ITERATIVE ALGORITHMS FOR FAMILIES OF VARIATIONAL INEQUALITIES FIXED POINTS AND EQUILIBRIUM PROBLEMS

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ABSTRACT. We introduce an iterative algorithm for finding a common element of the set of fixed points for an infinite family of nonexpansive mappings, the set of solutions of the variational inequalities for a family of α-inverse-strongly monotone mappings and the set of solutions of a system of equilibrium problems in a Hilbert space. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. Moreover, we apply our result to the problem of finding a common fixed point of a family of strictly pseudocontractive mappings.

1. Introduction

Let C be a closed convex subset of a Hilbert space H. Then, a mapping S of C into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$. We denote by $Fix(S)$ the set of fixed points of S.

Recall that a mapping $B : C \to H$ is called α-inverse-strongly monotone [3] if there exists a positive real number α such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in C.$$

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It is easy to see that if $B : C \to H$ is α-inverse-strongly monotone, then it is a $\frac{1}{\alpha}$-Lipschitzian mapping.

Let $B : C \to H$ be a mapping. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of this variational inequality is denoted by $VI(C, B)$.

For finding an element of $Fix(S) \cap VI(C, B)$, Takahashi and Toyoda [27] introduced the following iterative scheme:

$$x_{n+1} = \epsilon_n x_n + (1 - \epsilon_n) SP_C(x_n - \lambda_n B x_n), \quad n \geq 0,$$

where, $x_0 = x \in C$, $\{\epsilon_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$.

On the other hand, Moudafi [16] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on a Hilbert space H (i.e., $\|f(x) - f(y)\| \leq l\|x - y\|$, $x, y \in H$ and $0 \leq l < 1$). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$(1.1) \qquad x_{n+1} = (1 - \epsilon_n) S x_n + \epsilon_n f(x_n), \quad n \geq 0,$$

where, $\{\epsilon_n\}$ is a sequence in $(0, 1)$. It is proved [16, 30] that, under certain appropriate conditions imposed on $\{\epsilon_n\}$, the sequence $\{x_n\}$ generated by (1.1) strongly converges to the unique solution x^* in $\mathcal{F} := Fix(S)$ of the variational inequality

$$\langle (I - f) x^*, x - x^* \rangle \geq 0, \quad x \in \mathcal{F}.$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems and variational inequalities; see, e.g., [4, 10, , 21, 28, 29, 31, 32]. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$(1.2) \qquad \min_{x \in \mathcal{F}} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle,$$

where, $\mathcal{F}$ is the fixed point set of a nonexpansive mapping S on H and u is a given point in H. Assume A is strongly positive; that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \text{for all } x \in H.$$

Xu [28] (see also [31]) proved that the sequence $\{x_n\}$, defined by the iterative method

(1.3) $$x_{n+1} = (I - \epsilon_n A)Sx_n + \epsilon_n u, \quad n \geq 0,$$

with the initial guess $x_0 \in H$ chosen arbitrarily, converges strongly to the unique solution of the minimization problem (1.2) provided that the sequence $\{\epsilon_n\}$ satisfies certain conditions.

Marino and Xu [15] combined the iterative method (1.3) with the viscosity approximation method (1.1) and considered the following general iterative method:

(1.4) $$x_{n+1} = (I - \epsilon_n A)Sx_n + \epsilon_n \gamma f(x_n), \quad n \geq 0,$$

where, $0 < \gamma < \overline{\gamma}/\alpha$. They proved that if the sequence $\{\epsilon_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$, generated by (1.4), converges strongly to the unique solution of the variational inequality,

(1.5) $$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \mathcal{F},$$

which is the optimality condition for the minimization problem,

(1.6) $$\min_{x \in \mathcal{F}} \frac{1}{2}\langle Ax, x \rangle - h(x)$$

where, h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for $x \in H$).

Finding an optimal point in the intersection $\mathcal{F}$ of the fixed point sets of a family of nonexpansive mappings is a task frequently arising from various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, e.g., [1, 8]. A simple algorithmic solution to the problem of minimizing a quadratic function over $\mathcal{F}$ is of an extreme value in many applications including set theoretic signal estimation; see, e.g., [14, 33].

On the other hand, let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e., the set

$$EP(F) := \{x \in C : F(x, y) \geq 0 \; \forall y \in C\}.$$

Let $\mathcal{G} = \{F_i\}_{i \in I}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problems for $\mathcal{G} = \{F_i\}_{i \in I}$ is to determine common equilibrium points for $\mathcal{G} = \{F_i\}_{i \in I}$, i.e., the set

(1.7) $$EP(\mathcal{G}) := \{x \in C : F_i(x, y) \geq 0 \; \forall y \in C \; \forall i \in I\}.$$

Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems reduce into finding some element of $EP(F)$; see [2, 9, 11]. The formulation (1.7) extends this formalism to systems of such problems, covering, in particular, various forms of feasibility problems [9, 12].

Given any $r > 0$, the operator $J_r^F : H \to C$, defined by

$$J_r^F(x) := \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \; \forall y \in C\},$$

is called the resolvent of F (see [9, 12]).

It is shown [5, 7] that under suitable hypotheses on F, $J_r^F : H \to C$ is single-valued and firmly nonexpansive and satisfies $Fix(J_r^F) = EP(F)$, $\forall r > 0$.

Using this result, Takahashi and Takahashi [26] introduced a viscosity approximation method for finding a common element of $EP(F)$ and $Fix(S)$, where S is a nonexpansive mapping. Starting with an arbitrary element x_1 in H, they defined the sequence $\{x_n\}$ recursively by

$$(1.8) \qquad\qquad x_{n+1} = \epsilon_n f(x_n) + (1 - \epsilon_n) S J_{r_n}^F x_n.$$

They proved that, under certain appropriate conditions over ϵ_n and r_n, the sequences $\{x_n\}$ and $\{J_{r_n}^F x_n\}$ both converge strongly to $x^* = P_{Fix(S) \cap EP(F)} f(x^*)$.

By combining the schemes (1.4) and (1.8), Plubtieng and Punpaeng [17] proposed the following iterative scheme:

$$(1.9) \qquad\qquad x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S J_{r_n}^F x_n.$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{J_{r_n}^F x_n\}$ both converge strongly to the unique solution $x^* \in \mathcal{F} := Fix(S) \cap EP(F)$ of the variational inequality (1.5), being the optimality condition for the minimization problem (1.6). Note that the result in [15] is a particular case of this, corresponding to the choice $F(x, y) = 0$ (so that $J_{r_n}^F = I$).

Very recently, Colao et al. [6] proposed the following explicit scheme with respect to W-mappings for a finite family of nonexpansive mappings:

$$(1.10) \qquad\qquad x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1 - \beta)I - \epsilon_n A) W_n J_{r_n}^F x_n.$$

They proved, under weaker hypotheses, that both sequences $\{x_n\}$ and $\{J^F_{r_n} x_n\}$ converge strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality (1.5).

Here, motivated by Colao, et al. [6], Takahashi and Toyoda [27] and some of our previous results [19, 20], we introduce an iterative algorithm (Theorem 3.1) for finding a common element of the set of solutions of a system of equilibrium problems $EP(\mathcal{G})$ for a family $\mathcal{G} = \{F_i : i = 1, \ldots, M\}$ of bifunctions, the set of solutions of variational inequalities $VI(C, B_j)$ for a family $\{B_j : j = 1 \ldots N\}$ of α-inverse-strongly monotone mapping from C into H and the set of fixed point for an infinite family of nonexpansive mappings $\varphi = \{S_i : C \to C\}$, with respect to W-mappings defined in [22]. We prove the strong convergence of the proposed iterative process to the unique solution of the variational inequality (1.5). Our results cover all previous schemes specified by (1.1), (1.3), (1.4), (1.8), (1.9) and (1.10). Moreover, we apply our result to the problem of finding a common fixed point of a family of strictly pseudocontractive mappings (Theorem 3.2).

2. **Preliminaries**

Let C be a nonempty closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e., the set

$$EP(F) := \{x \in C : F(x, y) \geq 0 \; \forall y \in C\}.$$

Given any $r > 0$, the operator $J^F_r : H \to C$, defined by

$$J^F_r(x) := \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \; \forall y \in C\},$$

is called the resolvent of F; See [9, 12].

Lemma 2.1. *([9, 12]) Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ satisfy*

(A1) $F(x, x) = 0$, for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$.
(A3) for all $x, y, z \in C$,

$$\liminf_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) *for all* $x \in C$, $y \longmapsto F(x, y)$ *is convex and lower semicontinuous.*
Then,

(1) J_r^F *is single-valued;*
(2) J_r^F *is firmly nonexpansive, i.e.,*

$$\|J_r^F x - J_r^F y\|^2 \le \langle J_r^F x - J_r^F y, x - y \rangle, \text{ for all } x, y \in H;$$

(3) $Fix(J_r^F) = EP(F);$
(4) $EP(F)$ *is closed and convex.*

Recall that the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property,

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is known that P_C is a nonexpansive mapping and satisfies

(2.1) $$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

P_C is characterized as follows:

$$y = P_C x \iff \langle x - y, y - z \rangle \ge 0, \quad \forall z \in C.$$

In the context of the variational inequality problem, this implies:

(2.2) $$u \in VI(C, B) \iff u = P_C(u - \lambda Bu), \quad \forall \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$, we have $\langle f - g, x - y \rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \ge 0$, and $(y, g) \in G(T)$, we have $f \in Tx$. Let $B : C \to H$ be an inverse-strongly monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \ge 0, \forall u \in C\},$$

and define

$$Tv = \left\{ \begin{array}{ll} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{array} \right.$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$ (see [13, 18]).

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.2. *For all $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.3. *([29]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where, $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *([23]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n$ and $\limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n,$$

for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.5. *Let $\{S_i : C \to C\}$ be an infinite family of nonexpansive mappings and $\{\mu_i\}$ be a nonnegative real sequence with $0 \leq \mu_i < 1$, $\forall i \geq 1$. For any $n \geq 1$, define a mapping $W_n : C \to C$ as follows:*

$$U_{n,n+1} := I,$$
$$U_{n,n} := \mu_n S_n U_{n,n+1} + (1 - \mu_n)I,$$
$$U_{n,n-1} := \mu_{n-1} S_{n-1} U_{n,n} + (1 - \mu_{n-1})I,$$
$$\vdots$$
$$U_{n,k} := \mu_k S_k U_{n,k+1} + (1 - \mu_k)I,$$
$$U_{n,k-1} := \mu_{k-1} S_{k-1} U_{n,k} + (1 - \mu_{k-1})I,$$
$$\vdots$$
$$U_{n,2} := \mu_2 S_2 U_{n,3} + (1 - \mu_2)I,$$
(2.3) $$W_n := U_{n,1} = \mu_1 S_1 U_{n,2} + (1 - \mu_1)I.$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping, generated by $S_n, S_{n-1}, \ldots, S_1$ and $\mu_n, \mu_{n-1}, \ldots, \mu_1$.

The concept of W-mappings was introduced in [24, 25]. It is now a main tool in studying convergence of iterative methods to common fixed points of nonlinear mappings.

Lemma 2.6. *([22]) Let C be a nonempty closed convex subset of a Hilbert space H, $\varphi = \{S_i : C \to C\}$ be an infinite family of nonexpansive mappings with $Fix(\varphi) := \cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$, and $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$, $\forall i \geq 1$. Then,*

(1) W_n is nonexpansive and $Fix(W_n) = \cap_{i=1}^{n} Fix(S_i)$, for each $n \geq 1$;
(2) for each $x \in C$ and for each positive integer k, $\lim_{n \to \infty} U_{n,k} x$ exists;
(3) the mapping $W : C \to C$, defined by

$$W x := \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x,$$

is a nonexpansive mapping satisfying $Fix(W) = Fix(\varphi)$ and it is called the W-mapping, generated by $S_1, S_2, \ldots$ and $\mu_1, \mu_2, \ldots$;
(4) if K is any bounded subset of C, then

$$\lim_{n \to \infty} \sup_{x \in K} \|W x - W_n x\| = 0.$$

Lemma 2.7. *([15]) Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho \overline{\gamma}$.*

3. Main Results

The following is our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H, $\varphi = \{S_i : C \to C\}$ be an infinite family of nonexpansive mappings, $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), $\{B_k : k = 1 \ldots N\}$ be a finite family of α-inverse-strongly monotone mappings from C into H, and $\mathcal{F} := \cap_{k=1}^{N} VI(C, B_k) \cap Fix(\varphi) \cap EP(\mathcal{G}) \neq \emptyset$.*
Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, f be an l-contraction on H, for some $0 < l < 1$, $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{\lambda_{k,n}\}_{k=1}^{N}$ be sequences in $[a, b]$ with $0 < a \leq b < 2\alpha$, $\{r_{j,n}\}_{j=1}^{M}$ be sequences in $(0, \infty)$ and γ and β be two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \overline{\gamma}/l$. Assume:

(C1) $\lim_n \epsilon_n = 0$;

(C2) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(D1) $\liminf_n r_{j,n} > 0$, for every $j \in \{1, \ldots, M\}$;

(D2) $\lim_n r_{j,n}/r_{j,n+1} = 1$, for every $j \in \{1, \ldots, M\}$;

(D3) $\lim_n |\lambda_{k,n} - \lambda_{k,n+1}| = 0$, for every $k \in \{1, \ldots, N\}$.

For every $n \in \mathbb{N}$, let W_n be the W-mapping defined by (2.3). If $\{x_n\}$ is the sequence generated by $x_1 \in H$ and $\forall n \geq 1$,

$$(3.1) \quad \begin{cases} z_n = J_{r_{M,n}}^{F_M} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\[2mm] y_n = P_C(I - \lambda_{N,n}B_N) \ldots P_C(I - \lambda_{2,n}B_2)P_C(I - \lambda_{1,n}B_1)z_n, \\[2mm] x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n y_n, \end{cases}$$

then $\{x_n\}$ and $\{J_{r_{k,n}}^{F_k} x_n\}_{k=1}^M$ converge strongly to $x^* \in \mathcal{F} := \cap_{k=1}^N VI(C, B_k)$ $\cap Fix(\varphi) \cap EP(\mathcal{G})$, which is the unique solution of the variational inequality (1.5). Equivalently, we have $P_{\mathcal{F}}(I - A + \gamma f)x^* = x^*$.

Proof. Since $\epsilon_n \to 0$, we shall assume that $\epsilon_n \leq (1 - \beta)\|A\|^{-1}$ and $1 - \epsilon_n(\bar{\gamma} - l\gamma) > 0$. Observe that if $\|u\| = 1$, then

$$\langle ((1-\beta)I - \epsilon_n A)u, u \rangle = (1-\beta) - \epsilon_n \langle Au, u \rangle \geq (1 - \beta - \epsilon_n\|A\|) \geq 0.$$

By Lemma 2.7, we have

$$(3.2) \qquad \|(1-\beta)I - \epsilon_n A\| \leq 1 - \beta - \epsilon_n \bar{\gamma}.$$

Moreover, by taking

$$\mathcal{J}_n^k := J_{r_{k,n}}^{F_k} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}, \ \forall k \in \{1, \ldots, M\},$$

$$\mathcal{J}_n^0 := I,$$

and

$$\mathcal{P}_n^k := P_C(I - \lambda_{k,n}B_k) \ldots P_C(I - \lambda_{2,n}B_2)P_C(I - \lambda_{1,n}B_1), \ \forall k \in \{1, \ldots, N\},$$

$$\mathcal{P}_n^0 := I,$$

we write the scheme (3.1) as:

$$(3.3) \qquad x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n.$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded.

Proof of Step 1. Let $v \in \mathcal{F}$. Since, for each $k \in \{1, \ldots, M\}$, $J_{r_{k,n}}^{F_k}$ is nonexpansive, we have

$$\|\mathcal{J}_n^M x_n - v\| = \|\mathcal{J}_n^M x_n - \mathcal{J}_n^M v\| \leq \|x_n - v\|.$$

On the other hand, since for each $k \in \{1, \ldots N\}$, the mapping $B_k : C \to H$ is an α-inverse-strongly monotone mapping, for any $x, y \in C$ and $\lambda_{k,n} \in [a, b] \subset [0, 2\alpha]$, we have

$$\|(I - \lambda_{k,n} B_k)x - (I - \lambda_{k,n} B_k)y\|^2 = \|(x - y) - \lambda_{k,n}(B_k x - B_k y)\|^2$$

$$= \|x - y\|^2 - 2\lambda_{k,n}\langle x - y, B_k x - B_k y\rangle + \lambda_{k,n}^2\|B_k x - B_k y\|^2$$

$$\leq \|x - y\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\alpha)\|B_k x - B_k y\|^2 \leq \|x - y\|^2,$$

which implies that $P_C(I - \lambda_{k,n} B_k)$ is nonexpansive, and consequently that the mappings $\mathcal{P}_n^k$ are nonexpansive. From Lemma 2.6 and (2.2), we have $\mathcal{P}_n^N v = v = W_n v$. It follows that

$$\|x_{n+1} - v\| = \|((1 - \beta)I - \epsilon_n A)(W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M v)$$

$$\epsilon_n \gamma(f(x_n) - f(v)) + \epsilon_n(\gamma f(v) - Av) + \beta(x_n - v)\|$$

$$\leq (1 - \epsilon_n(\overline{\gamma} - l\gamma))\|x_n - v\| + \epsilon_n(\overline{\gamma} - l\gamma)\frac{\|\gamma f(v) - Av\|}{\overline{\gamma} - l\gamma},$$

which gives

$$\|x_n - v\| \leq \max\{\|x_1 - v\|, \frac{\|\gamma f(v) - Av\|}{\overline{\gamma} - l\gamma}\}, \quad \forall n \geq 1.$$

Step 2. Let $\{\omega_n\}$ be a bounded sequence in H. Then,

$$(3.4) \qquad \lim_{n \to \infty} \|\mathcal{J}_{n+1}^k w_n - \mathcal{J}_n^k \omega_n\| = 0,$$

for every $k \in \{1, \ldots, M\}$.

Proof of Step 2. From [6, Step 2], we have that

$$(3.5) \qquad \lim_n \|J_{r_{k,n+1}}^{F_k} \omega_n - J_{r_{k,n}}^{F_k} \omega_n\| = 0,$$

for for every $k \in \{1, \ldots, M\}$. Note that for every $k \in \{1, \ldots, M\}$, we have

$$\mathcal{J}_n^k = J_{r_{k,n}}^{F_k} \mathcal{J}_n^{k-1}.$$

So,

$$\|\mathcal{J}_{n+1}^k w_n - \mathcal{J}_n^k \omega_n\| \leq \|J_{r_{k,n+1}}^{F_k} \mathcal{J}_{n+1}^{k-1} w_n - J_{r_{k,n}}^{F_k} \mathcal{J}_{n+1}^{k-1} w_n\|$$

$$+ \|J_{r_{k,n}}^{F_k} J_{r_{k-1,n+1}}^{F_{k-1}} \mathcal{J}_{n+1}^{k-2} w_n - J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} \mathcal{J}_{n+1}^{k-2} w_n\| + \ldots$$

$$+ \|J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} \ldots J_{r_{3,n}}^{F_3} J_{r_{2,n+1}}^{F_2} J_{r_{1,n+1}}^{F_1} \omega_n - J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} \ldots J_{r_{3,n}}^{F_3} J_{r_{2,n}}^{F_2} J_{r_{1,n+1}}^{F_1} \omega_n\|$$

$$+ \|J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} \omega_n\|$$

$$\leq \|J^{F_k}_{r_{k,n+1}} \mathcal{J}^{k-1}_{n+1} w_n - J^{F_k}_{r_{k,n}} \mathcal{J}^{k-1}_{n+1} w_n\|$$

$$+ \|J^{F_{k-1}}_{r_{k-1,n+1}} \mathcal{J}^{k-2}_{n+1} w_n - J^{F_{k-1}}_{r_{k-1,n}} \mathcal{J}^{k-2}_{n+1} w_n\|$$

$$+ \dots + \|J^{F_2}_{r_{2,n+1}} J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n+1}} \omega_n\| + \|J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_1}_{r_{1,n}} \omega_n\|$$

$$= \sum_{j=1}^{k} \|J^{F_j}_{r_{j,n+1}} (\mathcal{J}^{j-1}_{n+1} w_n) - J^{F_j}_{r_{j,n}} (\mathcal{J}^{j-1}_{n+1} w_n)\|.$$

Now, apply (3.5) to conclude (3.4).

Step 3. Let $\{\omega_n\}$ be a bounded sequence in C. Then,

$$\lim_{n \to \infty} \|P_C(I - \lambda_{k,n+1} B_k) w_n - P_C(I - \lambda_{k,n} B_k) \omega_n\| = 0,$$

and

$$\lim_{n \to \infty} \|\mathcal{P}^k_{n+1} w_n - \mathcal{P}^k_n \omega_n\| = 0,$$

for every $k \in \{1, \dots, N\}$.

Proof of Step 3. Since $\{\omega_n\}$ is bounded and B_k, for $k \in \{1, \dots, N\}$, is a Lipschitzian mapping, we know that

$$L := \sup_n \{\|B_k \omega_n\|\} < \infty.$$

Now,

$$\|P_C(I - \lambda_{k,n+1} B_k) w_n - P_C(I - \lambda_{k,n} B_k) \omega_n\|$$

$$\leq \|(I - \lambda_{k,n+1} B_k) w_n - (I - \lambda_{k,n} B_k) \omega_n\|$$

$$= |\lambda_{k,n+1} - \lambda_{k,n}| \|B_k \omega_n\| \leq |\lambda_{k,n+1} - \lambda_{k,n}| L \to 0, \text{ as } n \to \infty.$$

From this and applying a technique similar to that used in proof of Step 2, it is easy to prove the second assertion.

Step 4. $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Proof of Step 4. Define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta x_n)/(1 - \beta)$ so that $x_{n+1} = \beta x_n + (1 - \beta) z_n$.

Now, compute

$$(3.6) \qquad \|z_{n+1} - z_n\| = \frac{1}{1-\beta} \|(x_{n+2} - \beta x_{n+1} - (x_{n+1} - \beta x_n))\|$$

$$= \frac{1}{1-\beta} \|\gamma(\epsilon_n f(x_{n+1}) - \epsilon_n f(x_n)) + ((1-\beta)I - \epsilon_{n+1} A) W_{n+1} \mathcal{P}^N_{n+1} \mathcal{J}^M_{n+1} x_{n+1}$$

$$- ((1-\beta)I - \epsilon_n A) W_n \mathcal{P}^N_n \mathcal{J}^M_n x_n\|$$

$$= \|\frac{\gamma}{1-\beta} (\epsilon_n f(x_{n+1}) - \epsilon_n f(x_n)) - \frac{1}{1-\beta} (\epsilon_{n+1} A W_{n+1} \mathcal{P}^N_{n+1} \mathcal{J}^M_{n+1} x_{n+1}$$

$$-\epsilon_n A W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n) + W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_{n+1}^M x_{n+1} - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|.$$

Since $\{x_n\}$ is bounded and by (3.6), we have for some big enough constant $K > 0$,

$$(3.7)\ \|z_{n+1}-z_n\| \leq \|W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_{n+1}^M x_{n+1} - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| + K(\epsilon_{n+1}+\epsilon_n)$$

$$\leq \|W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_{n+1}^M x_{n+1} - W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_{n+1}^M x_n\|$$

$$+\|W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_{n+1}^M x_n - W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_n^M x_n\|$$

$$+\|W_{n+1}\mathcal{P}_{n+1}^N \mathcal{J}_n^M x_n - W_{n+1}\mathcal{P}_n^N \mathcal{J}_n^M x_n\|$$

$$+\|W_{n+1}\mathcal{P}_n^N \mathcal{J}_n^M x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| + K(\epsilon_{n+1} + \epsilon_n)$$

$$\leq \|x_{n+1} - x_n\| + \|\mathcal{J}_{n+1}^M x_n - \mathcal{J}_n^M x_n\| + \|\mathcal{P}_{n+1}^N \mathcal{J}_n^M x_n - \mathcal{P}_n^N \mathcal{J}_n^M x_n\|$$

$$+\|W_{n+1}\mathcal{P}_n^N \mathcal{J}_n^M x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| + K(\epsilon_{n+1} + \epsilon_n).$$

Now, since $\epsilon_n \to 0$ and by Steps 2, 3 and Lemma 2.6, we immediately conclude from (3.7) that

$$\limsup_n(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|)$$

$$\leq \limsup_n\{\|\mathcal{J}_{n+1}^M x_n - \mathcal{J}_n^M x_n\| + \|\mathcal{P}_{n+1}^N \mathcal{J}_n^M x_n - \mathcal{P}_n^N \mathcal{J}_n^M x_n\|$$

$$+\|W_{n+1}\mathcal{P}_n^N \mathcal{J}_n^M x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| + K(\epsilon_{n+1} + \epsilon_n)\} \leq 0.$$

Apply Lemma 2.4 to get $\lim_n \|x_{n+1} - x_n\| = (1 - \beta)\lim_n \|x_n - z_n\| = 0$.

Step 5. $\lim_{n\to\infty}\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\| = 0$, $\forall k \in \{0, 1, \ldots, M - 1\}$.

Proof of Step 5. Let $v \in \mathcal{F}$ and $k \in \{0, 1, \ldots, M - 1\}$. Since $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive, we obtain

$$\|v - \mathcal{J}_n^{k+1}x_n\|^2 = \|J_{r_{k+1,n}}^{F_{k+1}}v - J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^k x_n\|^2$$

$$\leq \langle J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^k x_n - v, \mathcal{J}_n^k x_n - v\rangle$$

$$= \frac{1}{2}(\|J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^k x_n - v\|^2 + \|\mathcal{J}_n^k x_n - v\|^2 - \|\mathcal{J}_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^k x_n\|^2).$$

It follows that

$$(3.8)\qquad |\mathcal{J}_n^{k+1}x_n - v\|^2 \leq \|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1}x_n\|^2.$$

Set $t_n = \gamma f(x_n) - AW_n \mathcal{P}_n^N \mathcal{J}_n^M x_n$ and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k}\{\|t_n\|, \|x_k - v\|\}.$$

Using Lemma 2.2 and noting that $\|.\|^2$ is convex, we derive, using (3.8),

$$(3.9)\quad \|x_{n+1} - v\|^2 = \|(1 - \beta)(W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v) + \beta(x_n - v) + \epsilon_n t_n\|^2$$

$$\leq \|(1-\beta)(W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v) + \beta(x_n - v)\|^2 + 2\epsilon_n \langle t_n, x_{n+1} - v \rangle$$

$$\leq (1-\beta)\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n.$$

So,

$$\|x_{n+1} - v\|^2 \leq (1-\beta)\|\mathcal{J}_n^{k+1} x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n$$

$$\leq (1-\beta)(\|x_n - v\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2) + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n$$

$$= \|x_n - v\|^2 - (1-\beta)\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 + 2\lambda^2 \epsilon_n.$$

It follows, by Step 4 and condition (C1), that

$$\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 \leq \frac{1}{1-\beta}(\|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2 \epsilon_n)$$

$$\leq \frac{1}{1-\beta}(2\lambda\|x_n - x_{n+1}\| + 2\lambda^2 \epsilon_n) \to 0, \text{ as } n \to \infty.$$

Step 6. $\lim_{n\to\infty} \|x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| = 0$.
  Proof of Step 6. Observe that

$$\|x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|$$

$$= \|x_n - x_{n+1}\| + \|\epsilon_n(\gamma f(x_n) - A W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n) + \beta(x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n)\|$$

$$\leq \|x_n - x_{n+1}\| + \epsilon_n(\gamma\|f(x_n)\| + \|A W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|) + \beta\|x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|.$$

It follows from Step 4 that

$$\|x_n - W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|$$

$$\leq \frac{1}{1-\beta}(\|x_n - x_{n+1}\| + \epsilon_n(\gamma\|f(x_n)\| + \|A W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n\|)) \to 0, \text{ as } n \to \infty.$$

Step 7. $\lim_{n\to\infty} \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\| = 0, \forall k \in \{0, 1, \ldots, N-1\}$.
  Proof of Step 7. Since $\{B_k : k = 1 \ldots N\}$ are α-inverse-strongly monotone, by the assumptions imposed on $\{\lambda_n\}$, for given $v \in \mathcal{F}$ and $k \in \{0, 1, \ldots, N-1\}$, we have

$$\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\|^2 \leq \|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2$$
$$= \|P_C(I - \lambda_{k+1,n} B_{k+1}) \mathcal{P}_n^k \mathcal{J}_n^M x_n - P_C(I - \lambda_{k+1,n} B_{k+1})v\|^2$$
$$\leq \|(I - \lambda_{k+1,n} B_{k+1}) \mathcal{P}_n^k \mathcal{J}_n^M x_n - (I - \lambda_{k+1,n} B_{k+1})v\|^2$$
$$\leq \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - v\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\alpha)\|B_{k+1} \mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2$$
$$\leq \|x_n - v\|^2 + a(b - 2\alpha)\|B_{k+1} \mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2.$$

Thus, by (3.9), we have

$$
\begin{aligned}
\|x_{n+1} - v\|^2 &\le (1-\beta)\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\
&\le (1-\beta)\{\|x_n - v\|^2 + a(b - 2\alpha)\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2\} \\
&\quad + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\
&= (1-\beta)a(b - 2\alpha)\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2 \\
&\quad + \|x_n - v\|^2 + 2\lambda^2 \epsilon_n.
\end{aligned}
$$

So,

$$
\begin{aligned}
(1-\beta)a(2\alpha - b)&\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2 \\
&\le \|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2 \epsilon_n \\
&\le \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|) + 2\lambda^2 \epsilon_n.
\end{aligned}
$$

Since $0 < \beta < 1$, and $\|x_n - x_{n+1}\| \to 0$, we obtain:

$$
(3.10) \qquad |B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\| \to 0 \quad (n \to \infty).
$$

Again from (2.1) and the fact that $I - \lambda_{k+1,n}B_{k+1}$ is nonexpansive, we have

$$
\begin{aligned}
&\|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 \\
&= \|P_C(I - \lambda_{k+1,n}B_{k+1})\mathcal{P}_n^k \mathcal{J}_n^M x_n - P_C(I - \lambda_{k+1,n}B_{k+1})v\|^2 \\
&\le \langle (\mathcal{P}_n^k \mathcal{J}_n^M x_n - \lambda_{k+1,n}B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n) - (v - \lambda_{k+1,n}B_{k+1}v), \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v \rangle \\
&= \tfrac{1}{2}\{\|(\mathcal{P}_n^k \mathcal{J}_n^M x_n - \lambda_{k+1,n}B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n) - (v - \lambda_{k+1,n}B_{k+1}v)\|^2 \\
&\quad + \|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 \\
&\quad - \|(\mathcal{P}_n^k \mathcal{J}_n^M x_n - \lambda_{k+1,n}B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n) - (v - \lambda_{k+1,n}B_{k+1}v) - (\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v)\|^2\} \\
&\le \tfrac{1}{2}\{\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - v\|^2 + \|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 \\
&\quad - \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - \lambda_{k+1,n}(B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v)\|^2\} \\
&= \tfrac{1}{2}\{\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - v\|^2 + \|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 - \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|^2 \\
&\quad + 2\lambda_{k+1,n}\langle \mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n, B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v \rangle \\
&\quad - \lambda_{k+1,n}^2\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2\}.
\end{aligned}
$$

This implies:

$$
\begin{aligned}
\|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 &\le \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - v\|^2 - \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|^2 \\
&\quad + 2\lambda_{k+1,n}\langle \mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n, B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v \rangle \\
&\quad - \lambda_{k+1,n}^2\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2.
\end{aligned}
$$

Then, by (3.9), we have

$$
\begin{aligned}
\|x_{n+1} - v\|^2 &\le (1-\beta)\|W_n \mathcal{P}_n^N \mathcal{J}_n^M x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\
&\le (1-\beta)\|\mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\
&\le (1-\beta)\{\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - v\|^2 - \|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|^2
\end{aligned}
$$

$$+2\lambda_{k+1,n}\langle \mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n, B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\rangle$$

$$-\lambda_{k+1,n}^2 \|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|^2\} + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n$$

$$\leq \|x_n - v\|^2 - (1-\beta)\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|^2$$

$$+2(1-\beta)\lambda_{k+1,n}\langle \mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n, B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\rangle + 2\lambda^2\epsilon_n,$$

which implies:

$$(1-\beta)\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2\epsilon_n$$

$$+2(1-\beta)\lambda_{k+1,n}\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\|\|B_{k+1}\mathcal{P}_n^k \mathcal{J}_n^M x_n - B_{k+1}v\|.$$

Hence, it follows from Step 4 and (3.10) that

$$\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k+1} \mathcal{J}_n^M x_n\| \to 0.$$

Step 8. The weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Proof of Step 8. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From steps 5 and 7, we also obtain that

$$\mathcal{J}_{n_m}^k x_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \ldots, M\}$, and

$$\mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \ldots, N\}$. We need to show that $z_0 \in \mathcal{F}$. First, we prove $z_0 \in \cap_{i=1}^N VI(C, B_i)$. For this purpose, let $k \in \{1, \ldots, N\}$ and T_k be the maximal monotone mapping defined by

$$T_k x = \begin{cases} B_k x + N_C x, & x \in C; \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, u) \in G(T_k)$, $u - B_k x \in N_C x$. Since $\mathcal{P}_n^k \mathcal{J}_n^M x_n \in C$, by the definition of N_C, we have

(3.11) $\quad \langle x - \mathcal{P}_n^k \mathcal{J}_n^M x_n, u - B_k x\rangle \geq 0.$

On the other hand, since $\mathcal{P}_n^k \mathcal{J}_n^M x_n = P_C(\mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n - \lambda_{k,n} B_k \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n)$, we have

$$\langle x - \mathcal{P}_n^k \mathcal{J}_n^M x_n, \mathcal{P}_n^k \mathcal{J}_n^M x_n - (\mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n - \lambda_{k,n} B_k \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n)\rangle \geq 0.$$

So,

$$\langle x - \mathcal{P}_n^k \mathcal{J}_n^M x_n, \frac{\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n}{\lambda_{k,n}} + B_k \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n\rangle \geq 0.$$

By (3.11) and the α-inverse monotonicity of B_k, we have

$$\langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, u\rangle \geq \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, B_k x\rangle$$

$$\geq \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, B_k x \rangle$$

$$-\langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, \frac{\mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} - \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m}}{\lambda_{k,n_m}} + B_k \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m} \rangle$$

$$= \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, B_k x - B_k \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} \rangle$$

$$+ \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, B_k \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} - B_k \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m} \rangle$$

$$- \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, \frac{\mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} - \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m}}{\lambda_{k,n_m}} \rangle$$

$$\geq \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, B_k \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} - B_k \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m} \rangle$$

$$- \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, \frac{\mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} - \mathcal{P}_{n_m}^{k-1} \mathcal{J}_{n_m}^M x_{n_m}}{\lambda_{k,n_m}} \rangle.$$

Since $\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n\| \to 0$, $\mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m} \rightharpoonup z_0$ and $\{B_k : k = 1, \ldots, N\}$ are Lipschitz continuous, we have

$$\lim_{m \to \infty} \langle x - \mathcal{P}_{n_m}^k \mathcal{J}_{n_m}^M x_{n_m}, u \rangle = \langle x - z_0, u \rangle \geq 0.$$

Again, since T_k is maximal monotone, then $0 \in T_k z_0$. This shows that $z_0 \in VI(C, B_k)$. From this, it follows:

(3.12) $$z_0 \in \cap_{i=1}^N VI(C, B_i).$$

Note that by $(A2)$ and given $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$, we have

$$\frac{1}{r_{n,k+1}} \langle y - \mathcal{J}_n^{k+1} x_n, \mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n \rangle \geq F_{k+1}(y, \mathcal{J}_n^{k+1} x_n).$$

Thus,

(3.13) $$\langle y - \mathcal{J}_{n_m}^{k+1} x_{n_m}, \frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{n_m,k+1}} \rangle \geq F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}).$$

By condition $(A4)$, $F_i(y, .)$, $\forall i$, is lower semicontinuous and convex, and thus weakly semicontinuous. Step 5 and condition $\liminf_n r_{n,j} > 0$ imply that

$$\frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{n_m,k+1}} \to 0,$$

in norm. Therefore, letting $m \to \infty$ in (3.13), yields:

$$F_{k+1}(y, z_0) \leq \lim_m F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}) \leq 0,$$

for all $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$. Replacing y with $y_t := ty + (1-t)z_0$ with $t \in (0,1)$ and using $(A1)$ and $(A4)$, we obtain:

$$0 = F_{k+1}(y_t, y_t) \leq tF_{k+1}(y_t, y) + (1-t)F_{k+1}(y_t, z_0) \leq tF_{k+1}(y_t, y).$$

Hence, $F_{k+1}(ty + (1-t)z_0, y) \geq 0$, for all $t \in (0,1)$ and $y \in C$. Letting $t \to 0^+$ and using $(A3)$, we conclude $F_{k+1}(z_0, y) \geq 0$, for all $y \in C$ and $k \in \{0, \ldots, M-1\}$. Therefore,

$$(3.14) \qquad z_0 \in \bigcap_{k=1}^{M} EP(F_k) = EP(\mathcal{G}).$$

We next show $z_0 \in \cap_{i=1}^{\infty} Fix(S_i)$. By Lemma 2.6, we have, for every $z \in C$,

$$(3.15) \qquad\qquad W_{n_m} z \to Wz,$$

and $Fix(W) = \cap_{i=1}^{\infty} Fix(T_i)$. Assume that $z_0 \notin Fix(W)$. Then, $z_0 \neq Wz_0$. From Opial's property of Hilbert space, (3.12), (3.14), (3.15) and Step 6, we have

$$\liminf_m \|x_{n_m} - z_0\| < \liminf_m \|x_{n_m} - Wz_0\|$$

$$\leq \liminf_m (\|x_{n_m} - W_{n_m}\mathcal{P}_{n_m}^N \mathcal{J}_{n_m}^M x_{n_m}\|$$

$$+ \|W_{n_m}\mathcal{P}_{n_m}^N \mathcal{J}_{n_m}^M x_{n_m} - W_{n_m}\mathcal{P}_{n_m}^N \mathcal{J}_{n_m}^M z_0\| + \|W_{n_m}z_0 - Wz_0\|)$$

$$\leq \liminf_m (\|x_{n_m} - z_0\| + \|W_{n_m}z_0 - Wz_0\|) = \liminf_m \|x_{n_m} - z_0\|.$$

This is a contradiction. Therefore, z_0 belongs to $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.

Step 9. Let x^* be the unique solution of the variational inequality,

$$(3.16) \qquad \langle(\gamma f - A)x^*, x - x^*\rangle \leq 0, \ x \in \mathcal{F}.$$

Then,

$$(3.17) \qquad\qquad \limsup_{n \to \infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle \leq 0.$$

Proof of Step 9. Lemma 2.3 of Marino and Xu [15] guarantees that $P_{\mathcal{F}}(\gamma f + (I - A))$ has a unique fixed point x^*, which is the unique solution of (3.16). Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$(3.18) \qquad \lim_k\langle(\gamma f - A)x^*, x_{n_k} - x^*\rangle = \limsup_{n \to \infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle.$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ weakly converges to some $z \in C$. By Step 8, $z \in \mathcal{F}$. Thus, combining (3.16) and (3.18), we get

$$\limsup_{n \to \infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle \leq \langle (\gamma f - A)x^*, z - x^* \rangle \leq 0,$$

as required.

Step 10. The sequences $\{x_n\}$ and $\{J_{r_{k,n}}^{F_k} x_n\}_{k=1}^M$ converge strongly to x^*.
  Proof of Step 10. Taking

$$u_n = W_n \mathcal{P}_{n_m}^N \mathcal{J}_n^M x_n, \quad \forall n \geq 1,$$

we have $\|u_n - x^*\| \leq \|x_n - x^*\|$. By using lemmas 2.2 and 2.7, we have

$$\|x_{n+1} - x^*\|^2 = \|[((1-\beta)I - \epsilon_n A)(u_n - x^*) + \beta(x_n - x^*)]$$
$$+ \epsilon_n(\gamma f(x_n) - Ax^*)\|^2$$
$$\leq \|((1-\beta)I - \epsilon_n A)(u_n - x^*) + \beta(x_n - x^*)\|^2$$
$$+ 2\epsilon_n \langle \gamma f(x_n) - Ax^*, x_{n+1} - x^* \rangle$$
$$\leq (1-\beta)\|\frac{((1-\beta)I - \epsilon_n A)}{1-\beta}(u_n - x^*)\|^2 + \beta\|x_n - x^*\|^2$$
$$+ 2\epsilon_n \gamma \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle + 2\epsilon_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle$$
$$\leq \frac{\|(1-\beta)I - \epsilon_n A\|^2}{1-\beta}\|u_n - x^*\|^2 + \beta\|x_n - x^*\|^2$$
$$+ 2\epsilon_n \gamma l \|x_n - x^*\| \|x_{n+1} - x^*\| + 2\epsilon_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle$$
$$\leq \frac{\|(1-\beta)I - \epsilon_n A\|^2}{1-\beta}\|x_n - x^*\|^2 + \beta\|x_n - x^*\|^2$$
$$+ \epsilon_n \gamma l (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + 2\epsilon_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle$$
$$\leq \left(\frac{((1-\beta) - \overline{\gamma}\epsilon_n)^2}{1-\beta} + \beta + \epsilon_n \gamma l\right)\|x_n - x^*\|^2$$
$$+ \epsilon_n \gamma l \|x_{n+1} - x^*\|^2 + 2\epsilon_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle.$$

It follows that

$$\|x_{n+1} - x^*\|^2 \leq \left(1 - \frac{2(\overline{\gamma} - l\gamma)\epsilon_n}{1 - l\gamma\epsilon_n}\right)\|x_n - x^*\|^2$$

$$+ \frac{\epsilon_n}{1 - l\gamma\epsilon_n}\left(2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle + \frac{\overline{\gamma}^2 \epsilon_n}{1-\beta}\|x_n - x^*\|^2\right).$$

Now, from conditions (C1) and (C2), Step 8 and Lemma 2.3, we get $\|x_n - x^*\| \to 0$; namely, $x_n \to x^*$, in norm. Finally, noticing $\|J_{r_{k,n}}^{F_k} x_n -$

$x^*\| \leq \|x_n - x^*\|$, we have that, for all $k \in \{1, \ldots, M\}$, $J^{F_k}_{r_{k,n}} x_n \to x^*$, in norm. $\qquad\square$

A mapping $T : C \to H$ is called *strictly pseudocontractive* on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $B = I - T$, where, $T : C \to H$ is a strictly pseudocontractive mapping with k. It is known that B is $\frac{1-k}{2}$-inverse-strongly monotone and $B^{-1}(0) = Fix(T)$.

Now, using Theorem 3.1, we state a strong convergence theorem for a family of strictly pseudocontractive mappings as follows.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H, $\varphi = \{S_i : C \to C\}$ be an infinite family of nonexpansive mappings, $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy $(A1)$-$(A4)$, $\psi = \{T_j : j = 1 \ldots N\}$ be a finite family of strictly pseudocontractive mappings with $0 \leq k < 1$ from C into C, and $\mathcal{F} := Fix(\varphi) \cap EP(\mathcal{G}) \cap Fix(\psi) \neq \emptyset$.*

Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, f be an l-contraction on H for some $0 < l < 1$, $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{\lambda_{j,n}\}^N_{j=1}$ be sequences in $[a, b]$, with $0 < a \leq b < 1 - k$, $\{r_{j,n}\}^M_{j=1}$ be sequences in $(0, \infty)$ and γ and β be two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \overline{\gamma}/l$. Assume:

$(C1)$ $\lim_n \epsilon_n = 0$;
$(C2)$ $\sum_{n=1}^\infty \epsilon_n = \infty$;
$(D1)$ $\liminf_n r_{j,n} > 0$, for every $j \in \{1, \ldots, M\}$;
$(D2)$ $\lim_n r_{j,n}/r_{j,n+1} = 1$, for every $j \in \{1, \ldots, M\}$;
$(D3)$ $\lim_n |\lambda_{j,n} - \lambda_{j,n+1}| = 0$, for every $j \in \{1, \ldots, N\}$.

For every $n \in \mathbb{N}$, let W_n be the W-mapping defined by (2.3). If $\{x_n\}$ is the sequence generated by $x_1 \in H$ and $\forall n \geq 1$,

$$\begin{cases} z_n = J^{F_M}_{r_{M,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = ((1 - \lambda_{N,n})I + \lambda_{N,n}T_N) \ldots ((1 - \lambda_{2,n})I + \lambda_{2,n}T_2) \\ \qquad \times ((1 - \lambda_{1,n})I + \lambda_{1,n}T_1)z_n, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1 - \beta)I - \epsilon_n A)W_n y_n, \end{cases}$$

then $\{x_n\}$ and $\{J^{F_k}_{r_{k,n}} x_n\}^M_{k=1}$ converge strongly to $x^* \in \mathcal{F} := Fix(\varphi) \cap EP(\mathcal{G}) \cap Fix(\psi)$, which is the unique solution of the variational inequality (1.5). Equivalently, we have $P_\mathcal{F}(I - A + \gamma f)x^* = x^*$.

Proof. Put $B_j = I - T_j$, for every $j \in \{1, \ldots, N\}$. Then, B_j is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $Fix(T_j)$ is the solution set of $VI(C, B_j)$; i.e., $Fix(T_j) = VI(C, B_j)$. Therefore, by Theorem 3.1, the result follows. $\qquad\square$

Remark 3.3. *We may put*

$$v_n = P_C(I - \lambda_{N,n}(I - T_N)) \ldots P_C(I - \lambda_{2,n}(I - T_2))P_C(I - \lambda_{1,n}(I - T_1))u_n,$$

in the scheme of Theorem 3.2, and obtain a scheme for families of non-self strictly pseudocontractive mappings.

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