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Conference Paper

EXACT SOLUTIONS FOR FLOW OF A SISKO FLUID IN PIPE

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Abstract. By means of He's homotopy perturbation method (HPM) an approximate solution of velocity field is derived for the flow in straight pipes of non-Newtonian fluid obeying the Sisko model. The nonlinear equations governing the flow in pipe are formulated and analyzed, using homotopy perturbation method due to He. Furthermore, the obtained solutions for velocity field is graphically sketched and compared with Newtonian fluid to show the accuracy of this work. Volume flux, average velocity and pressure gradient are also calculated. Results reveal that the proposed method is very effective and simple for solving nonlinear equations like non-Newtonian fluids.

1. Introduction

Advances in technology have brought a wide range of rheologically complex fluids that are characterized by diverse and often significant deviations from simple Newtonian behavior. This has made Newtonian behavior often the exception rather than the rule in the process

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of industries. These conditions are typical of many industrial applications including most multi-phase mixtures (e.g., emulsions, suspensions, foams/froths, and dispersions) [1].
The equations modeling non-Newtonian incompressible fluid flow give rise to nonlinear differential equations. Usually, we encounter difficulties in finding their exact analytical solutions.Very recently, some promising approximate analytical solutions are proposed, such as homotopy perturbation method [2], and variation iteration method (VIM) [3]. A new perturbation method called homotopy perturbation method (HPM), due to HE, is, in fact, a coupling of the traditional method and homotopy in topology. Most recently Siddiqui [4] discussed the thin film flows of the Sisko and Olroyd 6 constant fluids on a moving belt. Siddiqui investigated the thin film flow of a third grade fluid down an inclined plane. Here, we investigate the behavior of Sisko fluid in pipe by using HPM.

2. Fundamentals of the homotopy perturbation method

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He considered the following non-linear differential equation:

$$A(U) = f(r), \qquad r \in \Omega, \tag{2.1}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts L and N. Therefore, (2.1) can be rewritten as follows:

$$M(u) + N(u) = f(r), \qquad r \in \Omega. \tag{2.3}$$

He constructed a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow \Re$, which satisfies

$$H(v,p) = (1-p)\left[M(v) - M(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \tag{2.4}$$

and is equivalent to:

$$H(v,p) = M(v) - M(u_0) + pM(u_0) + p\left[A(v) - f(r)\right] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. Obviously, we have

$$H(v,0) = M(v) - M(u_0) = 0, \tag{2.6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{2.7}$$

The changing process of p from zero to unity is just that of $H(v,p)$ from $M(v) - M(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $M(v) - M(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of (2.4) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots, \tag{2.8}$$

as $p \to 1$, (2.4) corresponds to the original one, (2.3) and (2.8) become the approximate solution of (2.3), i.e.,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots. \tag{2.9}$$

The convergence of the series in (2.9) is discussed by He [2].

3. Governing equations

The physical problem contains a straight pipe having a non-Newtonian fluid and the fluid moves in pipe with a constant rate of flow. A schematic of the coordinate system on the physical model is shown in Figure 1.

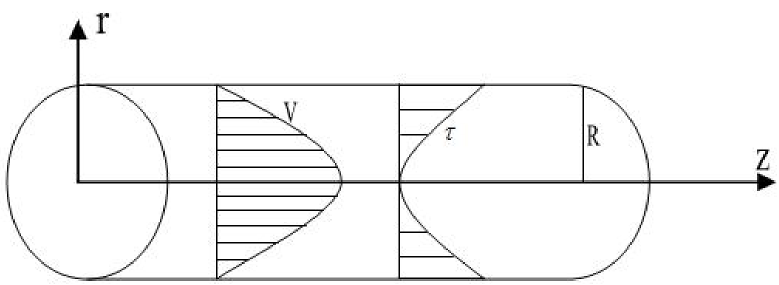

FIGURE 1. Physical model

For simplicity, some major approximations and assumptions are made:

i. The flow is in steady state.
ii. The flow is laminar and uniform.
iii. The gravity is negligible.

We choose a cylindrical-coordinate system. The only velocity component is in the z-direction:

$$\mathbf{V} = \mathbf{V}(0, 0, v(r)). \tag{3.1}$$

The valid conservation equations for this physical problem in the coordinate system can be written as follows.
Mass conservation:

$$\nabla \cdot \mathbf{V} = 0. \tag{3.2}$$

Conservation of momentum:

$$\mathbf{k} + \frac{1}{\rho}\nabla \cdot \mathbf{S} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\,\mathbf{V}, \tag{3.3}$$

where $\mathbf{k}$ is the body force and $\mathbf{T} = -P\mathbf{I} + \mathbf{S}$ is the stress tensor, P and $\mathbf{I}$ are pressure and unit tensor, respectively. For the Sisko model studied here, the shear stress ($\mathbf{S}$) function for a time-independent fluid takes the following form [5]:

$$\mathbf{S} = \left[m + \eta \left| \sqrt{\frac{1}{2} trac(\mathbf{A}_1^2)} \right|^{n-1} \right] \mathbf{A}_1, \tag{3.4}$$

where $\mathbf{A}_1$ is the rate of deformation tensor, m, η and n are constants defined differently for different fluids. We can find the rate of deformation tensors $\mathbf{A}_1$. The Rivlin-Ericksen tensor is given by

$$\mathbf{A}_1 = 2\mathbf{d} = \mathbf{L} + \mathbf{L}^{\mathbf{T}}, \tag{3.5}$$

in the cylindrical coordinates:

$$\mathbf{A}_1 = 2\mathbf{d} = \begin{bmatrix} 2\frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} & \frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \\ + & 2(\frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}) & \frac{\partial V_\theta}{\partial z} + \frac{1}{r}\frac{\partial V_z}{\partial \theta} \\ + & + & 2\frac{\partial V_z}{\partial z} \end{bmatrix}, \tag{3.6}$$

where + denotes a symmetric entry.
By interring the velocity field and simplifying we have

$$\left| \sqrt{\frac{1}{2} trac(\mathbf{A}_1^2)} \right|^{n-1} = (-\frac{\partial v}{\partial r})^{n-1} \Rightarrow \mathbf{S}_{rz} = m\frac{\partial v}{\partial r} - \eta(-\frac{\partial v}{\partial r})^{n}, \tag{3.7}$$

$$\nabla.\mathbf{S} = \frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{S}_{rz}), \tag{3.8}$$

$$\nabla.\mathbf{T} = -\frac{\partial P}{\partial z} + m\frac{\partial^2 v}{\partial r^2} - \eta\frac{\partial}{\partial r}(\frac{\partial v}{\partial r})^n + \frac{1}{r}(m\frac{\partial v}{\partial r} + \eta(\frac{\partial v}{\partial r})^n). \tag{3.9}$$

The acceleration vector, written $\frac{D\mathbf{V}}{Dt}$, is defined by

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V} = 0. \tag{3.10}$$

Then, by using (3.2), (3.9) and (3.10) in momentum (3.3), without body force, we can obtain the equation of the Sisko fluid for our problem as follow.

r-momentum:

$$-\frac{dP}{dr} = 0, \tag{3.11}$$

θ-momentum:

$$-\frac{dP}{d\theta} = 0, \tag{3.12}$$

z-momentum:

$$m\frac{d^2v}{dr^2} + \eta n(-\frac{dv}{dr})^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[m\frac{dv}{dr} - \eta(-\frac{dv}{dr})^n\right] - \frac{dP}{dz} = 0. \tag{3.13}$$

From (3.11) and (3.12), we deduce that $p = p(z)$. The boundary conditions will be

$$\begin{cases} \mathbf{S}_{rz} = 0 \quad at \quad r = 0 \\ v = 0 \quad at \quad r = R, \end{cases} \tag{3.14}$$

where $\mathbf{S}_{rz}$, the shear stress in (3.14) for the flow problem under consideration from (3.4), is given by (3.7). Substituting (3.7) in the second boundary condition of (3.14), we get

$$\frac{dv}{dr} = 0 \quad at \quad r = 0. \tag{3.15}$$

We obtain the same result in case of a Newtonian fluid. Thus, the flow of a Sisko fluid in pipe is governed by the system

$$m\frac{d^2v}{dr^2} + \eta n(-\frac{dv}{dr})^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[m\frac{dv}{dr} - \eta(-\frac{dv}{dr})^n\right] - \frac{dP}{dz} = 0, \tag{3.16}$$

$$\begin{cases} \frac{dv}{dr} = 0 \quad at \quad r = 0 \\ v = 0 \quad at \quad r = R. \end{cases} \tag{3.17}$$

Let $\bar{u}$ and R be the average velocity and radius of pipe, respectively:

$$\bar{\mu} = \eta(\frac{\bar{u}}{R})^{n-1}. \tag{3.18}$$

The dimension of $\bar{\mu}$ is the same as the dimension of Newtonian fluid viscosity. We introduce the following non-dimensional variables:

$$r^* = \frac{r}{R} \quad v^* = \frac{v}{\bar{u}} \quad \theta = \frac{\bar{\mu}}{m} \quad z^* = \frac{z}{R}. \tag{3.19}$$

In this case, the pressure gradient can be written as

$$\frac{dP}{dz} = -\frac{2m\,\bar{u}}{R^2}P_s, \tag{3.20}$$

where P_s is the non-dimensional steady state pressure gradient, the dimensionless form of (3.16) subject to (3.17), without the *, is:

$$\frac{d^2v}{dr^2} + n\theta(-\frac{dv}{dr})^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[\frac{dv}{dr} - \theta(-\frac{dv}{dr})^n\right] + 2P_s = 0, \tag{3.21}$$

$$\begin{cases} \frac{dv}{dr} = 0 & at \quad r = 0 \\ v = 0 & at \quad r = 1. \end{cases} \tag{3.22}$$

We note that (3.21) is a second order nonlinear differential equation with two boundary conditions. Next, we give the solution of (3.21) under the boundary conditions (3.22) by HPM.

4. Analysis of the Sisko fluids problem using homotopy perturbation method

To obtain the solution of (3.21), we use HPM. First, we consider operators L and N as follows:

$$L = \frac{d^2}{dr}(.), \tag{4.1}$$

and

(4.2)

$$N = n\theta\left(-\frac{d}{dr}()\right)^{(n-1)}\frac{d^2}{dr^2}() + \frac{1}{r}\left[\left(\frac{d}{dr}()\right) - \theta\left(-\frac{d}{dr}()\right)^n\right] + 2Ps.$$

Then, we construct the homotopy $v\left(r,p\right) : \Omega \times [0,1] \to \Re$, witch satisfies

$$H\left(v,p\right) = (1-p)\left(L(v) - L\left(u_0\right)\right) + p\left[L(v) + N\left(v\right)\right] = 0, \tag{4.3}$$

$$(4.4)\quad \Big(1-p\Big)\Big(\frac{d^2v}{dr^2}-\frac{d^2u_0}{dr^2}\Big)+p\Big[\frac{d^2v}{dr^2}+n\theta\Big(-\frac{dv}{dr}\Big)^{\left(n-1\right)}\frac{d^2v}{dr^2}+\frac{1}{r}\Big[\Big(\frac{dv}{dr}\Big)-\theta\Big(-\frac{dv}{dr}\Big)^{n}\Big]+2Ps\Big]=0,$$

where $p\in[0,1]$ is the embedding parameter and u_0 is the initial guess. Accordingly to HPM and with respect to boundary conditions (3.21), we assume that (4.4) has a solution of the form

$$(4.5)\qquad v\left(r,p\right)=v_0\left(r\right)+pv_1\left(r\right)+p^2v_2\left(r\right)+\ldots.$$

By substituting (4.5) into (4.4) and (4.3), and equating the same powers of p and choosing powers of p from zero to two with respect to our approximation scheme, we finally obtain the following systems of differential equations.

4.1. **Zeroth-order.** The differential equation of the zeroth-order with the boundary conditions is obtained as follows:

$$(4.6)\qquad L\left(v\right)-L\left(u_0\right)=0,$$

$$(4.7)\qquad \begin{cases}\frac{dv_0}{dr}=0 & at\quad r=0\\ v_0=0 & at\quad r=1.\end{cases}$$

We note that u_0 is the initial guess and with respect to our problem, it is considered to be a parabola,

$$(4.8)\qquad u_0=\frac{P_s}{2}\left(1-r^2\right).$$

It should be noted that the initial guess satisfies the boundary conditions. Since L is a linear operator, we conclude that

$$(4.9)\qquad v_0=u_0=\frac{P_s}{2}\left(1-r^2\right).$$

4.2. **First-order.** The first order equation is:

$$(4.10)\quad \frac{d^2v_1}{dr^2}+\frac{d^2v_0}{dr^2}+n\theta\left(-\frac{dv_0}{dr}\right)^{(n-1)}\frac{d^2v_0}{dr^2}+\frac{1}{r}\left[\left(\frac{dv_0}{dr}\right)-\theta\left(-\frac{dv_0}{dr}\right)^{n}\right]+2Ps=0,$$

subject to the boundary conditions:

$$\begin{cases} \frac{dv_1}{dr} = 0 & at \quad r = 0 \\ v_1 = 0 & at \quad r = 1. \end{cases} \tag{4.11}$$

For the first order solution, we substitute the zeroth-order solution v_0 into Eq.(4.10) and with some simplification along with the boundary conditions, we obtain the first order solution to be

$$v_1 = -\frac{P_s^n \theta}{n}\left(1 - r^{(n+1)}\right). \tag{4.12}$$

4.3. **Second-order.** The second order equation along with the boundary conditions is:

$$\frac{d^2 v_2}{dr^2} + n\theta\left(-1\right)^{(n-1)}\left[\left(n-1\right)\left(\frac{dv_0}{dr}\right)^{(n-2)}\left(\frac{dv_1}{dr}\right)\left(\frac{d^2 v_0}{dr^2}\right) + \left(\frac{dv_0}{dr}\right)^{(n-1)}\left(\frac{d^2 v_1}{dr^2}\right)\right] + \frac{1}{r}\left[\left(\frac{dv_1}{dr}\right) + \left(-1\right)^{(n+1)}\left(n-1\right)\theta\left(\frac{dv_0}{dr}\right)^{(n-1)}\left(\frac{dv_1}{dr}\right)\right] = 0, \tag{4.13}$$

$$\begin{cases} \frac{dv_2}{dr} = 0 & at \quad r = 0 \\ v_2 = 0 & at \quad r = 1. \end{cases} \tag{4.14}$$

The resulting differential equation subjected to the boundary condition, v_2, is obtained to be:

$$v_2 = \frac{P_s^{(2n-1)}\theta^2\left(2n^3 + n^2 - 1\right)}{\left(2n-1\right)\left(2n\right)}\left(1 - r^{(2n)}\right) + \frac{P_s^n \theta}{n^2}\left(1 - r^{(n+1)}\right). \tag{4.15}$$

Final solution of (3.21) by using HPM up to the second order is:

$$v\left(r\right) = \lim_{p\to 1} v\left(r, p\right) = v_0\left(r\right) + v_1\left(r\right) + v_2\left(r\right) + ..., \tag{4.16}$$

or

$$v\left(r\right) = \frac{P_s}{2}\left(1 - r^2\right) + \frac{P_s^{(2n-1)}\theta^2\left(2n^3 + n^2 - 1\right)}{\left(2n-1\right)\left(2n\right)}\left(1 - r^{(2n)}\right) + \frac{P_s^n \theta(1-n)}{n^2}\left(1 - r^{(n+1)}\right). \tag{4.17}$$

By back substitution of values of dimensionless parameters, we get the solution (4.17) in dimensional form as:

$$v(r) = \frac{-\frac{dP}{dz}}{4m}\left(R^2 - r^2\right) + \frac{\left(-\frac{dP}{dz}/2m\right)^{(2n-1)}\eta^2\left(2n^3+n^2-1\right)}{(2n-1)(2n)m^2}\left(R^{2n}-r^{2n}\right) + \frac{(-\frac{dP}{dz}/2m)^n\eta(1-n)}{n^2 m}\left(R^{n+1}-r^{n+1}\right). \tag{4.18}$$

5. Flow rate and average pipe velocity

The flow rate Q per unit width is given by

$$Q = \int_0^R v(r)2\pi r dr. \tag{5.1}$$

By substituting (4.18) in (5.1), we obtain Q up to second order as:

$$Q = \frac{-\frac{dP}{dz}\pi R^4}{8m} + \frac{\left(-\frac{dP}{dz}/2m\right)^{(2n-1)}\eta^2\left(2n^3+n^2-1\right)\pi R^{2n+2}}{2(2n-1)(n+1)m^2} + \frac{(-\frac{dP}{dz}/2m)^n\eta(1-n^2)\pi R^{n+3}}{(n+3)n^2 m}. \tag{5.2}$$

The average pipe velocity $\overline{V}$ is then given by

$$\overline{V} = \frac{Q}{\pi R^2}. \tag{5.3}$$

Therefore, the average velocity of a Sisko fluid is:

$$\overline{V} = \frac{-\frac{dP}{dz}R^2}{8m} + \frac{\left(-\frac{dP}{dz}/2m\right)^{(2n-1)}\eta^2\left(2n^3+n^2-1\right)R^{2n}}{2(2n-1)(n+1)m^2} + \frac{(-\frac{dP}{dz}/2m)^n\eta(1-n^2)R^{n+1}}{(n+3)n^2 m}. \tag{5.4}$$

From (5.4), it can be observed that if we access the pressure gradient by $\Delta P/L$, we obtain the pressure drop in pipe as the function of average velocity:

$$\frac{8\overline{V}}{D} = \frac{(D\Delta P/4L)}{m} + \frac{2\eta^2\left(2n^3+n^2-1\right)R^{2n}(D\Delta P/4L)^{(2n-1)}}{(2n-1)\,(n+1)\,m^{2n+1}} + \frac{4(D\Delta P/4L)^n\eta(1-n^2)R^{n+1}}{(n+3)n^2m^{n+1}}, \tag{5.5}$$

where D is the diameter and L is the pipe length.

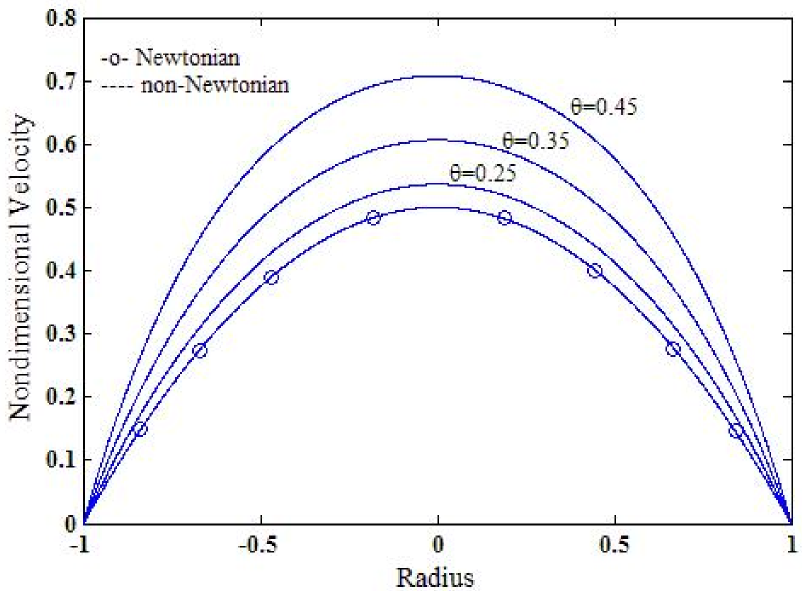

FIGURE 2. Dimensionless velocity profile in the pipe for Sisko fluid with various value of non-Newtonian parameter (θ) for a fixed value of P_s=1

6. Results and discussion

Figure 2 shows the fluid velocity changes in the pipe according to the radius for different ratios of nonlinear to linear viscosities (θ) in different fluids compared with the Newtonian fluid (θ=0) in the case that n=2. We can observe that as θ is increased, the fluid velocity becomes larger. Figure 2 also indicates that with increasing n with θ=0.2, the velocity

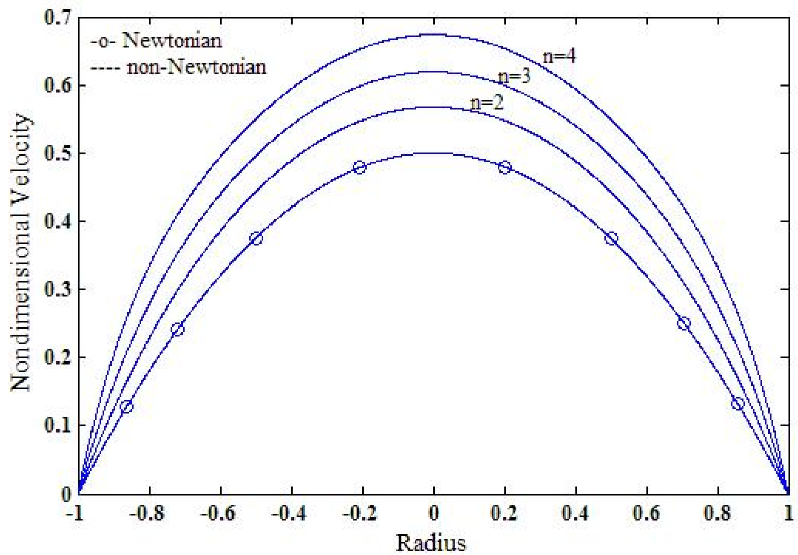

FIGURE 3. Dimensionless velocity profile in the pipe for Sisko fluid with various value of non-Newtonian parameter n for fixed value P_s=1

increases and turns away from the Newtonian case. This increase in the fluid velocity is the result of increasing the shear stress. By keeping aloof from the center of the pipe, the shear stress becomes larger for the reason of increasing the velocity gradient and the growth of the pipe on the wall to the maximum amount. The increasing of n and θ also strengthen the nonlinear behavior of fluid and the shear stress.

7. Conclusion

We considered the flow problem of non-Newtonian fluids, namely the Sisko fluids in pipe. We applied the homotopy perturbation method to obtain the velocity profile, the shear stress and pressure gradient. Predicting the decrease in pressure of the pipe can help a lot in modeling and analyzing the fluid problems, which were nearly impossible in the past.

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