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APPROXIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

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ABSTRACT. Let C be a nonempty closed convex subset of a complete CAT(0) space and $T: C \to C$ be a generalized nonexpansive mapping with $F(T) = \{x \in C : T(x) = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

 $x_{n+1} = t_n T[s_n T x_n \oplus (1-s_n) x_n] \oplus (1-t_n) x_n,$

for all $n \geq 1$, where $\{t_n\}$ and $\{s_n\}$ are real sequences in [0, 1] such that one of the following two conditions is satisfied: (i) $t_n \in [a, b]$ and $s_n \in [0, 1]$, for some a, b with $0 < a \leq b < 1$, (ii) $t_n \in [a, 1]$ and $s_n \in [a, b]$, for some a, b with $0 < a \leq b < 1$. Then, the sequence $\{x_n\}$, Δ -converges to a fixed point of T. Our results extend the ones in Laokul and Panyanak [T. Laokul and B. Panyanak, Int. J. Math. Anal. **3** (2009) 1305–1315.] and also the ones in Nanjaras et al. [B. Nanjaras, B. Panyanak and W. Phuangrattana, Nonlinear Anal. Hybrid Syst. **4** (2010) 25-31.].

1. Introduction

Recently, Suzuki [17] introduced condition (C) as follows. Condition(C): Let T be a mapping on a subset C of Banach space E.

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Then, T is said to satisfy condition (C) (or generalized nonexpansive mapping) if

$$\frac{1}{2}||x - Tx|| \le ||x - y|| \text{ implies } ||Tx - Ty|| \le ||x - y||,$$

for all $x, y \in C$.

Proposition 1.1. Every nonexpansive mapping satisfies condition (C), but the inverse is not true.

Example 1.2. Define a mapping T on [0,3] by

$$T(x) = \begin{cases} 0 & ifx \neq 3, \\ 1 & ifx = 3. \end{cases}$$

Then, T satisfies condition (C), but T is not nonexpansive.

The purpose of this paper is to study the iterative scheme defined as follows.

Let C be a nonempty closed convex subset of a complete CAT(0) space and $T: C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

(1.1)
$$x_{n+1} = t_n T[s_n T x_n \oplus (1 - s_n) x_n] \oplus (1 - t_n) x_n,$$

for all $n \ge 1$, where, $\{t_n\}$ and $\{s_n\}$ are real sequences in [0, 1] such that one of the following two conditions is satisfied:

(1.2)

We show that the sequence $\{x_n\}$, defined by (1.1), Δ -converges to a fixed point of T.

2.
$$CAT(0)$$
 Spaces

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that c(0) = x, c(l) = y, and d(c(t), c(t)) = |t - t|, for all $t, t \in [0, l]$. In particular, c is an isometry and d(x, y) = l. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by [x, y]. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining

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x to y, for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Yincludes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of \triangle) and a geodesic segment between each pair of vertices (the edges of \triangle). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a CAT(0) space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let \triangle be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for \triangle . Then, \triangle is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}, d(x, y) \leq d_{E^2}(\overline{x}, \overline{y})$. It is known that in a CAT(0) space, the distance function is convex [1].

Complete CAT(0) spaces are often called Hadamard spaces. Finally, we observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

(2.1)
$$d(x, \frac{y_1 \oplus y_2}{2})^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2,$$

because equality holds in the Euclidean metric. In fact (see [1, page 163]), a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1) (which is known as the CN inequality of Bruhat and Tits [2]).

The following lemmas can be found in [4].

Lemma 2.1. Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x,z) = td(x,y)$$
 and $d(y,z) = (1-t)d(x,y).$

We use the notation $(1-t)x \oplus ty$ for this unique z.

Lemma 2.2. Let (X, d) be a CAT(0) space. Then,

 $d((1-t)x \oplus ty, z)^2 \le (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2,$ for all $t \in [0, 1]$ and $x, y, z \in X$.

The following result is of Xu [18].

Lemma 2.3. Let R > 1 be a fixed number and X be a Banach space. Then, X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g: [0, \infty) \to [0, \infty)$ with g(0) = 0 such that

 $||\lambda x + (1 - \lambda)y||^2 \le \lambda ||x||^2 + (1 - \lambda)||y||^2 - \lambda (1 - \lambda)g(||x - y||),$ for all $x, y \in B_R(0) = \{x \in X : ||x|| \le R\}$ and $\lambda \in [0, 1].$

Therefore, by Lemma 2.2, it turns out that CAT(0) spaces offer nice examples of uniformly convex metric spaces. It is worth mentioning that the results in CAT(0) spaces can be applied to any $CAT(\kappa)$ space with $\kappa \leq 0$, since any $CAT(\kappa)$ space is a $CAT(\kappa)$ space, for every $\kappa \geq \kappa$ (see [1, page 165]).

Now, we recall some definitions from [15]. Let X be a complete CAT(0) space and (x_n) be a bounded sequence in X. For $x \in X$, set

 $r(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n).$

The asymptotic radius $r((x_n))$ of (x_n) is given by

 $r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},\$

and the asymptotic center $A((x_n))$ of (x_n) is the set

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

Definition 2.4. (see [9, Definition 3.1]) A sequence (x_n) in a CAT(0) X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of (u_n) , for every sequence (u_n) of (x_n) . In this case, we write $\Delta - \lim_n x_n = x$ and call x the Δ -lim of (x_n) .

It is known that in a CAT(0) space, $A((x_n))$ consists of exactly one point [6]. Also, every CAT(0) space has the *Opial* property, i.e., if (x_n) is a sequence in K and Δ -lim $x_n = x$, then for each $y \neq x \in K$,

$$\limsup_{n} d(x_n, x) < \limsup_{n} d(x_n, y).$$

Lemma 2.5. [9] Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.

Lemma 2.6. [5] Let C be a closed convex subset of a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in C. Then, the asymptotic center of $\{x_n\}$ is in C.

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Lemma 2.7. [17] Let C be a closed convex subset of a complete CAT(0) space X, and $T: C \to C$ be a generalized nonexpansive mapping. Then,

$$d(x, Ty) \le 3d(x, Tx) + d(x, y)$$

for all $x, y \in C$.

The following result is a consequence of Lemma 2.9 in [10].

Lemma 2.8. Let X be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in [b, c], for some $b, c \in (0, 1)$, and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n d(x_n, x)} \leq r, \limsup_{n d(y_n, x)} \leq r$, and $\lim_{n d((1 - t_n)x_n \oplus t_ny_n, x)} = r$, for some $r \geq 0$. Then,

$$\lim_{n \to \infty} d(x_n, y_n) = 0$$

3. Main Result

Here, our main result is presented.

Theorem 3.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T: C \to C$ be a generalized nonexpansive mapping. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, $\lim_{n\to\infty} d(x_n, x^*)$ exists, for all $x^* \in F(T)$.

Proof. Set $y_n = s_n T x_n \oplus (1-s_n) x_n$. Since T is generalized nonexpansive and $x^* \in F(T)$,

$$\frac{1}{2}d(x^*, Tx^*) = 0 \le d(x^*, y_n),$$

and

$$\frac{1}{2}d(x^*, Tx^*) = 0 \le d(x^*, x_n),$$

for all $n \ge 1$. It implies $d(Tx^*, Ty_n) \le d(x^*, y_n)$ and $d(Tx^*, Tx_n) \le d(x^*, x_n)$. So,

$$d(x_{n+1}, x^*) = d(t_n T[s_n T x_n \oplus (1 - s_n) x_n] \oplus (1 - t_n) x_n, x^*)$$

$$\leq t_n d(T y_n, x^*) + (1 - t_n) d(x_n, x^*)$$

$$\leq t_n d(y_n, x^*) + (1 - t_n) d(x_n, x^*)$$

$$\leq t_n (s_n d(T x_n, x^*) + (1 - s_n) d(x_n, x^*)) + (1 - t_n) d(x_n, x^*)$$

$$\leq d(x_n, x^*).$$

This implies $d(x_n, x^*)$ is decreasing and bounded below, and so $\lim_n d(x_n, x^*)$ exists.

Theorem 3.2. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a generalized nonexpansive mapping. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, F(T) is nonempty if and only if $\{x_n\}$ is bounded and $\lim_n d(Tx_n, x_n) = 0$.

Proof. Suppose that F(T) is nonempty and $x^* \in F(T)$. Then, by Theorem 3.1, $\lim_n d(x_n, x^*)$ exists and $\{x_n\}$ is bounded. Set

$$(3.1) c = \lim d(x_n, x^*)$$

and $y_n = s_n T x_n \oplus (1 - s_n) x_n$, for all $n \ge 1$. Since

$$\frac{1}{2}d(x^*, Tx^*) = 0 \le d(x^*, y_n),$$

and

$$\frac{1}{2}d(x^*, Tx^*) = 0 \le d(x^*, x_n),$$

for all $n \ge 1$, then $d(Tx^*, Ty_n) \le d(x^*, y_n)$ and $d(Tx^*, Tx_n) \le d(x^*, x_n)$. Thus,

$$d(Ty_n, x^*) \leq d(y_n, x^*) \\ = d(s_n Tx_n \oplus (1 - s_n)x_n, x^*) \\ \leq s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*) \\ \leq s_n d(x_n, x^*) + (1 - s_n)d(x_n, x^*) \\ = d(x_n, x^*).$$

Therefore,

(3.2)
$$\limsup_{n} d(Ty_n, x^*) \le \limsup_{n} d(y_n, x^*) \le c.$$

Furthermore, we have

(3.3)
$$\lim_{n} d(t_n T y_n \oplus (1 - t_n) x_n, x^*) = \lim_{n} d(x_{n+1}, x^*) = c.$$

Case 1: $0 < a \le t_n \le b < 1$ and $0 \le s_n \le b < 1$. By (3.1), (3.2), (3.3) and Lemma 2.8, we have $\lim_n d(Ty_n, x_n) = 0$. Since for each $s_n \in [0, b]$,

then we have

$$s_n d(x_n, Tx_n) \le d(y_n, x_n).$$

Since T is generalized nonexpansive, by choosing $s_n = \frac{1}{2}$, we obtain $d(Tx_n, Ty_n) \leq d(x_n, y_n)$, and so it follows:

$$d(Tx_n, x_n) \leq d(Tx_n, Ty_n) + d(Ty_n, x_n) \\\leq d(x_n, y_n) + d(Ty_n, x_n) \\= d(s_n Tx_n \oplus (1 - s_n)x_n, x_n) + d(Ty_n, x_n) \\\leq s_n d(Tx_n, x_n) + d(Ty_n, x_n).$$

Thus, we have $(1-b)d(Tx_n, x_n) \leq (1-s_n)d(Tx_n, x_n) \leq d(Ty_n, x_n)$. Therefore, $\lim_{n \to \infty} d(Tx_n, x_n) \leq \frac{1}{(1-b)} \lim_{n \to \infty} d(Ty_n, x_n) = 0$.

Case 2: $0 < a \le t_n \le 1$ and $0 < a \le s_n \le b < 1$. Since we have $d(Tx_n, x^*) \leq d(x_n, x^*)$, for all $n \geq 1$, we get (3.4)

4)
$$\limsup_{n} d(Tx_n, x^*) \le c$$

Now,

$$\begin{aligned} d(x_{n+1}, x^*) &\leq t_n d(Ty_n, x^*) + (1 - t_n) d(x_n, x^*) \\ &\leq t_n d(y_n, x^*) + (1 - t_n) d(x_n, x^*) \\ &= t_n d(y_n, x^*) + d(x_n, x^*) - t_n d(x_n, x^*), \end{aligned}$$

which implies

$$\frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \le d(y_n, x^*) - d(x_n, x^*).$$

Taking liminf from both sides of the above inequality, we have

 $\liminf \frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \le \liminf (d(y_n, x^*) - d(x_n, x^*)).$ Since $\lim d(x_{n+1}, x^*) = \lim d(x_n, x^*) = c$, then $0 < \liminf (d(y_n, x^*) - d(x_n, x^*)).$

$$0 \le \liminf(d(y_n, x^*) - d(x_n, x^*))$$

On the other hand, since $d(y_n, x^*) - d(x_n, x^*) \le 0$, $\liminf(d(y_n, x^*) - d(x_n, x^*)) \le 0$. $d(x_n, x^*) \leq 0$. Therefore, $\liminf(d(y_n, x^*) - d(x_n, x^*)) = 0$. This shows

$$\begin{array}{rcl}
0 &=& \liminf(d(y_n, x^*) - d(x_n, x^*)) \\
\leq&& \liminf d(y_n, x^*) - \liminf d(x_n, x^*)
\end{array}$$

Therefore, $\liminf d(x_n, x^*) \leq \liminf d(y_n, x^*)$. This means that $c \leq c$ $\liminf_n d(y_n, x^*)$. By combining this inequality and (3.2), we have

$$c \le \liminf_n d(y_n, x^*) \le \limsup_n d(y_n, x^*) \le c.$$

Therefore,

(3.5)
$$c = \lim_{n} d(y_n, x^*) = \lim_{n} d(s_n T x_n \oplus (1 - s_n) x_n, x^*).$$

By (3.5), (3.4), (3.1) and Lemma 2.8, we have $\lim_n d(Tx_n, x_n) = 0$. Conversely, suppose that $\{x_n\}$ is bounded and $\lim_n d(x_n, Tx_n) = 0$. Let $A((x_n)) = \{x\}$. Then, $x \in C$, by Lemma 2.6. Since T is generalized nonexpansive, we have, by Lemma 2.7,

$$d(x_n, Tx) \le 3d(x_n, Tx_n) + d(x_n, x),$$

which implies

$$\limsup_{n} d(x_n, Tx) \leq \limsup_{n} [3d(x_n, Tx_n) + d(x_n, x)]$$

=
$$\limsup_{n} d(x_n, x).$$

By the uniqueness of asymptotic centers, we get Tx = x. Therefore, x is a fixed point of T.

Theorem 3.3. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and $T: C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is defined by (1.1), where $\{t_n\}$ and $\{s_n\}$ are given by (1.2). Then, $\{x_n\}$, Δ -converges to a fixed point of T.

Proof. Theorem 3.2 guarantees that $\{x_n\}$ is bounded and

$$\lim_{n} d(x_n, Tx_n) = 0.$$

Let $W_w(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset F(T)$.

Let $u \in W_w(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A((u_n)) = \{u\}$. By Lemmas 2.5 and 2.6, there exists a subsequence v_n of u_n such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_n d(v_n, Tv_n) = 0$ and T is generalized nonexpansive, then, by Lemma 2.7,

$$d(v_n, Tv) \le 3d(v_n, Tv_n) + d(v_n, v).$$

By taking lim and *Opial* property, we obtain $v \in F(T)$. Now, we claim that u = v. If not, by Theorem 3.1, $\lim_{n \to \infty} d(x_n, v)$ exists, and thus by the uniqueness of asymptotic centers,

$$\limsup_{n} d(v_n, v) < \limsup_{n} d(v_n, u)$$

$$\leq \limsup_{n} d(u_n, u)$$

$$< \limsup_{n} d(u_n, v)$$

$$= \limsup_{n} d(x_n, v)$$

$$= \limsup_{n} d(v_n, v),$$

which is a contradiction. So, $u = v \in F(T)$. In order to show $\{x_n\}$, Δ -converges to a fixed point of T, it suffices to show that $W_w(x_n)$ consists

of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By lemmas 2.5 and 2.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Let $A((u_n)) = \{u\}$ and $A((x_n)) = \{x\}$. We have seen that v = uand $v \in F(T)$. Therefore, we can complete the proof by showing that v = x. If not, since $\{d(x_n, v)\}$ is convergent by the last argument, then, by the uniqueness of asymptotic centers,

$$\limsup_{n} d(v_n, v) < \limsup_{n} d(v_n, x)$$

$$\leq \limsup_{n} d(x_n, x)$$

$$< \limsup_{n} d(x_n, v)$$

$$= \limsup_{n} d(u_n, v),$$

which is a contradiction, and hence the conclusion follows.

We recall (see [16]), a mapping $T: C \to C$ is said to satisfy *condi*tion (I), if there exists a nondecreasing function $f: [0,\infty] \to [0,\infty)$ with f(0) = 0 and f(r) > 0, for all r > 0, such that $d(x,Tx) \ge f(d(x,F(T)))$, for all $x \in C$, where, $d(x,F(T)) = \inf_{z \in F(T)} d(x,z)$.

Theorem 3.4. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and $T: C \to C$ be a generalized nonexpansive mapping satisfying condition (I) with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is defined by (1.1), where $\{t_n\}$ and $\{s_n\}$ are given by (1.2). Then, $\{x_n\}$ converges strongly to some fixed point of T.

Proof. First, we show that F(T) is closed. Let $\{x_n\}$ be a sequence in F(T) converging to some point $z \in C$. Since

$$\frac{1}{2}d(x_n, Tx_n) = 0 \le d(x_n, z),$$

we have

$$\limsup_{n} d(x_n, Tz) = \limsup_{n} d(Tx_n, Tz)$$

$$\leq \limsup_{n} d(x_n, z)$$

$$= 0.$$

That is, $\{x_n\}$ converges to Tz. This implies Tz = z. Therefore, F(T) is closed. By Theorem 3.2, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. It follows from condition (I) that

$$\lim_{n \to \infty} f\left(d\left(x_n, F(T)\right)\right) \le \lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

Then, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f: [0,\infty] \to [0,\infty)$ is a nondecreasing function satisfying f(0) = 0, f(r) > 0, for all $r \in (0,\infty)$,

we obtain $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) \le \frac{1}{2^k},$$

for all integer $k \ge 1$ and some sequence $\{p_k\}$ in F(T). Again, by Theorem 3.1,

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) \le \frac{1}{2^k}.$$

Hence,

$$\begin{split} d(p_{k+1},p_k) &\leq d(p_{k+1},x_{n_{k+1}}) + d(x_{n_{k+1}},p_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}, \end{split}$$

which implies $\{p_k\}$ is a Cauchy sequence. Since F(T) is closed, then $\{p_k\}$ converges strongly to a point p in F(T). It is readily seen that $\{x_{n_k}\}$ converges strongly to p. Since $\lim_n d(x_n, p)$ exists, it must be the case that $\lim_{n\to\infty} d(x_n, p) = 0$.

Remark 3.5. Since every nonexpansive mapping is a generalized nonexpansive mapping, one can state all the above results for nonexpansive mappings and obtain the results in [10]. Also, by setting $s_n = 0$, one can obtain the results in [13].

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