k) >< \widetilde{\Phi}(x-k), g(x) > \\ &\quad + \textstyle\sum_{\ell=1}^L \sum_{j\in Z}\sum_{k\in Z} < f,\Psi^{\ell,s}_{j,k} >< \widetilde{\Psi}^{\ell,-s}_{j,k}, g >, \end{aligned}$$

then we say that $X^s(\Phi;\Psi^1,\cdots,\Psi^L)$ and $X^{-s}(\widetilde{\Phi};\widetilde{\Psi}^1,\cdots,\widetilde{\Psi}^L)$ generate a pair of dual d-multiwavelet frames in $(H^s(\mathbb{R}),H^{-s}(\mathbb{R}))$.

Concerning the two-direction refinable functions, we introduce the concept of two-direction frames in Sobolev spaces.

Assume $\phi(x)\in H^s(\mathbb{R})$ and $\widetilde{\phi}(x)\in H^{-s}(\mathbb{R})$ are a pair of two-direction refinable functions. They satisfy the following refinement equations:

$$\phi(x) = d\sum_k p_k^+\phi(dx-k) + d\sum_k p_k^-\phi(k-dx),$$

$$\widetilde{\phi}(x) = d\sum_k \widetilde{p}_k^+ \widetilde{\phi}(dx-k) + d\sum_k \widetilde{p}_k^- \widetilde{\phi}(k-dx).$$

Moreover, $\{\psi^\ell(x)\}_{\ell=1}^L$ and $\{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L$ are two sets of functions, and there exist $4L$ sequences $\{q_{\ell,k}^+\}_k$, $\{\widetilde{q}_{\ell,k}^+\}_k$, $\{q_{\ell,k}^-\}_k$ and $\{\widetilde{q}_{\ell,k}^-\}_k$ such that

$$\psi^\ell(x) = d\sum_k p_{\ell,k}^+ \phi(dx-k) + d\sum_k p_{\ell,k}^- \phi(k-dx), \qquad \ell = 1,\cdots,L,$$

and

$$\widetilde{\psi}^\ell(x) = d\sum_k \widetilde{p}_{\ell,k}^+ \phi(dx-k) + d\sum_k \widetilde{p}_{\ell,k}^- \phi(k-dx), \qquad \ell = 1,\cdots,L.$$

If $X^s(\Phi;\Psi^1,\cdots,\Psi^L)$ and $X^{-s}(\widetilde{\Phi};\widetilde{\Psi}^1,\cdots,\widetilde{\Psi}^L)$ generate a pair of dual d-multiwavelet frames, then we say that $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ can generate a pair of dual two-direction frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, where,

$$\Phi(x) = [\phi(x),\phi(-x)]^T, \widetilde{\Phi}(x) = [\widetilde{\phi}(x),\widetilde{\phi}(-x)]^T, \Psi^\ell(x) = [\psi^\ell(x),\psi^\ell(-x)]^T,$$

and $\widetilde{\Psi}^\ell(x) = [\widetilde{\psi}^\ell(x),\widetilde{\psi}^\ell(-x)]^T$.

2. Main Result

Theorem 2.1. *Let $\phi(x) \in H^s(\mathbb{R})$ and $\widetilde{\phi}(x) \in H^s(\mathbb{R})$, with $s > 0$, satisfy* (1.1) *and*

$$\widetilde{\phi}(x) = d\sum_k \widetilde{p}_k^+ \widetilde{\phi}(dx-k) + d\sum_k \widetilde{p}_k^- \widetilde{\phi}(k-dx).$$

Obviously, $\widetilde{\phi}(x) \in H^{-s}(\mathbb{R})$. Moreover, assume that

$$\begin{array}{ll} (1+e^{-iw}+\cdots+e^{-i(d-1)w})|P^+(w) & (1+e^{-iw}+\cdots+e^{-i(d-1)w})|P^-(w), \\ (1+e^{-iw}+\cdots+e^{-i(d-1)w})|\widetilde{P}^+(w) & (1+e^{-iw}+\cdots+e^{-i(d-1)w})|\widetilde{P}^-(w). \end{array} \tag{2.1}$$

Then, there exist $2d$ two-direction functions $\{\psi_\ell(x)\}_{\ell=1}^d \subset V_1$ and $\{\widetilde{\psi}_\ell(x)\}_{\ell=1}^d \subset \widetilde{V}_1$ such that $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ can generate a pair of dual two-direction frames in $(H^{s^*}(\mathbb{R}), H^{-s^*}(\mathbb{R}))$, where, $s^* = \min\{1,s\}$, $V_j = span\{d^{\frac{j}{2}}\phi(d^jx-k), d^{\frac{j}{2}}\phi(-d^jx-k)\}$ and $\widetilde{V}_j = span\{d^{\frac{j}{2}}\widetilde{\phi}(d^jx-k), d^{\frac{j}{2}}\widetilde{\phi}(-d^jx$

$-k)\}$. *Furthermore, construct*

$$\begin{cases} \Phi(x) = (\phi_1(x), \phi_2(x))^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi(x) \\ \phi(-x) \end{bmatrix}, \\ \widetilde{\Phi}(x) = (\widetilde{\phi}_1(x), \widetilde{\phi}_2(x))^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{\phi}(x) \\ \widetilde{\phi}(-x) \end{bmatrix}, \\ \Psi^\ell(x) = (\psi_1^\ell(x), \psi_2^\ell(x))^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \psi^\ell(x) \\ \psi^\ell(-x) \end{bmatrix}, \\ \widetilde{\Psi}^\ell(x) = (\widetilde{\psi}_1^\ell(x), \widetilde{\psi}_2^\ell(x))^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{\psi}^\ell(x) \\ \widetilde{\psi}^\ell(-x) \end{bmatrix}, \end{cases} \quad \ell = 1, \cdots, d.$$

Then $\Phi(x)$, $\Psi^\ell(x)$ *and* $\widetilde{\Psi}^\ell(x)$ *are symmetric. Moreover,* $X^{s^*}(\Phi;\Psi^1,\cdots,\Psi^L)$ *and* $X^{-s^*}(\widetilde{\Phi};\widetilde{\Psi}^1,\cdots,\widetilde{\Psi}^L)$ *generate a pair of dual symmetric* d*- multi-wavelet frames in* $(H^{s^*}(\mathbb{R}), H^{-s^*}(\mathbb{R}))$.

3. Dual Two-direction Frames

In this section, we shall study how to construct dual two-direction frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$. We begin with giving the following definition.

Definition 3.1. *If a* 2×2 *matrix* $(A_{i,j}(w))$ *of* 2π*-periodic trigonometric polynomials satisfies* $A_{1,1}(w) = \overline{A_{2,2}(w)}$ *and* $A_{1,2}(w) = \overline{A_{2,1}(w)}$*, then we say that* $A(w)$ *is a two-direction matrix. We call* $A_{1,1}(w)$ *and* $A_{2,2}(w)$ *a pair of dual entries. The appellation also makes sense for* $A_{1,2}(w)$ *and* $A_{2,1}(w)$.

Remark 3.2. *If* $A(w)$ *and* $B(w)$ *are two-direction matrices, then* $A(w)\pm B(w)$ *and* $A(w)B(w)$ *are also two-direction matrices.*

Next, we shall establish **Oblique Extension Principle** (OEP) for dual d-multiwavelet frames in dual Sobolev spaces. It is easily proved by the same way as in [3, Theorem 3.1] and [9, Theorem 2.4, Corollary 2.5]. In fact, this was also mentioned in [10, Section 1] and [11, Theorem 2.9].

Lemma 3.3. *Let two refinable function vectors* $\Phi(x) \in (H^s(\mathbb{R}))^r$ *and* $\widetilde{\Phi}(x) \in (H^s(\mathbb{R}))^r$ *be defined by* (1.5) *and* (1.6)*, respectively. Suppose*

there exist $r \times r$ matrices of 2π-periodic trigonometric polynomials a^1, $\cdots$, a^d, $b^1, \cdots, b^d, \Theta$ such that

$$\widehat{\Phi}(0)^*\Theta(0)\widehat{\widetilde{\Phi}}(0) = 1, \qquad P(0)^*\Theta(0)\widehat{\widetilde{\Phi}}(0) = \Theta(0)\widehat{\widetilde{\Phi}}(0)$$

and

$$\begin{bmatrix} a^1(w)^* & a^2(w)^* & \cdots & a^d(w)^* \\ a^1(w+\frac{2\pi}{d})^* & a^2(w+\frac{2\pi}{d})^* & \cdots & a^d(w+\frac{2\pi}{d})^* \\ \vdots & \vdots & \ddots & \vdots \\ a^1(w+\frac{2\pi(d-1)}{d})^* & a^2(w+\frac{2\pi(d-1)}{d})^* & \cdots & a^d(w+\frac{2\pi(d-1)}{d})^* \end{bmatrix}$$

$$\times \begin{bmatrix} b^1(w) \\ b^2(w) \\ \vdots \\ b^d(w) \end{bmatrix} = M(w),$$

where,

$$M(w) = \begin{bmatrix} \Theta(\xi) - P(w)^*\Theta(d\xi)\widetilde{P}(w) \\ -P(w+\frac{2\pi}{d})^*\Theta(d\xi)\widetilde{P}(w) \\ \vdots \\ -P(w+\frac{2\pi(d-1)}{d})^*\Theta(d\xi)\widetilde{P}(w) \end{bmatrix}. \tag{3.1}$$

Construct function vectors

$\Psi^\ell(x) = (\psi_1^\ell(x), \cdots, \psi_r^\ell(x))^T, \widetilde{\Psi}^\ell(x) = (\widetilde{\psi}_1^\ell(x), \cdots, \widetilde{\psi}_r^\ell(x))^T, \ell = 1, \cdots, d,$

through

$$\widehat{\Psi}^\ell(w) = a^\ell(\frac{w}{d})\widehat{\phi}(\frac{w}{d}), \quad \widehat{\widetilde{\Psi}^\ell}(w) = b^\ell(\frac{w}{d})\widehat{\widetilde{\phi}}(\frac{w}{d}).$$

If

$$[\widehat{\widetilde{\Psi}^\ell}]^{(j)}(0) = 0, \ell = 1, \cdots, d, j = 0, \cdots, \alpha - 1; \alpha \in \mathbb{N},$$

then $X^{s^*}(\Phi; \Psi^1, \cdots, \Psi^L)$ and $X^{-s^*}(\widetilde{\Phi}; \widetilde{\Psi}^1, \cdots, \widetilde{\Psi}^L)$ can generate a pair of dual d-multiwavelet frames in $(H^{s^*}(\mathbb{R}), H^{-s^*}(\mathbb{R}))$, with $s^* = \min\{\alpha + 1, s\}$.

Remark 3.4. *We can see from Lemma 3.1 that we do not have any requirement on the vanishing moment of $\Psi^\ell(x)$, $\ell = 1, \cdots, d$, which is the most significant difference from that of multiwavelet frames in*

$L^2(\mathbb{R})$ and which makes the construction of dual multiwavelet frames much easier.

Next, based on Lemma 3.3, we shall study how to explicitly construct pairs of dual two-direction wavelet frames in dual Sobolev spaces from two-direction refinable functions $\phi \in H^s(\mathbb{R})$ and $\widetilde{\phi} \in H^s(\mathbb{R})$, with $s > 0$. For two-direction matrices $a^\ell(w)$, $\ell = 1, \cdots, d$, denote

(3.2)
$$E(w) := \begin{bmatrix} a^1(w)^* & a^2(w)^* & \cdots & a^d(w)^* \\ a^1(w+\frac{2\pi}{d})^* & a^2(w+\frac{2\pi}{d})^* & \cdots & a^d(w+\frac{2\pi}{d})^* \\ \vdots & \vdots & \ddots & \vdots \\ a^1(w+\frac{2\pi(d-1)}{d})^* & a^2(w+\frac{2\pi(d-1)}{d})^* & \cdots & a^d(w+\frac{2\pi(d-1)}{d})^* \end{bmatrix}.$$

From now on, we require that $E(w)$ is strongly invertible, which means that $E(w)^{-1}$ is also a matrix of 2π-trigonometric polynomials.

Theorem 3.5. *Let $\phi(x)$ and $\widetilde{\phi}(x)$ be as in Theorem 2.1. Suppose their mask symbols are $\{P^+(w), P^-(w)\}$ and $\{\widetilde{P}^+(w), \widetilde{P}^-(w)\}$. Denote*

$$P(w) := \begin{bmatrix} P^+(w) & P^-(w) \\ P^-(w) & P^+(w) \end{bmatrix}, \widetilde{P}(w) := \begin{bmatrix} \widetilde{P}^+(w) & \widetilde{P}^-(w) \\ \widetilde{P}^-(w) & \widetilde{P}^+(w) \end{bmatrix},$$
$$a^\ell(w) = \begin{bmatrix} a^\ell_{1,1}(w) & a^\ell_{1,2}(w) \\ a^\ell_{1,2}(w) & a^\ell_{1,1}(w) \end{bmatrix}, \ell = 1, \cdots, d,$$

and construct

$$\begin{bmatrix} b^1(w) \\ \vdots \\ b^d(w) \end{bmatrix} := E^{-1}(w)M(w),$$

with the strongly invertible matrix $E(w)$ being defined in (3.2) *and $M(w)$ in* (3.1)*. Consequently, $b^\ell(w) = (b^\ell_{i,j}(w))_{i,j}$ are two-direction matrices, $\ell = 1, \cdots, d$. Moreover, assume that*

$$[1,1]\Theta(0)[1,1]^T = 1, P^*(0)\Theta(0)[1,1]^T = \Theta(0)[1,1]^T$$

and construct $\{\psi^\ell(x)\}_{\ell=1}^d$ and $\{\widetilde{\psi}^\ell(x)\}_{\ell=1}^d$ through

$$\widehat{\psi}^\ell(dw) = a^\ell_{1,1}(w)\widehat{\phi}(w) + a^\ell_{1,2}(w)\widehat{\phi}(-w)$$

and

$$\widehat{\widetilde{\psi}}^\ell(dw) = b^\ell_{1,1}(w)\widehat{\widetilde{\phi}}(w) + b^\ell_{1,2}(w)\widehat{\widetilde{\phi}}(-w). \tag{3.3}$$

Then, $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ *can generate a pair of dual two-direction frames in* $(H^{s^*}(\mathbb{R}),\ H^{-s^*}(\mathbb{R}))$*, with* $s^* = \min\{1, s\}$.

Proof. Rewrite $E(w)$ as:

$$E(w) = \begin{bmatrix} \overline{a_{11}^1(w)} & a_{12}^1(w) & \cdots & a_{12}^d(w) \\ \overline{a_{12}^1(w)} & a_{11}^1(w) & \cdots & a_{11}^d(w) \\ \overline{a_{11}^1(w+\frac{2\pi}{d})} & a_{12}^1(w+\frac{2\pi}{d}) & \cdots & a_{12}^d(w+\frac{2\pi}{d}) \\ \cdots & \cdots & \cdots & \cdots \\ \overline{a_{12}^1(w+\frac{2\pi(d-1)}{d})} & a_{11}^1(w+\frac{2\pi(d-1)}{d}) & \cdots & a_{11}^d(w+\frac{2\pi(d-1)}{d}) \end{bmatrix}.$$

It is easy to see that $E(w)$ consists of blocks of two-direction matrices. Denote

$$E_{2i-1,2j-1}^{-1}(w) =: \begin{bmatrix} e_{2i-1,2j-1}(w) & e_{2i-1,2j}(w) \\ e_{2i,2j-1}(w) & e_{2i,2j}(w) \end{bmatrix},$$

where, $e_{i,j}$ is the (i,j)-entry of $E^{-1}(w)$. Next, we prove that $E_{2i-1,2j-1}^{-1}(w)$ is a two-direction matrix. Without loss of generality, we just prove that $E_{1,1}^{-1}(w)$ is a two-direction matrix. Let $A_{i,j}(w)$ be the algebraic cofactor of $E_{i,j}(w)$ with $E_{i,j}(w)$ being the (i,j)-entry of $E(w)$. Assume p is the set of all arrangements of $\{2, 3, \cdots, 2d\}$, and τ the number of inverse order. Then,

$$A_{11}(w) = \begin{vmatrix} a_{11}^1(w) & \cdots & \overline{a_{12}^d(w)} & a_{11}^d(w) \\ a_{12}^1(w+\frac{2\pi}{d}) & \cdots & \overline{a_{11}^d(w+\frac{2\pi}{d})} & a_{12}^d(w+\frac{2\pi}{d}) \\ \cdots & \cdots & \cdots & \cdots \\ a_{11}^1(w+\frac{2\pi(d-1)}{d}) & \cdots & \overline{a_{12}^d(w+\frac{2\pi(d-1)}{d})} & a_{11}^d(w+\frac{2\pi(d-1)}{d}) \end{vmatrix}$$

$$= \sum_{i_2 i_3 \cdots i_{2d}} \sum_{j_2 j_3 \cdots j_{2d}} (-1)^{\tau(i_2 i_3 \cdots i_{2d}) + \tau(j_2 j_3 \cdots j_{2d})} E_{i_2 j_2}(w) E_{i_3 j_3}(w) \cdots E_{i_{2d} j_{2d}}(w),$$

where, the arrangements $i_2 i_3 \cdots i_{2d}, j_2 j_3 \cdots j_{2d} \in p$.

Let q be the set of all arrangements of $\{1, 3, 4, \cdots, 2d\}$. Then,

$$A_{22}(w) = \begin{vmatrix} \overline{a_{11}^1(w)} & \cdots & \overline{a_{11}^d(w)} & a_{12}^d(w) \\ \overline{a_{11}^1(w+\frac{2\pi}{d})} & \cdots & \overline{a_{11}^d(w+\frac{2\pi}{d})} & a_{12}^d(w+\frac{2\pi}{d}) \\ \cdots & \cdots & \cdots & \cdots \\ \overline{a_{12}^1(w+\frac{2\pi(d-1)}{d})} & \cdots & \overline{a_{12}^d(w+\frac{2\pi(d-1)}{d})} & a_{11}^d(w+\frac{2\pi(d-1)}{d}) \end{vmatrix}$$

$$= \sum_{k_1 k_3 \cdots k_{2d}} \sum_{\ell_1 \ell_3 \cdots \ell_{2d}} (-1)^{\tau(i_1 i_3 \cdots i_{2d}) + \tau(j_1 j_3 \cdots j_{2d})} E_{k_1 \ell_1}(w) E_{k_3 \ell_3}(w) \cdots E_{k_{2d} \ell_{2d}}(w),$$

where, the arrangements $k_1 k_3 \cdots k_{2d}, \ell_1 \ell_3 \cdots \ell_{2d} \in q$. We know that any pair of dual entries is on a diagonal. On the other hand, we observe that

if any two entries of $E(w)$ on the same row (column), then its dual entries are also on the same row (column). Therefore, $\forall i_2 i_3 \cdots i_{2d}, j_2 j_3 \cdots j_{2d} \in p$, from the structure of $E(w)$, it is easy to see that there exist two arrangements $k_1 k_3 \cdots k_{2d}, \ell_1 \ell_3 \cdots \ell_{2d} \in q$ such that

$$\overline{E}_{i_2 j_2}(w) = E_{k_1 \ell_1}(w), \overline{E}_{i_3 j_3}(w) = E_{k_3 \ell_3}(w), \cdots, \overline{E}_{i_{2d} j_{2d}}(w) = E_{k_{2d} \ell_{2d}}(w).$$

We know that for any pair of dual entries $E_{i,j}(w)$ and $E_{k,\ell}(w)$, it is straightforward to see $|i-k| = 1$ and $|j-\ell| = 1$. Thus,

$$\begin{cases} |\tau(k_1 i_3 \cdots i_{2d}) - \tau(i_2 i_3 \cdots i_{2d})| = 1, \\ |\tau(k_1 k_3 i_4 \cdots i_{2d}) - \tau(k_1 i_3 i_4 \cdots i_{2d})| = 1, \\ \cdots \qquad \cdots \\ |\tau(k_1 k_3 \cdots k_{2d}) - \tau(k_1 k_3 \cdots i_{2d})| = 1. \end{cases}$$

So, $(-1)^{\tau(k_1 k_3 \cdots k_{2d})} = (-1)^{2d-1}(-1)^{\tau(i_2 i_3 \cdots i_{2d})}$. Similarly, $(-1)^{\tau(\ell_1 \ell_3 \cdots \ell_{2d})} = (-1)^{2d-1}(-1)^{\tau(j_2 j_3 \cdots j_{2d})}$. Therefore, $(-1)^{\tau(k_1 k_3 \cdots k_{2d})} \times (-1)^{\tau(\ell_1 \ell_3 \cdots \ell_{2d})} = (-1)^{\tau(i_2 i_3 \cdots i_{2d})} \times (-1)^{\tau(j_2 j_3 \cdots j_{2d})}$. On the other hand, $|p| = |q| = (2d-1)!$. From the discussion above, we know that $A_{11}(w) = \overline{A_{22}(w)}$. Similarly, we get $A_{12}(w) = \overline{A_{21}(w)}$. Thus, $E_{1,1}^{-1}(w)$ is a two-direction matrix. Without loss of generality, we point out that $E_{i,j}^{-1}(w)$ $i, j = 1, \cdots, d$, is a two-direction matrix.

Note that $\widetilde{\phi} \in H^s(\mathbb{R})$. It is straightforward to see that $\widetilde{\phi} \in H^{-\mu}(\mathbb{R})$, for $\forall \mu \in \mathbb{R}^+$. To prove that $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ can generate a pair of dual two-direction frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, by Lemma 3.1, it suffices to prove $\widehat{\widetilde{\psi}^\ell}(0) = 0,\ \ell = 1, \cdots, d$. Since $\widehat{\phi}(0) = \widehat{\widetilde{\phi}}(0) = 1$,

$$P(0)[1,1]^T = \widetilde{P}(0)[1,1]^T = [1,1]^T,$$

which, together with (2.1), leads to

(3.4)
$$(\Theta(0) - P(0)^* \Theta(0) \widetilde{P}(0))[1,1]^T = \Theta(0)[1,1]^T - P(0)^* \Theta(0)[1,1]^T = O$$

and

(3.5)
$$P(\frac{2k\pi}{d})^* \Theta(0) \widetilde{P}(0)[1,1]^T = P(\frac{2k\pi}{d})^* \Theta(0)[1,1]^T = O,\ k = 1, \cdots, d-1.$$

It is easy to verify from (3.4) and (3.5) that $\widehat{\widetilde{\psi}^\ell}(0) = 0,\ \ell = 1, \cdots, d$. Therefore, by Lemma 3.1, $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ can generate a pair of two-direction frames in $(H^{s^*}(\mathbb{R}), H^{-s^*}(\mathbb{R}))$. $\square$

Note 1 (I) In Theorem 3.5, for the convenience of computation, we can set $\phi(x) = \widetilde{\phi}(x)$. (II) From Theorem 3.5, we know that the strongly invertible matrix $E(w)$ plays a key role in constructing dual two-direction frames.

Next, we just consider the case of $d = 2$. Other cases can be discussed similarly. There are many ways of constructing $E(w)$. We shall give Theorem 3.6 as a special way of constructing $E(w)$, based on which, the dual two-direction frames $\{\psi^\ell(x)\}_{\ell=1}^2$ and $\{\widetilde{\psi}^\ell(x)\}_{\ell=1}^2$ can be constructed through (3.3).

Theorem 3.6. *In Theorem 3.2, if we set dilation factor to $d = 2$, and select $a^\ell(w)$ as*

$$a^1(w) = \begin{bmatrix} a + bz + cz^2 & d + fz + gz^2 \\ d + fz^{-1} + gz^{-2} & a + bz^{-1} + cz^{-2} \end{bmatrix},$$

$$a^2(w) = \begin{bmatrix} h + pz + qz^2 & m + nz + kz^2 \\ m + nz^{-1} + kz^{-2} & h + pz^{-1} + qz^{-2}, \end{bmatrix}.$$

Where, $a, b, c, d, f, g, h, p, q, m, n,$ and k are parameters. satisfying

$$\begin{vmatrix} f & b & n & p \\ b & f & p & n \\ a & d & h & m \\ g & c & k & q \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} b & f & p & n \\ f & b & n & p \\ c & g & q & k \\ g & c & k & q \end{vmatrix} + \begin{vmatrix} b & f & p & n \\ f & b & n & p \\ a & d & h & m \\ d & a & m & h \end{vmatrix} \neq 0,$$

then $E(w)$ is strongly invertible.

Proof. It is only necessary to prove that $\det(E(w))$ is a real number. In fact,

$$\det(E(w)) = \begin{vmatrix} a + bz^{-1} + cz^{-2} & d + fz^{-1} + gz^{-2} & h + pz^{-1} + qz^{-2} & m + nz^{-1} + kz^2 \\ d + fz + gz^2 & a + bz + cz^2 & m + nz + kz^2 & h + pz + qz^2 \\ a - bz^{-1} + cz^{-2} & d - fz^{-1} + gz^{-2} & h - pz^{-1} + qz^{-2} & m - nz^{-1} + kz^{-2} \\ d - fz + gz^2 & a - bz + cz^2 & m - nz + kz^2 & h - pz + qz^2 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} b & f & p & n \\ f & b & n & p \\ a - bz^{-1} + cz^{-2} & d - fz^{-1} + gz^{-2} & h - pz^{-1} + qz^{-2} & m - nz^{-1} + kz^{-2} \\ d - fz + gz^2 & a - bz + cz^2 & m - nz + kz^2 & h - pz + qz^2 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} b & f & p & n \\ f & b & n & p \\ c & g & q & k \\ g & c & k & q \end{vmatrix} + 4 \times \begin{vmatrix} b & f & p & n \\ f & b & n & p \\ a & d & h & m \\ d & a & m & h \end{vmatrix} \neq 0.$$

□

Corollary 3.7. *If a^1 and a^2 are chosen in Theorem 3.3, then $E^{-1}(w) = (e_{i,j}(w))_{i,j}$, where,*

$$e_{1,1}(w) = \overline{e_{2,2}(w)} = \frac{2}{\det(E(w))}\Big\{(-gnh + bhq + gpm + ank - bmk - apq)z^{-1} +$$

$$(ap^2 - fpm + fnh - an^2 + bnm - bph) +$$

$$(bq^2 - cpq + anm - bk^2 - bm^2 + dpm - dnh + cnk + gpk - aph - gnq + bh^2)z +$$

$$(cp^2 - cn^2 - fpk + fnq + bnk - bpq)z^2 + (-cph - dnq + cnm + dpk - bmk + bhq)z^3\Big\},$$

$$e_{1,2}(w) = \overline{e_{2,1}(w)} = \frac{-2}{\det(E(w))}\Big\{(-anq + apk - fmk + fhq + gnm - gph)z^{-3} +$$

$$(bnq - bpk - gn^2 - fqp + fkn + gp^2)z^{-2} +$$

$$(-gpq - fm^2 - fk^2 + dnm + cpk + apm + fh^2 - cnq + fq^2 - anh - dph + gnk)z^{-1} +$$

$$(-fhp + fmn + bnh - dn^2 - bpm + dp^2) + (-fmk - cnh + fhq + cpm - dpq + dnk)z\Big\},$$

$$e_{1,3}(w) = \overline{e_{2,3}(w)} = \frac{2}{\det(E(w))}\Big\{(gnh - gpm - bhq + apq + bmk - ank)z^{-1} +$$

$$(ap^2 + bnm - fpm - bph - an^2 + fnh) +$$

$$(bm^2 - bq^2 + bk^2 - cnk - dpm + aph + cpq - anm + gnq - gpk - bh^2)z +$$

$$dnh + (cp^2 - bpq - cn^2 + fnq + bnk - fpk)z^2 +$$

$$(-bhq + cph - dpk - cnm + bmk + dnq)z^3\Big\},$$

$$e_{1,4}(w) = \overline{e_{2,4}(w)} = \frac{-2}{\det(E(w))}\Big\{(gnm - anq + fhq - fmk + apk - gph)z^{-3} +$$

$$(gn^2 - bnq - fnk + bpk + fpq - gp^2)z^{-2} -$$

$$(fk^2 + fq^2 - fm^2 - dph + dnm - gpq + fh^2 - cnq + apm + cpk + gnk - anh)z^{-1} +$$

$$(-dp^2 - bnh + bpm - fnm + dn^2 + fph) + (dnk - dpq + cpm - fmk + fhq - cnh)z\Big\},$$

$$e_{3,1}(w) = \overline{e_{4,2}(w)} = \frac{2}{\det(E(w))}\Big\{(-cbh + apc + fgh - fka + dbk - gpd)z^{-1} +$$

$$(fna + b^2h - f^2h - dbn + dpf - bpa) -$$

$$(fma + pa^2 - pg^2 + fgq - fkc - abh + dbm + c^2p - cbq + gbk - d^2p + fdh)z +$$

$$(-f^2q+fnc-gbn-bpc+b^2q+gpf)z^2+(-fmc+fdq-abq+apc+gbm-gpd)z^3\Big\},$$

$$e_{3,2}(w)=\overline{e_{4,1}(w)}=\frac{-2}{\det(E(w))}\Big\{(-cfh+dfk-gnd+anc+bgh-bka)z^{-3}+$$
$$(-f^2k+fng+kb^2-cbn-bgp+cpf)z^{-2}+$$
$$(c^2n-ng^2-bma+dfm-cfq+gfk+bgq-afh+na^2+bdh-d^2n-bkc)z^{-1}+$$
$$(-abn-f^2m+apf+mb^2-bdp+fnd)+(gfm+anc-afq-gnd-bmc+bdq)z\Big\},$$

$$e_{3,3}(w)=\overline{e_{4,4}(w)}=\frac{2}{\det(E(w))}\Big\{(-apc-fgh-dbk+cbh+gpd+fka)z^{-1}+$$
$$(b^2h-dbn-f^2h+dpf-bpa+fna)-$$
$$(fgq+fma+cbq-c^2p+pg^2+d^2p+abh-pa^2-gbk-fdh-dbm+fkc)z+$$
$$(gpf-f^2q+b^2q+fnc-gbn-bpc)z^2+(-gbm+abq-apc+gpd-fdq+fmc)z^3\Big\},$$

$$e_{3,4}(w)=\overline{e_{4,3}(w)}=\frac{-2}{\det(E(w))}\Big\{(gnd+cfh-dfk-anc+bka-bgh)z^{-3}+$$
$$(-bpg-cbn-f^2k+kb^2+fng+cpf)z^{-2}-$$
$$(na^2-bdh+bma+afh+cfq+ng^2+d^2n-dfm-c^2n-gfk-bgq+bkc)z^{-1}+$$
$$(-f^2m+apf-bpd+fnd-abn+b^2m)+(-gfm+afq-anc+bmc-bdq+gnd)z\Big\}.$$

Remark 3.8. *By the same method used in [3, Lemma 3.2], we can construct dual two-direction wavelet frames* $(\{\psi^\ell(x)\}_{\ell=1}^L, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L)$ *in* $(H^s(\mathbb{R}),\, H^{-s}(\mathbb{R}))$*, with* $\{\widetilde{\psi}^\ell(x)\}_{\ell=1}^L$ *having high vanishing moments.*

4. Proof of Theorem 2.1

Proof. Since $(\{\psi^\ell(x)\}_{\ell=1}^d, \{\widetilde{\psi}^\ell(x)\}_{\ell=1}^d)$ is a pair of dual two-direction frames in $(H^{s^*}(\mathbb{R}), H^{-s^*}(\mathbb{R}))$,

$$\begin{aligned}
< f,g > \;=\; & \sum_k \Big(< f(x),\phi(x-k) >< \widetilde{\phi}(x-k), g(x) > \\
+ \; & < f(x),\phi(-x-k) >< \widetilde{\phi}(-x-k), g(x) > \Big) \\
+ \; & \sum_{\ell=1}^d \sum_{j\in Z}\sum_{k\in Z} \Big(< f(x), \psi_{j,k}^{\ell,s^*}(x) >< \widetilde{\psi}_{j,k}^{\ell,-s^*}(x), g(x) > \\
+ \; & < f(x), \psi_{j,k}^{\ell,s^*}(-x) >< \widetilde{\psi}_{j,k}^{\ell,-s^*}(-x), g(x) > \Big), \\
& \forall\ f\in H^{-s^*}(\mathbb{R}), g\in H^{s^*}(\mathbb{R}),
\end{aligned}$$

where, $\psi_{j,k}^{\ell,s^*}(x) = d^{\frac{1}{2}(j-s^*)}\psi^\ell(d^jx-k)$ and $\widetilde{\psi}_{j,k}^{\ell,-s^*}(x) = d^{\frac{1}{2}(j+s^*)}\widetilde{\psi}^\ell(d^jx-k)$. It is easy to check that

$$\begin{aligned}
&\textstyle\sum_{n=1}^2 \Big[< f(x), \psi_{n,j,k}^{\ell,s^*}(x) >< \widetilde{\psi}_{n,j,k}^{\ell,-s^*}(x), g(x) > \Big] \\
&= \textstyle\sum_{n=1}^2 \Big[< f(x), \frac{d^{1/2(j-s^*)}}{2}(\psi^\ell(d^jx-k) + (-1)^n\psi^\ell(-d^jx+k)) > \\
&\qquad\qquad \times < \frac{d^{1/2(j+s^*)}}{2}(\widetilde{\psi}^\ell(d^jx-k) + (-1)^n\widetilde{\psi}^\ell(-d^jx+k)), g(x) > \Big] \\
&= \textstyle\sum_{\ell=1}^d \sum_{j\in Z}\sum_{k\in Z} \Big[< f(x), \psi_{j,k}^{\ell,s^*}(x) >< \widetilde{\psi}_{j,k}^{\ell,-s^*}(x), g(x) > \\
&\qquad\qquad + < f(x), \psi_{j,k}^{\ell,s^*}(-x) >< \widetilde{\psi}_{j,k}^{\ell,-s^*}(-x), g(x) > \Big].
\end{aligned}$$

Similarly, we can check that

$$\begin{aligned}
&\textstyle\sum_{n=1}^2 < f(x), \phi_n(x-k) >< \widetilde{\phi}_n(x-k), g(x) > \\
&=< f(x), \phi(x-k) >< \widetilde{\phi}(x-k), g(x) > \\
&\quad + < f(x), \phi(-x-k) >< \widetilde{\phi}(-x-k), g(x) > .
\end{aligned}$$

From the discussion above, it is easy to see that $X^{s^*}(\Phi;\Psi^1,\cdots,\Psi^L)$ and $X^{-s^*}(\widetilde{\Phi};\widetilde{\Psi}^1,\cdots,\widetilde{\Psi}^L)$ generate a pair of dual symmetric d-multiwavelet frames in $(H^{s^*}(\mathbb{R}),\ H^{-s^*}(\mathbb{R}))$. □

5. Example

Let $\phi(x) = \widetilde{\phi}(x)$ be a two-direction refinable function with the dilation factor 2 satisfying the following refinable function,

$$\phi(x) = 0.102\phi(2x-2)+0.3062\phi(2x-3)+0.3062\phi(2x-4)+0.102\phi(2x-5)$$
$$+0.0406\phi(-2x)+0.3418\phi(-2x-1)+0.546\phi(-2x-2)+0.25\phi(-2x-3).$$

According to [8, 12], we know that $\phi(x)$ provides an approximation order 2 and $\phi \in W^{1.9499}$, where, W^γ denotes the Sobolev smoothness. In Theorem 3.3, select $a=0, b=0, c=\frac{7}{2}, d=0, f=0, g=\frac{3}{2}, h=0; p=\frac{7}{2}, q=0, m=0, k=0, n=\frac{3}{2}$ and

$$\Theta(w) \equiv \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}.$$

According to Theorem 3.6 and Corollary 3.7, we get

$$\begin{cases}
\psi^1(x) = 7/2\phi(x-2) + 3/2\phi(-x+2), \\
\psi^2(x) = 7/2\phi(x-1) + 3/2\phi(-x+1), \\
\widehat{\widetilde{\psi}^1}(2w) = b_{1,1}^1(w)\widehat{\widetilde{\phi}}(w) + b_{1,2}^1(w)\widehat{\widetilde{\phi}}(-w), \\
\widehat{\widetilde{\psi}^2}(2w) = b_{1,1}^2(w)\widehat{\widetilde{\phi}}(w) + b_{1,2}^2(w)\widehat{\widetilde{\phi}}(-w),
\end{cases}$$

where,

$$\begin{aligned} b^1_{1,1}(w) &= -8.2483616625z - 0.00114345z^{-3} + 3.666624675z^{-2} \\ &\quad -2.985966425z^2 - 7.2344746625z^4 - 3.60042445z^5 \\ &\quad -10.9854916625z^3 - 1.633553475z^{-1} - 0.38024655z^7 \\ &\quad -0.10222065z^8 - 1.1903217125z^6 - 4.502862475, \end{aligned}$$

$$\begin{aligned} b^1_{1,2}(w) &= -0.00553905z^3 + 8.524990175z^2 - 0.716040575z \\ &\quad -1.951958875 - 5.9467932125z^{-1} - 6.578244725z^{-2} \\ &\quad -11.4539477125z^{-3} - 9.8056408125z^{-4} - 5.96696005z^{-5} \\ &\quad -2.5172651625z^{-6} - 0.75994065z^{-7} - 0.02110185z^{-8}, \end{aligned}$$

$$\begin{aligned} b^2_{1,1}(w) &= -0.10222065z^9 - 0.32994315z^8 - 1.1243771625z^7 \\ &\quad -2.286751425z^6 - 3.3535539125z^5 - 4.94099875z^4 \\ &\quad -6.39896885z^3 - 8.7656382125z^2 - 1.050391775z \\ &\quad -6.8726085375 - 0.235492075z^{-1} - 1.652137725z^{-2} \\ &\quad -0.084216825z^{-3} - 0.00114345z^{-4}, \end{aligned}$$

$$\begin{aligned} b^2_{1,2}(w) &= -3.2873747375 + 6.947540825z - 9.2278012z^{-4} \\ &\quad -8.7316757z^{-3} - 5.7251032625z^{-2} - 0.611878125z^2 \\ &\quad -0.00553905z^4 - 0.222284025z^3 + 0.184557125z^{-1} \\ &\quad -2.2871501625z^{-7} - 0.77547015z^{-8} - 5.216054725z^{-6} \\ &\quad -8.2191074625z^{-5} - 0.02110185z^{-9}. \end{aligned}$$

According to Theorem 3.5, $\{\psi^\ell(x)\}_{\ell=1}^d$ and $\{\widetilde{\psi}^\ell(x)\}_{\ell=1}^d$ can generate a pair of dual two-direction frames in $(H^1(\mathbb{R}), H^{-1}(\mathbb{R}))$. For $\ell = 1, 2$, construct

$$\Phi(x) = (\phi_1(x), \phi_2(x))^T = (\frac{\phi(x) + \phi(-x)}{2}, \frac{\phi(x) - \phi(-x)}{2})^T,$$

$$\widetilde{\Phi}(x) = (\widetilde{\phi}_1(x), \widetilde{\phi}_2(x))^T = (\frac{\widetilde{\phi}(x) + \widetilde{\phi}(-x)}{2}, \frac{\widetilde{\phi}(x) - \widetilde{\phi}(-x)}{2})^T,$$

$$\Psi^\ell(x) = (\Psi^\ell_1(x), \Psi^\ell_2(x))^T = (\frac{\psi^\ell(x) + \psi^\ell(-x)}{2}, \frac{\psi^\ell(x) - \psi^\ell(-x)}{2})^T,$$

and

$$\widetilde{\Psi}^\ell(x) = (\widetilde{\Psi}^\ell_1(x), \widetilde{\Psi}^\ell_2(x))^T = (\frac{\widetilde{\psi}^\ell(x) + \widetilde{\psi}^\ell(-x)}{2}, \frac{\widetilde{\psi}^\ell(x) - \widetilde{\psi}^\ell(-x)}{2})^T.$$

According to Theorem 2.1, $X^1(\Phi; \Psi^1, \Psi^2)$ and $X^{-1}(\widetilde{\Phi}; \widetilde{\Psi}^1, \widetilde{\Psi}^2)$ generate a pair of dual symmetric d-multiwavelet frames in $(H^1(\mathbb{R}), H^{-1}(\mathbb{R}))$.

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