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LINEAR PRESERVING GD-MAJORIZATION FUNCTIONS FROM $\mathbf{M}_{n,m}$ TO $\mathbf{M}_{n,k}$

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ABSTRACT. Let $\mathbf{M}_{n,m}$ be the vector space of all $n \times m$ real matrices. For $A, B \in \mathbf{M}_{n,m}$, it is said that B is *gd-majorized* by A (written $A \succ_{gd} B$) if for every $x \in \mathbb{R}^n$ there exists a g-doubly stochastic matrix D_x such that $Bx = D_x(Ax)$. Here, we show that if $A \succ_{gd} B$, then there exists a g-doubly stochastic matrix D (independent of x) such that $B = DA$. Also, the possible structures of linear preserving gd-majorization functions from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$ are found. Finally, all linear strongly preserving gd-majorization functions from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$ are characterized.

1. Introduction

Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ be a linear function and let $\sim$ be a relation on both $\mathbf{M}_{n,m}$ and $\mathbf{M}_{n,k}$. We say that T preserves $\sim$ when $X \sim Y$ implies $TX \sim TY$; if in addition, $TX \sim TY$ implies $X \sim Y$, we say that T strongly preserves $\sim$. For $x, y \in \mathbb{R}^n$, it is said that x is vector majorized by y (written $y \succ x$) if there exists a doubly stochastic matrix D such that $x = Dy$. For given $X, Y \in \mathbf{M}_{n,m}$, it is said that X is directionally majorized by Y (written $Y \succ_d X$) if $Yv \succ Xv$, for all $v \in \mathbb{R}^m$. The linear preservers of $\succ_d$ on $\mathbf{M}_{n,m}$ have been characterized in [7]. Some

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types of majorization and their linear preservers are presented in [1], [5] and [6]. Throughout the paper, the notation $\mathbf{M}_n$ is fixed for the algebra of all $n \times n$ real matrices. The space $\mathbf{M}_{n,1}$ of all $n \times 1$ real vectors is denoted by the usual notation $\mathbb{R}^n$. The collection of all $n\times n$ permutation matrices is denoted by $\mathcal{P}_n$. The notation $X = [x_1|\cdots|x_m]$ is used for an $n \times m$ matrix with $x_j \in \mathbb{R}^n$ as the jth column of X $(1 \leq j \leq m)$. The letters $\mathbf{J}$ and e stand for the square matrix and the vector, which respectively all of their entries are 1, and the dimensions of the matrix $\mathbf{J}$ and the vector e are understood from the context. The standard basis of $\mathbb{R}^n$ is denoted by $\{\epsilon_1, ..., \epsilon_n\}$. The notation A^t stands for the transpose of a given matrix A . For a given vector $x \in \mathbb{R}^n$, $tr(x)$ is the sum of all components of x. Now, we state an extension of a familiar result [7, Theorem 2] about linear functions preserving directional majorization from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$.

Proposition 1.1. *[2, Theorem 1.3] A linear function $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,k}$ preserves directional majorization if and only if one of the following holds.*

(i) There exist $A_1, \ldots, A_m \in \boldsymbol{M}_{n,k}$ such that $T(X) = \sum_{j=1}^{m}(trx_j)A_j$, *where,* $X = [x_1|\cdots|x_m]$.

(ii) There exist $R, S \in \boldsymbol{M}_{m,k}$ and $P \in \mathcal{P}_n$ such that $T(X) = PXR + \boldsymbol{J}XS$.

A (not necessarily nonnegative) matrix $D \in \mathbf{M}_n$ with the properties $De{=}e$ and $D^te = e$ is said to be a g-doubly stochastic matrix. This generalization of stochastic matrices was introduced in [4]. We denote the set of all $n \times n$ g-doubly stochastic matrices by $\mathbf{GD}_n$. For matrices $A, B \in \mathbf{M}_{n,m}$, it is said that B is gs-majorized by A (written $A \succ_{gs} B$) if there exists an $n \times n$ g-doubly stochastic matrix D such that $B{=}DA$. In [3], the authors found the possible structures of all linear operators preserving $\succ_{gs}$ on $\mathbf{M}_{n,m}$ as follows.

Proposition 1.2. *[3, Theorem 3.3] Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator that preserves $\succ_{gs}$. Then, one of the following holds.*

(i) There exist $A_1, \cdots, A_m \in \mathbf{M}_{n,m}$ such that $T(X){=}\sum_{j=1}^m tr(x_j)A_j$, where, $X = [x_1|\ldots|x_m]$.

(ii) There exist $S \in \mathbf{M}_m$, $a_1, \ldots, a_m \in \mathbb{R}^m$ and invertible matrices $D_1, \ldots, D_m \in \mathbf{GD}_n$ such that $T(X){=}[D_1Xa_1|\cdots|D_mXa_m] + \boldsymbol{J}XS$.

For $A, B \in \mathbf{M}_{n,m}$, it is said that B is gd-majorized by A (written $A \succ_{gd} B$) if $Ax \succ_{gs} Bx$, for all $x \in \mathbb{R}^m$. In fact, $A \succ_{gd} B$ if and only if, for every $x \in \mathbb{R}^m$, there exists a g-doubly stochastic matrix D_x such that $Bx = D_x(Ax)$. Here we prove the following theorem which gives the possible structures of all linear functions preserving $\succ_{gd}$ from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$.

Theorem 1.3. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ be a linear function that preserves $\succ_{gd}$. Then, one of the following holds.*

(i) There exist $A_1, \ldots, A_m \in \mathbf{M}_{n,k}$ such that $T(X) = \sum_{j=1}^{m} tr(x_j)A_j$, where, $X = [x_1|\ldots|x_m]$.

(ii) There exist $R, S \in \mathbf{M}_{m,k}$ and an invertible matrix $D \in \boldsymbol{GD}_n$ such that $T(X) = DXR + \boldsymbol{J}XS$.

(iii) There exist $S \in \mathbf{M}_{m,k}$, $a \in \mathbb{R}^m$, $r_1, \ldots, r_k \in \mathbb{R}$ and invertible matrices $D_1, \ldots, D_k \in \boldsymbol{GD}_n$ such that $T(X) = [r_1D_1Xa|\ldots|r_kD_kXa] + \boldsymbol{J}XS$.

2. Gd-Majorization

In this section, we present some properties of $\succ_{gd}$ and then show that the relation implies $\succ_{gs}$ on $\mathbf{M}_{n,m}$.

Lemma 2.1. *Let x and y be two distinct vectors in $\mathbb{R}^n$. Then, $x \succ_{gs} y$ if and only if $x \notin span\{e\}$ and $tr(x) = tr(y)$.*

Proposition 2.2. *Let $A = [a_1|\cdots|a_m], B = [b_1|\cdots|b_m] \in \mathbf{M}_{n,m}$. Then, B is gd-majorized by A if and only if the following conditions hold.*

(a) For every i $(1 \le i \le m)$, $tr(a_i) = tr(b_i)$; in other words, $A^te = B^te$.

(b) For every $x \in \mathbb{R}^m$ such that $Ax \in span\{e\}$, $Ax = Bx$.

Proof. It is clear that $A \succ_{gd} B$ implies the conditions (a) and (b). Conversely, assume (a) and (b) hold. For $x \in \mathbb{R}^m$, if $Ax \in span\{e\}$, then $Bx = Ax$, and hence $Ax \succ_{gs} Bx$. If $Ax \notin span\{e\}$, since $tr(b_i) = tr(a_i)$ for every i $(1 \le i \le n)$, then $tr(Bx) = tr(Ax)$. So, $Ax \succ_{gs} Bx$, by Lemma 2.1, and therefore $A \succ_{gd} B$. $\square$

Remark 2.3. *Let $X, Y \in \mathbf{M}_{n,m}$, $A, B \in \boldsymbol{GD}_n$, $C \in \mathbf{M}_m$ and $\alpha, \beta \in \mathbb{R}$ such that A, B and C are invertible and $\alpha \neq 0$. Then, the following conditions are equivalent:*

(1) $X \succ_{gd} Y$.

(2) $AX \succ_{gd} BY$.

(3) $\alpha X + \beta\boldsymbol{J}_{nm} \succ_{gd} \alpha Y + \beta\boldsymbol{J}_{nm}$, where $\boldsymbol{J}_{nm} \in \mathbf{M}_{n,m}$ is the matrix with all entries equal to one.

(4) $XC \succ_{gd} YC$.

Now, we show that $\succ_{gs}$ coincides with $\succ_{gd}$ on $\mathbf{M}_{n,m}$.

Lemma 2.4. *Let* $A, B \in \mathbf{M}_n$*. If* A *is invertible and* $A \succ_{gd} B$*, then* $A \succ_{gs} B$*.*

Proof. Put $D = BA^{-1}$. Since $DA = B$, it is enough to show that D is a g-doubly stochastic matrix. By invertibility of A, there exists a unique $x_0 \in \mathbb{R}^n$ such that $Ax_0 = e$, and hence $Bx_0 = e$, by Proposition 2.2. So, $De = (BA^{-1})e = B(A^{-1}e) = Bx_0 = e$. On the other hand, $A^t e = B^t e$, by Proposition 2.2, and hence $D^t e = (BA^{-1})^t(e) = (A^{-1})^t(B^t e) = (A^{-1})^t A^t e = e$. $\square$

Lemma 2.5. *Let* $A = [C|D]$ *and* $B = [E|F] \in \mathbf{M}_{n,m}$*, where* $C, E \in \mathbf{M}_{n,k}$ *and* $D, F \in \mathbf{M}_{n,(m-k)}$*. Suppose that the columns of* D *are generated by the columns of* C*. If* $A \succ_{gd} B$ *and* $C \succ_{gs} E$*, then* $A \succ_{gs} B$*.*

Proof. Since $C \succ_{gs} E$, then there exits a g-doubly stochastic matrix $R \in \mathbf{GD}_n$ such that $RC = E$, and hence $Rc_i = e_i$ $(1 \leq i \leq k)$, where c_i and e_i are the ith columns of C and E, respectively. We claim that $RA = B$. Suppose that d is the first column of D. Then, there exist scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that $d = \sum_{i=1}^{k} \alpha_i c_i$. Put $x_0 = (\alpha_1, \ldots, \alpha_k, -1, 0, \ldots, 0)^t \in \mathbb{R}^m$. Then, $Ax_0 = 0$, and hence $Bx_0 = 0$ by, Proposition 2.2. So, $f = \sum_{i=1}^{k} \alpha_i e_i$, where f is the first column of F. Thus, $Rd = \sum_{i=1}^{k} \alpha_i Rc_i = \sum_{i=1}^{k} \alpha_i e_i = f$. This argument is valid for other columns of D and F, and hence $RA = B$. $\square$

Theorem 2.6. *The concepts of gs and gd-majorization on* $\mathbf{M}_{n,m}$ *are the same.*

Proof. It is clear that $\succ_{gs}$ implies $\succ_{gd}$, and so we prove only the converse. Let $A, B \in \mathbf{M}_{n,m}$ and $A \succ_{gd} B$. By Remark 2.3, $A \succ_{gd} B$ if and only if $AP \succ_{gd} BP$, for every permutation matrix $P \in \mathbf{M}_n$. Then, without loss of generality, we can assume that $A = [C|D]$, where, $C = [c_1|\cdots|c_k] \in \mathbf{M}_{n,k}$ is a full rank matrix, $D = [d_1|\cdots|d_{m-k}] \in \mathbf{M}_{n,(m-k)}$ and $d_1, \cdots, d_{m-k} \in span\{c_1, \cdots, c_k\}$ (C or D can be vacuous). It is clear that $k \leq n$. Choose some vectors $c_{k+1}, \cdots, c_n \in \mathbb{R}^n$ such that $C' = [c_1|\cdots|c_n] \in \mathbf{M}_n$ is an invertible matrix. Consider the matrices $E \in \mathbf{M}_{n,k}$ and $F \in \mathbf{M}_{n,(m-k)}$ such that $B = [E|F]$. Since C' is invertible, then there exists a unique vector $x = (x_1, \cdots, x_n)^t \in \mathbb{R}^n$ such that $C'x = e$. Put $y = (x_1, \cdots, x_k)^t \in \mathbb{R}^k$. Since $A \succ_{gd} B$, then

$C \succ_{gd} E$, by Proposition 2.2, and hence $Cy \succ_{gs} Ey$. So, there exists a g-doubly stochastic matrix $R \in \mathbf{M}_n$ such that $RCy = Ey$. Put $E' = [E|Rc_{k+1}|\cdots|Rc_n] \in \mathbf{M}_n$. Then, $E'x = x_1e_1 + \cdots + x_ke_k + x_{k+1}Rc_{k+1} + \cdots + x_nRc_n$, where e_i is the ith column of E. Since $RCy = Ey$ then $x_1Rc_1 + \cdots + x_kRc_k = x_1e_1 + \cdots + x_ke_k$, then $E'x = x_1Rc_1 + \cdots + x_nRc_n = R(C'x) = Re = e$. On the other hand, $tr(e_i) = tr(c_i)$, for every i $(1 \leq i \leq k)$, and $tr(c_i) = tr(Rc_i)$, for every i $(k+1 \leq i \leq n)$. Then, $C' \succ_{gd} E'$, by Proposition 2.2. Therefore, $C' \succ_{gs} E'$, by Lemma 2.4, and hence $C \succ_{gs} E$. Since $d_1, \cdots, d_{m-k} \in span\{c_1, \cdots, c_k\}$ and $A \succ_{gd} B$, we get $A \succ_{gs} B$, by Lemma 2.5. □

3. **Linear Preservers**

In this section, we prove the following statements which shed light on the structure of linear functions preserving $\succ_{gd}$ from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$.

Theorem 3.1. *Let $T : \mathbf{M}_{n,m} \to \mathbb{R}^n$ be a linear function. Then, T preserves $\succ_{gs}$ if and only if one of the following holds.*

(a) There exist $a_1, \ldots, a_m \in \mathbb{R}^n$ such that $T(X) = \sum_{j=1}^{m} tr(x_j)a_j$, where, $X = [x_1|\ldots|x_m]$.

(b) There exist $a,\ b \in \mathbb{R}^m$ and an invertible matrix $A \in \boldsymbol{GD}_n$ such that $T(X) = AXa + \boldsymbol{J}Xb$.

Proof. The fact that each of the conditions (a) or (b) is sufficient for T to be a preserver of $\succ_{gs}$ is easy to prove. So, we prove the necessity of the conditions. Define $T' : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by $T'(X) = [T(X)|0]$, where 0 denotes an $n \times (m-1)$ zero block. Clearly, T' is a linear function which preserves $\succ_{gs}$. Then, by Proposition 1.2, T' has one of the following forms.

(i) $T'(X) = \sum_{j=1}^{m} tr(x_j)B_j$, for some $B_1, \ldots, B_m \in \mathbf{M}_{n,m}$. So, $T(X) = \sum_{j=1}^{m} tr(x_j)a_j$, where a_j is the first column of B_j, for every j $(1 \leq j \leq m)$, and hence (a) holds.

(ii) $T'(X) = [D_1Xa_1|\ldots|D_mXa_m] + \mathbf{J}XS$, for some $S \in \mathbf{M}_m$, $a_1, \ldots, a_m \in \mathbb{R}^n$ and invertible matrices $D_1, \ldots, D_m \in \mathbf{GD}_n$. So, $TX = D_1Xa_1 + \mathbf{J}Xb$, where b is the first column of S, and hence (b) holds. □

Lemma 3.2. *[3, Lemma 3.1] Let $A \in \boldsymbol{GD}_n$ be invertible. Then, the following conditions are equivalent.*

(*a*) $A = \alpha I + \beta \boldsymbol{J}$, *for some* $\alpha, \beta \in \mathbb{R}$.
(*b*) $(x + Ay) \succ_{gs} (Dx + ADy)$, *for all* $D \in \boldsymbol{GD}_n$ *and for all* $x, y \in \mathbb{R}^n$.

Remark 3.3. *Assume that* T_1 *and* T_2 *are of the form* (a) *and* (b) *in Theorem 3.1, respectively. Then,* $T_1 = T_2$ *if and only if* $a = 0$ *and* $a_j = \lambda_j e$, *for every* j $(1 \leq j \leq m)$, *where* a, a_j $(1 \leq j \leq m)$ *and* $b = (\lambda_1, \ldots, \lambda_m)^t$ *are as in Theorem 3.1.*

Lemma 3.4. *Let* $T_1, T_2 : \mathbf{M}_{n,m} \to \mathbb{R}^n$ *be two linear preservers of* $\succ_{gs}$ *such that* $T_1 + T_2$ *preserves* $\succ_{gs}$. *If* $T_1(X) = DXa + \boldsymbol{J}Xb$, *for some* $a, b \in \mathbb{R}^m$, $a \neq 0$ *and an invertible matrix* $D \in \boldsymbol{GD}_n$, *then* $T_2(X) = D'Xc + \boldsymbol{J}Xd$, *for some* $c, d \in \mathbb{R}^m$ *and an invertible matrix* $D' \in \boldsymbol{GD}_n$.

Proof. Since $T_1 + T_2$ preserves $\succ_{gs}$, then $T_1 + T_2$ is of the form (a) or (b) in Theorem 3.1. Now, consider two cases.

Case 1: Suppose that $T_1 + T_2$ is of the form (a). Since T_2 preserves $\succ_{gs}$, it is of the form (a) or (b) in Theorem 3.1. Assume, if possible, T_2 is of the form (a). Then, $T_1 = (T_1 + T_2) - T_2$ is of the form (a), as well. So, by Remark 3.3, we obtain $a = 0$, which is a contradiction. Therefore, T_2 is of the form (b).

Case 2: Suppose that $T_1 + T_2$ is of the form (b). So, $(T_1 + T_2)(X) = BXa' + \mathbf{J}Xb'$, for some $a', b' \in \mathbb{R}^m$ and invertible matrix $B \in \mathbf{GD}_n$. Assume, if possible, T_2 is of the form (a) and is not of the form (b). Then, by Theorem 3.1 and Remark 3.3, there exist (not all in $span\{e\}$) $a_1, \ldots, a_m \in \mathbb{R}^n$ such that $T_2(X) = \sum_{j=1}^{m} tr(x_j)a_j$. Without loss of generality, suppose that $a_1 \notin span\{e\}$. Put $X := [e|0|\ldots|0] \in \mathbf{M}_{n,m}$. So,

$$na_1 = \sum_{j=1}^{m} tr(x_j)a_j = T_2(X)$$

$$= (T_1 + T_2 - T_1)(X) \tag{3.1}$$

$$= [a'_1 + nb'_1 - a_1 - nb_1]e\ , \tag{3.2}$$

where, a'_1, b'_1, a_1 and b_1 are the first entry of a', b', a and b, respectively, which is a contradiction. Therefore, T_2 is of the form (b), and hence there exist $c, d \in \mathbb{R}^m$ and an invertible matrix $D' \in \mathbf{GD}_n$ such that $T_2(X) = D'Xc + \mathbf{J}Xd$. □

Now, we can prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that T preserves $\succ_{gd}$. Then, for every i $(1 \leq i \leq k)$, $T_i = E_i \circ T : \mathbf{M}_{n,m} \to \mathbb{R}^n$ preserves $\succ_{gd}$, where,

$E_i : \mathbf{M}_{n,k} \to \mathbb{R}^n$ is defined by $E_i(A) = A\epsilon_i$. Thus, T_i is of the form (a) or (b) in Theorem 3.1. Now, consider two cases.
Case 1: Assume T_i is of the form (a), for every i $(1 \leq i \leq k)$. Then, $T_i(X) = \sum_{j=1}^{m} tr(x_j)a_j^i$, for some $a_j^i \in \mathbb{R}^n$. Put $A_j := [a_j^1|\dots|a_j^k]$, for every j $(1 \leq j \leq m)$. So, $T(X) = \sum_{j=1}^{m} tr(x_j)A_j$, and hence the condition (i) holds.

Case 2: Assume there exists p $(1 \leq p \leq k)$ such that $T_p(X) = D_pXa_p + \mathbf{J}Xb_p$, for some $a_p, b_p \in \mathbb{R}^k$, $a_p \neq 0$ and an invertible matrix $D_p \in \mathbf{GD}_n$. Since T preserves $\succ_{gd}$, so $T_p + T_j$ preserves $\succ_{gd}$, for every j $(1 \leq j \leq k)$. Then, by Lemma 3.4, $T_j(X) = D_jXa_j + \mathbf{J}Xb_j$, for some $a_j, b_j \in \mathbb{R}^m$, and an invertible matrix $D_j \in \mathbf{GD}_n$. So,

$$\begin{aligned} T(X) &= [T_1(X)|\cdots|T_k(X)] \\ &= [D_1Xa_1 + \mathbf{J}Xb_1|\cdots|D_mXa_m + \mathbf{J}Xb_k] \\ &= [D_1Xa_1|\cdots|D_kXa_k] + \mathbf{J}X[b_1|\cdots|b_k]. \end{aligned}$$

If $rank[a_1|\cdots|a_k] \geq 2$, then, without loss of generality, we may assume that $rank[a_1|a_2] = 2$. Since for every $X \in \mathbf{M}_{n,m}$ and every $D \in \mathbf{GD}_n$, $X \succ_{gs} DX$, then $(T_1 + T_2)X \succ_{gs} (T_1 + T_2)(DX)$, and hence $D_1Xa_1 + D_2Xa_2 \succ_{gs} D_1DXa_1 + D_2DXa_2$. So, for every $D \in \mathbf{GD}_n$,

$$Xa_1 + (D_1^{-1}D_2)Xa_2 \succ_{gs} DXa_1 + (D_1^{-1}D_2)DXa_2, \forall X \in \mathbf{M}_{n,m}.$$

Since a_1 and a_2 are linearly independent, we may put some suitable X in the above relation and obtain the following:

$$x + (D_1^{-1}D_2)y \succ_{gs} Dx + (D_1^{-1}D_2)Dy \ , \ \forall x, y \in \mathbb{R}^m \ , \forall D \in \mathbf{GD}_n \ .$$

Then, by Lemma 3.2 , $D_1^{-1}D_2 = \lambda_1 I + \mu_1\mathbf{J}$, and hence $D_2 = \lambda_1 D_1 + \mu_1\mathbf{J}$, for some $\lambda_1, \mu_1 \in \mathbb{R}$. For every i $(2 \leq i \leq k)$, with $a_i \neq 0$, it is clear that $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent, and so, by a similar argument as above, $D_i = \lambda_i D_1 + \mu_i\mathbf{J}$, for some $\lambda_i, \mu_i \in \mathbb{R}$. Set $D := D_1$. Then, for every i $(1 \leq i \leq k)$, $D_i = \lambda_i D + \mu_i\mathbf{J}$, for some $\lambda_i, \mu_i \in \mathbb{R}$, and hence $T(X) = DXR + \mathbf{J}XS$, where, $R = [\lambda_1 a_1|\cdots|\lambda_k a_k]$ and $S = [\mu_1 a_1 + b_1|\cdots|\mu_k a_k + b_k]$. Therefore, the condition (ii) holds. If $rank[a_1|\dots|a_k] \leq 1$, then there exist $a \in \mathbb{R}^m$ and $r_1, \dots, r_k \in \mathbb{R}$ such that for every i $(1 \leq i \leq k)$, $a_i = r_i a$. Therefore, $T(X) = [r_1D_1Xa|\dots|r_kD_kXa] + \mathbf{J}XS$, where, $S = [b_1|\dots|b_k]$, and hence the condition (iii) holds. $\square$

It is easy to show that if T is of the form (i) or (ii) in Theorem 1.3, then T preserves $\succ_{gd}$. The following example shows that there is a linear function of the form (iii) not preserving $\succ_{gd}$.

Example 3.5. *Suppose that* $T : \mathbf{M}_{3,2} \to \mathbf{M}_{3,2}$ *is defined by* $T(X) = [X\epsilon_1|PX\epsilon_1]$*, where,* $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. *So,* T *is of the form* (iii) *in Theorem 1.3. Put* $A := \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 0 \end{pmatrix}$ *and* $B := \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$. *It is easy to show that* $B \succ_{gd} A$ *and* $TB \not\succ_{gd} TA$. *Then,* T *does not preserve* $\succ_{gd}$.

It is clear that the form (ii) is a special case of the form (iii) in Theorem 1.3 (put $D_1 = \cdots = D_k := D$ and $R := [r_1 a| \cdots |r_k a]$). The following example shows that there is a linear function preserving $\succ_{gd}$, which is of the form (iii) but is not of the form (ii).

Example 3.6. *[3, Example 3.5] Let* $T : \mathbf{M}_{3,2} \to \mathbf{M}_{3,2}$ *be defined by* $T(X) = [X\epsilon_1|PX\epsilon_1]$*, where,* $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *Then,* T *preserves* $\succ_{gd}$ *and* T *is not of the form* (ii) *in Theorem 1.3.*

Now, we state the following lemma which characterizes all strong linear preservers of $\succ_{gd}$ from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$.

Lemma 3.7. *[2, Lemma 2.4] Let* $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ *be a linear function of the form* $T(X) = XR + \mathbf{J}XS$*, for some* $R, S \in \mathbf{M}_{m,k}$. *Then,* T *is injective if and only if* R *and* $R + nS$ *are full-rank matrices.*

Proof. It is easy to see that the matrix representation of T with respect to the standard bases of $M_{n,m}$ and $M_{n,k}$ is similar to the following block matrix:

$$\begin{pmatrix} R+nS & & & * \\ & R & & \\ & & \ddots & \\ 0 & & & R \end{pmatrix} \in M_{nk,nm} .$$

Therefore, T is injective if and only if R and $R + nS$ are full-rank matrices. □

If T is a strong linear preserver of $\succ_{gd}$ and $T(A) = 0$, then $T(0) \succ_{gd} T(A)$. So, $0 \succ_{gd} A$, and hence $A = 0$.

Remark 3.8. *Every strong linear preserver of $\succ_{gd}$ from $M_{n,m}$ to $M_{n,k}$ is injective.*

If $m = 1$, then the following theorem is obtained from Theorem 3.1. So, in the proof we may assume $m \geq 2$.

Theorem 3.9. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,k}$ be a linear function. Then, T strongly preserves $\succ_{gd}$ if and only if there exist an invertible matrix $D \in \boldsymbol{GD}_n$ and matrices $R, S \in \boldsymbol{M}_{m,k}$ such that R and $R + nS$ are full-rank matrices and $TX = DXR + \boldsymbol{J}XS$.*

Proof. If T is of the form $TX = DXR + \mathbf{J}XS$, for some invertible matrix $D \in \mathbf{GD}_n$ and full-rank matrices $R, R + nS \in \mathbf{M}_{m,k}$, then it is easy to show that T is a strong linear preserver of $\succ_{gd}$. Conversely, assume T is a strong linear preserver of $\succ_{gd}$. So, T is of the form (i), (ii) or (iii) in Theorem 1.3. If T is of the form (i), then T is not injective, which is a contradiction. If T is of the form (iii), then we can choose $0 \neq b \in (span\{a\})^{\perp}$, by the assumption $m \geq 2$. Put $X_0 := [b \mid -b \mid 0 \mid \cdots \mid 0]^t \in \mathbf{M}_{n,m}$. So, $X_0 \neq 0$ and $T(X_0) = 0$, which is a contradiction. Therefore, T is of the form (ii), and by Lemma 3.7, R and $R + nS$ are full-rank matrices. $\square$

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References

[1] A. Armandnejad, Right GW-majorization on $\mathbf{M}_{n,m}$, *Bull. Iranian Math. Soc.* **35** (2009) 69-76.

[2] A. Armandnejad and H. R. Afshin, Linear functions preserving multivariate and directional majorization, *Iran. J. Math. Sci. Inform.* **5** (2010) 1-5.

[3] A. Armandnejad and A. Salemi, The structure of linear preservers of gs-majorization, *Bull. Iranian Math. Soc.* **32** (2006) 31-42.

[4] H. Chiang and C.-K. Li, Generalized doubly stochastic matrices and linear preservers, *Linear Multilinear Algebra* **53** (2005) 1-11.

[5] A. M. Hasani and M. Radjabalipuor, The structure of linear operators strongly preserving majorizations of matrices, *Electron. J. Linear Algebra* **15** (2006) 260-268.

[6] F. Khalooei and A. Salemi, The structure of linear preservers of left matrix majorization on $\mathbb{R}^p$, *Electron. J. Linear Algebra* **18** (2009) 88-97.
[7] C.-K. Li and E. Poon, Linear operators preserving directional majorization, *Linear Algebra Appl.* **325** (2001) 141-146.

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