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APPROXIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) = \{x \in C : T(x) = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

$$x_{n+1} = t_n T[s_n T x_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n,$$

for all $n \geq 1$, where $\{t_n\}$ and $\{s_n\}$ are real sequences in $[0, 1]$ such that one of the following two conditions is satisfied:
(i) $t_n \in [a, b]$ and $s_n \in [0, 1]$, for some a, b with $0 < a \leq b < 1$,
(ii) $t_n \in [a, 1]$ and $s_n \in [a, b]$, for some a, b with $0 < a \leq b < 1$.
Then, the sequence $\{x_n\}$, Δ-converges to a fixed point of T. Our results extend the ones in Laokul and Panyanak [T. Laokul and B. Panyanak, *Int. J. Math. Anal.* **3** (2009) 1305–1315.] and also the ones in Nanjaras et al. [B. Nanjaras, B. Panyanak and W. Phuangrattana, *Nonlinear Anal. Hybrid Syst.* **4** (2010) 25-31.].

1. **Introduction**

Recently, Suzuki [17] introduced condition (C) as follows.
Condition(C): Let T be a mapping on a subset C of Banach space E.

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Then, T is said to satisfy condition (C) (or generalized nonexpansive mapping) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \text{ implies } ||Tx - Ty|| \leq ||x - y||,$$

for all $x, y \in C$.

Proposition 1.1. *Every nonexpansive mapping satisfies condition (C), but the inverse is not true.*

Example 1.2. *Define a mapping T on $[0, 3]$ by*

$$T(x) = \begin{cases} 0 & if x \neq 3, \\ 1 & if x = 3. \end{cases}$$

Then, T satisfies condition (C), but T is not nonexpansive.

The purpose of this paper is to study the iterative scheme defined as follows.
Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

(1.1) $$x_{n+1} = t_n T[s_n T x_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n,$$

for all $n \geq 1$, where, $\{t_n\}$ and $\{s_n\}$ are real sequences in $[0, 1]$ such that one of the following two conditions is satisfied:
(1.2)
(i) $t_n \in [a, b]$ and $s_n \in [0, 1]$, for some a, b with $0 < a \leq b < 1$,
(ii) $t_n \in [a, 1]$ and $s_n \in [a, b]$, for some a, b with $0 < a \leq b < 1$.

We show that the sequence $\{x_n\}$, defined by (1.1), Δ-converges to a fixed point of T.

2. $CAT(0)$ Spaces

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(\acute{t}) = |t - \acute{t}|$, for all $t, \acute{t} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining

x to y, for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$, $d(x, y) \leq d_{E^2}(\bar{x}, \bar{y})$. It is known that in a $CAT(0)$ space, the distance function is convex [1].

Complete $CAT(0)$ spaces are often called Hadamard spaces. Finally, we observe that if x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2, \tag{2.1}$$

because equality holds in the Euclidean metric. In fact (see [1, page 163]), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies inequality (2.1) (which is known as the CN inequality of Bruhat and Tits [2]).

The following lemmas can be found in [4].

Lemma 2.1. *Let (X, d) be a $CAT(0)$ space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for this unique z.

Lemma 2.2. *Let (X, d) be a $CAT(0)$ space. Then,*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

The following result is of Xu [18].

Lemma 2.3. *Let $R > 1$ be a fixed number and X be a Banach space. Then, X is uniformly convex if and only if there exists a continuous,* *strictly increasing, and convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

for all $x, y \in B_R(0) = \{x \in X : ||x|| \leq R\}$ *and* $\lambda \in [0, 1]$.

Therefore, by Lemma 2.2, it turns out that $CAT(0)$ spaces offer nice examples of uniformly convex metric spaces. It is worth mentioning that the results in $CAT(0)$ spaces can be applied to any $CAT(\kappa)$ space with $\kappa \leq 0$, since any $CAT(\kappa)$ space is a $CAT(\acute{\kappa})$ space, for every $\acute{\kappa} \geq \kappa$(see [1, page 165]).

Now, we recall some definitions from [15].
Let X be a complete $CAT(0)$ space and (x_n) be a bounded sequence in X. For $x \in X$, set

$$r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is the set

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

Definition 2.4. (see [9, Definition 3.1]) A sequence (x_n) in a $CAT(0)$ X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of (u_n), for every sequence (u_n) of (x_n). In this case, we write $\Delta-\lim_n x_n = x$ and call x the Δ-lim of (x_n).

It is known that in a $CAT(0)$ space, $A((x_n))$ consists of exactly one point [6]. Also, every $CAT(0)$ space has the *Opial* property, i.e., if (x_n) is a sequence in K and Δ-$\lim x_n = x$, then for each $y(\neq x) \in K$,

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

Lemma 2.5. [9] *Every bounded sequence in a complete* $CAT(0)$ *space always has a* Δ*-convergent subsequence.*

Lemma 2.6. [5] *Let* C *be a closed convex subset of a complete* $CAT(0)$ *space and* $\{x_n\}$ *be a bounded sequence in* C*. Then, the asymptotic center of* $\{x_n\}$ *is in* C.

Lemma 2.7. [17] *Let C be a closed convex subset of a complete $CAT(0)$ space X, and $T : C \to C$ be a generalized nonexpansive mapping. Then,*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y),$$

for all $x, y \in C$.

The following result is a consequence of Lemma 2.9 in [10].

Lemma 2.8. *Let X be a complete $CAT(0)$ space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$, for some $b, c \in (0, 1)$, and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_n d(x_n, x) \leq r, \limsup_n d(y_n, x) \leq r$, and $\lim_n d((1 - t_n)x_n \oplus t_n y_n, x) = r$, for some $r \geq 0$. Then,*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

3. Main Result

Here, our main result is presented.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a generalized nonexpansive mapping. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, $\lim_{n\to\infty} d(x_n, x^*)$ exists, for all $x^* \in F(T)$.*

Proof. Set $y_n = s_n T x_n \oplus (1 - s_n)x_n$. Since T is generalized nonexpansive and $x^* \in F(T)$,

$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, y_n),$$

and

$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, x_n),$$

for all $n \geq 1$. It implies $d(Tx^*, Ty_n) \leq d(x^*, y_n)$ and $d(Tx^*, Tx_n) \leq d(x^*, x_n)$. So,

$$\begin{aligned} d(x_{n+1}, x^*) = \;& d(t_n T[s_n T x_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n, x^*) \\ \leq \;& t_n d(Ty_n, x^*) + (1 - t_n)d(x_n, x^*) \\ \leq \;& t_n d(y_n, x^*) + (1 - t_n)d(x_n, x^*) \\ \leq \;& t_n(s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*)) + (1 - t_n)d(x_n, x^*) \\ \leq \;& d(x_n, x^*). \end{aligned}$$

This implies $d(x_n, x^*)$ is decreasing and bounded below, and so $\lim_n d(x_n, x^*)$ exists. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a generalized nonexpansive mapping. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.1), where sequences $\{t_n\}, \{s_n\}$ are given by (1.2). Then, $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_n d(Tx_n, x_n) = 0$.*

Proof. Suppose that $F(T)$ is nonempty and $x^* \in F(T)$. Then, by Theorem 3.1, $\lim_n d(x_n, x^*)$ exists and $\{x_n\}$ is bounded. Set

$$c = \lim_n d(x_n, x^*) \tag{3.1}$$

and $y_n = s_n Tx_n \oplus (1 - s_n)x_n$, for all $n \geq 1$. Since

$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, y_n),$$

and

$$\frac{1}{2} d(x^*, Tx^*) = 0 \leq d(x^*, x_n),$$

for all $n \geq 1$, then $d(Tx^*, Ty_n) \leq d(x^*, y_n)$ and $d(Tx^*, Tx_n) \leq d(x^*, x_n)$. Thus,

$$\begin{aligned} d(Ty_n, x^*) \leq & \ d(y_n, x^*) \\ = & \ d(s_n Tx_n \oplus (1 - s_n)x_n, x^*) \\ \leq & \ s_n d(Tx_n, x^*) + (1 - s_n) d(x_n, x^*) \\ \leq & \ s_n d(x_n, x^*) + (1 - s_n) d(x_n, x^*) \\ = & \ d(x_n, x^*). \end{aligned}$$

Therefore,

$$\limsup_n d(Ty_n, x^*) \leq \limsup_n d(y_n, x^*) \leq c. \tag{3.2}$$

Furthermore, we have

$$\lim_n d(t_n Ty_n \oplus (1 - t_n)x_n, x^*) = \lim_n d(x_{n+1}, x^*) = c. \tag{3.3}$$

Case 1 : $0 < a \leq t_n \leq b < 1$ and $0 \leq s_n \leq b < 1$.
By (3.1), (3.2), (3.3) and Lemma 2.8, we have $\lim_n d(Ty_n, x_n) = 0$. Since for each $s_n \in [0, b]$,

$$\begin{aligned} d(Tx_n, x_n) \leq & \ d(Tx_n, y_n) + d(y_n, x_n) \\ \leq & \ (1 - s_n) d(x_n, Tx_n) + d(y_n, x_n), \end{aligned}$$

then we have

$$s_n d(x_n, Tx_n) \leq d(y_n, x_n).$$

Since T is generalized nonexpansive, by choosing $s_n = \frac{1}{2}$, we obtain $d(Tx_n, Ty_n) \leq d(x_n, y_n)$, and so it follows:

$$\begin{aligned} d(Tx_n, x_n) \leq & \ d(Tx_n, Ty_n) + d(Ty_n, x_n) \\ \leq & \ d(x_n, y_n) + d(Ty_n, x_n) \\ = & \ d(s_nTx_n \oplus (1-s_n)x_n, x_n) + d(Ty_n, x_n) \\ \leq & \ s_nd(Tx_n, x_n) + d(Ty_n, x_n). \end{aligned}$$

Thus, we have $(1-b)d(Tx_n, x_n) \leq (1-s_n)d(Tx_n, x_n) \leq d(Ty_n, x_n)$. Therefore, $\lim_n d(Tx_n, x_n) \leq \frac{1}{(1-b)} \lim_n d(Ty_n, x_n) = 0$.

Case 2 : $0 < a \leq t_n \leq 1$ and $0 < a \leq s_n \leq b < 1$.
Since we have $d(Tx_n, x^*) \leq d(x_n, x^*)$, for all $n \geq 1$, we get

$$\limsup_n d(Tx_n, x^*) \leq c. \tag{3.4}$$

Now,

$$\begin{aligned} d(x_{n+1}, x^*) \leq & \ t_nd(Ty_n, x^*) + (1-t_n)d(x_n, x^*) \\ \leq & \ t_nd(y_n, x^*) + (1-t_n)d(x_n, x^*) \\ = & \ t_nd(y_n, x^*) + d(x_n, x^*) - t_nd(x_n, x^*), \end{aligned}$$

which implies

$$\frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq d(y_n, x^*) - d(x_n, x^*).$$

Taking lim inf from both sides of the above inequality, we have

$$\liminf \frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq \liminf(d(y_n, x^*) - d(x_n, x^*)).$$

Since $\lim d(x_{n+1}, x^*) = \lim d(x_n, x^*) = c$, then

$$0 \leq \liminf(d(y_n, x^*) - d(x_n, x^*)).$$

On the other hand, since $d(y_n, x^*) - d(x_n, x^*) \leq 0$, $\liminf(d(y_n, x^*) - d(x_n, x^*)) \leq 0$. Therefore, $\liminf(d(y_n, x^*) - d(x_n, x^*)) = 0$. This shows

$$\begin{aligned} 0 = & \ \liminf(d(y_n, x^*) - d(x_n, x^*)) \\ \leq & \ \liminf d(y_n, x^*) - \liminf d(x_n, x^*). \end{aligned}$$

Therefore, $\liminf \ d(x_n, x^*) \ \leq \ \liminf d(y_n, x^*)$. This means that $c \leq \liminf_n d(y_n, x^*)$. By combining this inequality and (3.2), we have

$$c \leq \liminf_n d(y_n, x^*) \leq \limsup_n d(y_n, x^*) \leq c.$$

Therefore,

$$c = \lim_n d(y_n, x^*) = \lim_n d(s_nTx_n \oplus (1-s_n)x_n, x^*). \tag{3.5}$$

By (3.5), (3.4), (3.1) and Lemma 2.8, we have $\lim_n d(Tx_n, x_n) = 0$.
Conversely, suppose that $\{x_n\}$ is bounded and $\lim_n d(x_n, Tx_n) = 0$. Let $A((x_n)) = \{x\}$. Then, $x \in C$, by Lemma 2.6. Since T is generalized nonexpansive, we have, by Lemma 2.7,

$$d(x_n, Tx) \leq 3d(x_n, Tx_n) + d(x_n, x),$$

which implies

$$\begin{aligned} \limsup_n d(x_n, Tx) \leq & \ \limsup_n [3d(x_n, Tx_n) + d(x_n, x)] \\ = & \ \limsup_n d(x_n, x). \end{aligned}$$

By the uniqueness of asymptotic centers, we get $Tx = x$. Therefore, x is a fixed point of T. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is defined by (1.1), where $\{t_n\}$ and $\{s_n\}$ are given by (1.2). Then, $\{x_n\}$, Δ-converges to a fixed point of T.*

Proof. Theorem 3.2 guarantees that $\{x_n\}$ is bounded and

$$\lim_n d(x_n, Tx_n) = 0.$$

Let $W_w(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset F(T)$.
Let $u \in W_w(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A((u_n)) = \{u\}$. By Lemmas 2.5 and 2.6, there exists a subsequence v_n of u_n such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_n d(v_n, Tv_n) = 0$ and T is generalized nonexpansive, then, by Lemma 2.7,

$$d(v_n, Tv) \leq 3d(v_n, Tv_n) + d(v_n, v).$$

By taking lim and *Opial* property, we obtain $v \in F(T)$. Now, we claim that $u = v$. If not, by Theorem 3.1, $\lim_n d(x_n, v)$ exists, and thus by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_n d(v_n, v) < & \ \limsup_n d(v_n, u) \\ \leq & \ \limsup_n d(u_n, u) \\ < & \ \limsup_n d(u_n, v) \\ = & \ \limsup_n d(x_n, v) \\ = & \ \limsup_n d(v_n, v), \end{aligned}$$

which is a contradiction. So, $u = v \in F(T)$. In order to show $\{x_n\}$, Δ-converges to a fixed point of T, it suffices to show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By lemmas 2.5 and 2.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Let $A((u_n)) = \{u\}$ and $A((x_n)) = \{x\}$. We have seen that $v = u$ and $v \in F(T)$. Therefore, we can complete the proof by showing that $v = x$. If not, since $\{d(x_n, v)\}$ is convergent by the last argument, then, by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_n d(v_n, v) &< \limsup_n d(v_n, x) \\ &\leq \limsup_n d(x_n, x) \\ &< \limsup_n d(x_n, v) \\ &= \limsup_n d(u_n, v), \end{aligned}$$

which is a contradiction, and hence the conclusion follows. □

We recall (see [16]), a mapping $T : C \to C$ is said to satisfy *condition* (I), if there exists a nondecreasing function $f : [0, \infty] \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r > 0$, such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in C$, where, $d(x, F(T)) = \inf_{z \in F(T)} d(x, z)$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $T : C \to C$ be a generalized nonexpansive mapping satisfying condition (I) with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is defined by (1.1), where $\{t_n\}$ and $\{s_n\}$ are given by (1.2). Then, $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ converging to some point $z \in C$. Since

$$\frac{1}{2} d(x_n, Tx_n) = 0 \leq d(x_n, z),$$

we have

$$\begin{aligned} \limsup_n d(x_n, Tz) &= \limsup_n d(Tx_n, Tz) \\ &\leq \limsup_n d(x_n, z) \\ &= 0. \end{aligned}$$

That is, $\{x_n\}$ converges to Tz. This implies $Tz = z$. Therefore, $F(T)$ is closed. By Theorem 3.2, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. It follows from condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Then, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0, \infty] \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$, for all $r \in (0, \infty)$, we obtain $\lim_{n\to\infty} d\,(x_n, F(T)) = 0$. Hence, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) \le \frac{1}{2^k},$$

for all integer $k \ge 1$ and some sequence $\{p_k\}$ in $F(T)$. Again, by Theorem 3.1,

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) \le \frac{1}{2^k}.$$

Hence,

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}, \end{aligned}$$

which implies $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, then $\{p_k\}$ converges strongly to a point p in $F(T)$. It is readily seen that $\{x_{n_k}\}$ converges strongly to p. Since $\lim_n d(x_n, p)$ exists, it must be the case that $\lim\limits_{n\to\infty} d(x_n, p) = 0$. □

Remark 3.5. *Since every nonexpansive mapping is a generalized nonexpansive mapping, one can state all the above results for nonexpansive mappings and obtain the results in* [10]*. Also, by setting $s_n = 0$, one can obtain the results in* [13].

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