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APPROXIMATION OF STOCHASTIC PARABOLIC DIFFERENTIAL EQUATIONS WITH TWO DIFFERENT FINITE DIFFERENCE SCHEMES

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Abstract. We focus on the use of two stable and accurate explicit finite difference schemes in order to approximate the solution of stochastic partial differential equations of Itö type, in particular, parabolic equations. The main properties of these deterministic difference methods, i.e., convergence, consistency, and stability, are separately developed for the stochastic cases.

1. Introduction

Stochastic partial differential equations, or SPDEs, describe the dynamics of stochastic processes defined on space-time continuum. These equations have been widely used to model many applications in engineering and mathematical sciences.
A number of numerical methods have been developed to solve stochastic partial differential equations. We provide two methods for solving linear parabolic SPDEs based on the Saul'yev method and a higher order finite

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difference scheme. The deterministic theory of these methods are important subjects in order to approximate the solutions of partial differential equations because of their advantages in terms of numerical stability and computation accuracy, respectively. For instance, the Saul'yev scheme converts a seemingly implicit scheme into a explicit scheme. The main advantage of this scheme is that it is unconditionally stable and explicit in nature. This method is very useful in higher dimensions since it reduces the required computation by a large amount. The other method that we will investigate refers to the undetermined coefficients which enable us to obtain higher order approximations with better accuracy.
Hence, it is natural to verify the stochastic difference scheme, and approximate the stochastic partial differential equations. In this article, we try to extend the deterministic Saul'yev method and propose a finite difference scheme for solving parabolic differential equations to the stochastic case in order to approximate the solutions of parabolic Itô equations of the form

$$
\begin{aligned}
u_t(x,t) &= \gamma u_{xx}(x,t) + \sigma u(x,t) dW(t), \quad 0 \le t \le T \\
u(x,0) &= u_0(x), \qquad\qquad 0 \le x \le 1 \\
u(0,t) &= u(1,t) = 0,
\end{aligned}
\tag{1.1}
$$

where t is the time variable, x is the space coordinate, ∂_t and ∂_x denote derivatives with respect to t and x, respectively. The function $u(x,t)$, for example, describes the electrical potential current along cylindrical cable. The random noise $dW(t)$ is Gaussian with zero mean.
The remainder of our work is structured as follows: A review of the Saul'yev and the higher order finite difference schemes for deterministic parabolic differential equations and their stability conditions are stated in Section 2. In Section 3, we extend these two finite difference methods to the stochastic case for approximation of stochastic linear parabolic differential equations. In addition, convergence, consistency and stability, important properties of a deterministic difference scheme, are developed for the stochastic schemes. Numerical results are given in Section 4.

2. **Finite difference deterministic schemes**

Several physical phenomena are modeled by diffusion equations. Problems of this type arise in chemical diffusion, heat conduction, medical science, biochemistry and certain biological processes.

In this section, the numerical solution of the diffusion equation based on the deterministic Saul'yev and the higher order finite difference schemes is considered.
First, consider the following parabolic partial differential equation:

$$\begin{cases} u_t(x,t) = \gamma u_{xx}(x,t) \\ u(x,0) = u_0(x)\ t \in [0\ ,\ T]\ ,\ x \in [0\ ,\ X]. \end{cases} \tag{2.1}$$

The finite difference methods are universal applicable numerical methods for the solution of partial differential equations (PDEs). Basically, these schemes discretize the continuous space and time into an evenly distributed grid system, and the values of the state variables are evaluated at each node of the grid. Introducing a uniform space grid Δx and a uniform time grid Δt, one gets a time-space lattice, for which one can attempt to approximate the solution of the above equation at the points of the lattice. Notationally, u_k^n will be defined as a function at the point $(k\Delta x, n\Delta t)$ or at the lattice point (k,n), where n and k are integers. The function u_k^n will be an approximation of the solution at the point $(k\Delta x, n\Delta t)$, where we set $u_k^0 = u_0(k\Delta x)$.
In the following, the formulations of the Saul'yev and the higher order finite difference schemes are reviewed:

2.1. **The Saul'yev scheme.** Alternating direction explicit finite-difference methods make use of two approximations that are implemented for computations proceeding in alternating directions, e.g., from left to right and from right to left, whith each approximation being explicit in its respective direction of computations [1], [2]. First, we sketch the idea of deterministic Saul'yev methods.
Alternating direction explicit methods were first introduced by Saul'yev for solving initial value problems involving the one-dimensional heat diffusion equation. The principle is to employ two finite difference equations, of which one is explicit when the computation proceeds in one directions, while the other is explicit for calculations carried out in the opposite direction. Applied to the parabolic equations, the Saul'yev technique is unconditionally stable and because it is explicit, it does not require the solution of large system of simultaneous equations at each time step, unlike most other unconditionally stable methods.
In applying the Saul'yev method to the one dimensional diffusion equation, the time derivative is approximated by the usual forward-difference

expression

$$U_t(x,t) \approx \frac{U_k^{n+1} - U_k^n}{\Delta t},$$

and the space derivative is approximated alternately by

$$U_{xx}(x,t) \approx \frac{U_{k-1}^{n+1} - U_k^{n+1} - U_k^n + U_{k+1}^n}{\Delta x^2},$$

with the calculations proceeding in the direction of increasing x, i.e., from left to right, and

$$U_{xx}(x,t) \approx \frac{U_{k+1}^{n+1} - U_k^{n+1} - U_k^n + U_{k-1}^n}{\Delta x^2},$$

with the calculations proceeding in the direction of increasing x, i.e., from right to left.
The difference equations that approximate the diffusion equation are

$$(1+\gamma\rho)U_k^{n+1} = (1-\gamma\rho)U_k^n + \gamma\rho(U_{k-1}^{n+1} + U_{k+1}^n) \tag{2.2}$$

$$(1+\gamma\rho)U_k^{n+1} = (1-\gamma\rho)U_k^n + \gamma\rho(U_{k+1}^{n+1} + U_{k-1}^n), \tag{2.3}$$

where $\rho = \frac{\Delta t}{\Delta x^2}$.
The Saul'yev scheme can be shown to be unconditionally stable using the Von Neumann method of stability analysis as follows: Let $\hat{U}_k^n = \hat{U}(k\Delta x, n\Delta t)$ be the numerical solution of the finite difference equations (2.2) and (2.3) and $\tilde{U}_n(\kappa)$ be its Fourier transform. Then, the transform of $\hat{U}_{k+1}^n$ is $e^{i\kappa\Delta x}\tilde{U}^n(\kappa)$. Taking the Fourier transform of (2.2) and (2.3) gives, in each case, an equation of the form

$$\tilde{U}_k^{n+1}(\kappa) = A(\kappa\Delta x)\tilde{U}_n(\kappa),$$

where $A(\kappa\Delta x)$ is the amplification factor of the computation. For a fixed $\gamma\rho$, the condition for stability is that $|A| \leq 1$. In this way, we find that the amplification factor for the left to right step is

$$A_1 = \frac{1 - \gamma\rho + \gamma\rho e^{i\kappa\Delta x}}{1 + \gamma\rho - \gamma\rho e^{-i\kappa\Delta x}},$$

while the one for the right to left step is

$$A_2 = \frac{1 - \gamma\rho + \gamma\rho e^{-i\kappa\Delta x}}{1 + \gamma\rho - \gamma\rho e^{i\kappa\Delta x}}.$$

Since $|A_1| \leq 1$ and $|A_2| \leq 1$, for all ρ, κ, and Δx, the scheme is unconditionally stable in both cases.

2.2. **Higher order finite difference scheme.** The method of undetermined coefficient enables us to find approximation of the derivatives to any order desired. It can be shown that it is not possible to approximate u_{xx} to the fourth order using only the points $x = k\Delta x$ and $x = (k \pm 1)\Delta x$. As we develop the difference approximation of u_{xx}, we will see that it is possible to obtain a fourth order approximation of u_{xx} if we use the points $x = k\Delta x$, $x = (k \pm 1)\Delta x$ and $x = (k \pm 2)\Delta x$. Hence, we consider the approximation of u_{xx} as

$$\Delta^4 u_k = c_1 u_{k-2} + c_2 u_{k-1} + c_3 u_k + c_4 u_{k+1} + c_5 u_{k+2}$$

where $c_1, \cdots, c_5$ are yet to be determined. Expanding $\Delta^4 u_k$ in a Taylor's series expansion about $x = k\Delta x$, regrouping the terms in the expansion, and continuing the calculations, the five coefficients are obtained [5]. In fact, $\Delta^4 u_k$ is an order Δx^4 approximation of u_{xx}, and thus we have

$$u_{xx}(x,t) \approx \frac{1}{\Delta x^2}(-\frac{1}{12}u(x-2\Delta x,t) + \frac{4}{3}u(x-\Delta x,t) - \frac{5}{2}u(x,t) + \frac{4}{3}u(x+\Delta x,t) - \frac{1}{12}u(x+2\Delta x,t)), \tag{2.4}$$

with the truncation error being $O(\Delta x^4)$.
Again, we approximate the time derivative by the forward-difference expression

$$u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}.$$

Hence, for (2.1), a finite difference method will be

$$\begin{aligned} u_k^{n+1} &= u_k^n + \gamma\rho\Delta^4 u_n^k \\ &= u_k^n + \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right), \end{aligned}$$

where $\rho = \frac{\Delta t}{\Delta x^2}$. This method is consistent and conditionaly stable with $\gamma\rho \leq \frac{2}{5}$ and $\gamma > 0$. Therefore, it is a conditionaly convergent method, by the Lax-Richtmyer theorem.

3. Stochastic difference schemes

Consider the following stochastic partial differential equation of the first order,

$$u_t(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)dW(t), \tag{3.1}$$

with $u(x,0) = u_0(x), t \in [0,T], x \in R^1$. This equation should be read as

$$u(x,t) - u_0(x) - \gamma \int_0^t u_{xx}(x,s)ds - \sigma \int_0^t u(x,s)dW(s) = 0, \tag{3.2}$$

where the stochastic integral is the usual Itô-Integral with respect to an R^1-valued Wiener's process $(W(t), F_t)_{t\in[0,T]}$ defined on a complete probability space (Ω, F, P), adapted to standard filtration$(F_t)_{t\in[0,T]}$. Now, we introduce the following difference equation for the stochastic Saul'yev scheme:

$$u_k^{n+1}=u_k^n+\gamma\frac{\Delta t}{\Delta x^2}[u_{k+1}^n-u_k^n-u_k^{n+1}+u_{k-1}^{n+1}]+\sigma u_k^n[W((n+1)\Delta t)-W(n\Delta t)]. \tag{3.3}$$

This equation can be written as:

$$\begin{aligned}(1+\gamma\rho)u_k^{n+1} &= (\gamma\rho)u_{k+1}^n + (1-\gamma\rho)u_k^n + (\gamma\rho)u_{k-1}^{n+1} \\ &+ \sigma u_k^n[W((n+1)\Delta t) - W(n\Delta t)].\end{aligned} \tag{3.4}$$

In a similar way, the stochastic higher order finite difference scheme can be considered to be

$$\begin{aligned}u_k^{n+1} = u_k^n + \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right) \\ + \sigma u_k^n\left[W((n+1)\Delta t) - W(n\Delta t)\right].\end{aligned} \tag{3.5}$$

we intend to approximate the solution of (3.2) by the random variable u_k^n defined by (3.3) and (3.5), which are respectively the stochastic version of Saul'yev and the higher order finite difference methods.

Remark 3.1. *For all the proposed schemes, the increments of Wiener's process are independent of the state u_k^n.*

3.1. General concerns.

Convergence, consistency and stability are important properties of interest in deterministic theory for the stochastic case and we aim to appropriate them to the stochastic case. Convergence is the most important property of a scheme to be useful. It means that the solution of the difference scheme approximates the solution of the corresponding differential equation and that the approximation improves as the grid spacings Δt and Δx tend to zero. To get a higher degree of generality in the following definitions, it is useful to introduce the following notations. We consider a stochastic partial differential equation, say

$$Lv = G,$$

where L denotes the differential operator and $G \in L^2(R)$ is an inhomogeneity. Furthermore, we have an initial condition. Let us assume that an approximate solution u_k^n and an inhomogeneity G_k^n are obtained such that $L_k^n u_k^n = G_k^n$. As before, n corresponds to the time step and k refers to the spatial mesh point. For convergency, stability and consistency, we will need a norm. Hence, for a sequence $x = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$, the sup-norm is defined as $\|x\|_\infty = \sqrt{\sup_k |x_k|^2}$[3].

We refer to the paper by Roth [3], for the following definitions of a stochastic difference scheme (SDS).

Definition 3.2. *(Convergence of an SDS)*
A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating the stochastic partial differential equation $Lv = G$ is convergent in mean square at time t, if, as $(n+1)\Delta t$ converges to t,

$$E\|u^{n+1} - v^{n+1}\|^2 \to 0, \ for \ (n+1)\Delta t = t, \ \Delta x \to 0 \ and \ \Delta t \to 0,$$

where u^{n+1} and v^{n+1} are infinite dimensional vectors

$$u^{n+1} = (..., u_{k-2}^{n+1}, u_{k-1}^{n+1}, u_k^{n+1}, u_{k+1}^{n+1}, u_{k+2}^{n+1}, ...)^T$$

and

$$v^{n+1} = (..., v_{k-2}^{n+1}, v_{k-1}^{n+1}, v_k^{n+1}, v_{k+1}^{n+1}, v_{k+2}^{n+1}, ...)^T.$$

Definition 3.3. *(Consistency of an SDS)*
The finite stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the stochastic partial differential equation $Lv = G$ at point (x,t), if for any continuously differentiable function $\Phi = \Phi(x,t)$,

$$E\|(L\Phi \ - G)_k^n - [L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n]\|^2 \longrightarrow 0$$

in mean square, as $\Delta x \to 0, \Delta t \to t$, and $(k\Delta x, (n+1)\Delta t)$ converges to (x,t).

Remark 3.4. *consistency implies that the solution of the stochastic partial differential equation, if it is smooth, is an approximate solution of the finite difference. Consistency is a necessary criterion for a scheme to be convergent, but it is not sufficient.*

Definition 3.5. *(Stability of an SDS)*
A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and constants K and β such that

$$E\|u^{n+1}\|^2 \leq Ke^{\beta t}E\|u^0\|^2, \tag{3.6}$$

for all $0 \leq t = (n+1)\Delta t, 0 \leq \Delta x \leq \overline{\Delta x_0}$ and $0 \leq \Delta t \leq \overline{\Delta t_0}$.

Definition 3.6. *A difference scheme is called unconditionally stable if no restriction on the relationship between Δx and Δt are needed and it is called conditionally stable otherwise.*

Remark 3.7. *one interpretation of stability of a difference scheme is that, for stable difference schemes, small errors in the initial conditions causes small errors in the solutions. The definition allows the errors to grow, but limits them to grow not faster than exponentially.*

Remark 3.8. *The reason to consider numerical schemes and also stochastic numerical schemes is to approximate the solution on a computer. This does not make sense, if there appear infinitely many spatial steps. Therefore, if we want to approximate the solution on a time interval $[0,T]$ and a space interval $[0,1]$, we introduce a uniform grid, with the grid spacing $\Delta x = \frac{1}{M}$ such that $x_k = k\Delta x$, $k = 0, ..., M$. Then, the space is an $M-1$, M or $M+1$ dimensional space, depending on the boundary value conditions are used at each end of interval. Of course, $\Delta t \longrightarrow 0$ and Δt must approach zero in such a way that Δt and Δx satisfy their stability conditions.*

3.2. Stability analysis for stochastic difference schemes.

3.2.1. Stability of the stochastic Saul'yev scheme.

Theorem 3.9. *The stochastic Saul'yev scheme with*

$$(n+1)\Delta t = t$$

and $0 \leq \gamma(\Delta t/\Delta x^2) =: \gamma\rho \leq 1$ is stable with respect to $||.||_\infty = \sqrt{\sup_k |.|^2}$.

Proof. Applying $E|.|^2$ to (3.4) and using the independence of the Wiener increments, we get

$$\begin{aligned} E\ &|(1+\gamma\rho)u_k^{n+1} - (\gamma\rho)u_{k-1}^{n+1}|^2 \\ &= E|(\gamma\rho)u_{k+1}^n + (1-\gamma\rho)u_k^n + \sigma u_k^n(W((n+1)\Delta t) - W(n\Delta t)|^2 \\ &= E|(\gamma\rho)u_{k+1}^n + (1-\gamma\rho)u_k^n|^2 + \sigma^2\Delta t E|u_k^n|^2 \end{aligned}$$

$$
\begin{aligned}
&= ((\gamma\rho)^2 E|u_{k+1}^n|^2 + 2|\gamma\rho||1-\gamma\rho|E|u_{k+1}^n.u_k^n| + (1-\gamma\rho)^2 E|u_k^n|^2) \\
&+ \sigma^2 \Delta t E|u_k^n|^2 \\
&\leq ((|\gamma\rho| + |1-\gamma\rho|)^2 \sup_k E|u_k^n|^2) + \sigma^2 \Delta t \sup_k E|u_k^n|^2.
\end{aligned}
$$

Therefore,

(3.7)

$$
E|(1+\gamma\rho)u_k^{n+1} - (\gamma\rho)u_{k-1}^{n+1}|^2 \leq ((|\gamma\rho| + |1-\gamma\rho|)^2 + \sigma^2\Delta t) \sup_k E|u_k^n|^2.
$$

Now, with $0 \leq \gamma(\Delta t/\Delta x^2) =: \gamma\rho \leq 1$, we get

$$
E|(1+\gamma\rho)u_k^{n+1} - \gamma\rho u_{k-1}^{n+1}|^2 \leq (1+\sigma^2\Delta t) \sup_j E|u_j^n|^2,
$$

This holds for all k, and so, we have

$$
\begin{aligned}
\sup_k E|(1+\gamma\rho)u_k^{n+1} - (\gamma\rho)u_{k-1}^{n+1}|^2 &\leq (1+\sigma^2\Delta t)\|u^n\|_\infty^2 \\
(|1+\gamma\rho| - |\gamma\rho|)^2 \sup_k E|u_k^{n+1}|^2 &\leq (1+\sigma^2\Delta t)\|u^n\|_\infty^2 \\
\sup_k E|u_k^{n+1}|^2 &\leq (1+\sigma^2\Delta t)\|u^n\|_\infty^2.
\end{aligned}
$$

With the assumption $(n+1)\Delta t = t$, we obtain:

$$
\begin{aligned}
E\|u^{n+1}\|_\infty^2 &\leq (1+\frac{\sigma^2 t}{n+1})E\|u^n\|_\infty^2 \\
E\|u^{n+1}\|_\infty^2 &\leq (1+\frac{\sigma^2 t}{n+1})^{n+1}E\|u^0\|_\infty^2 \\
E\|u^{n+1}\|_\infty^2 &\leq e^{\sigma^2 t}E\|u^0\|_\infty^2.
\end{aligned}
$$

So, the left to right Stochastic Saul'yev Scheme is stable for $0 \leq \gamma\rho \leq 1$, according to Definition 3.5 , with $K = 1$ and $\beta = \sigma^2$.

3.2.2. **Stability of stochastic higher order finite difference scheme.**

Theorem 3.10. *The stochastic scheme in the form (3.5), with assumptions $\gamma\rho < \frac{2}{5}$ and $\gamma > 0$, is stable in mean square with respect to $\|.\|_\infty$.*

Proof. Consider

(3.8)

$$
\begin{aligned}
E|u_k^{n+1}|^2 = E|u_k^n + \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right) \\
+\sigma u_k^n \left(W((n+1)\Delta t) - W(n\Delta t)\right)|^2.
\end{aligned}
$$

since $\{W(.,t) - W(.,s)\}$ is normally distributed with mean zero and variance $t-s$, and increments of the Wiener process are independent of u_k^n, we will have
(3.9)
$$E|u_k^{n+1}|^2 = E|u_k^n + \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)|^2 + \sigma^2\Delta t E|u_k^n|^2.$$

By using $\gamma\rho < \frac{2}{5}$ and $\gamma > 0$, we have

$$\begin{aligned} E|u_k^{n+1}|^2 &\leq (1-\frac{5}{2}\gamma\rho)^2 \sup_k E(u_k^n)^2 + \frac{4}{144}(\gamma\rho)^2 \sup_k E(u_k^n)^2 \\ &+ \frac{64}{9}(\gamma\rho)^2 \sup_k E(u_k^n)^2 + 4(1-\frac{5}{2}\gamma\rho)(\frac{\gamma\rho}{12}) \sup_k E(u_k^n)^2 \\ &+ 4(1-\frac{5}{2}\gamma\rho)(\frac{4}{3}\gamma\rho) \sup_k E(u_k^n)^2 + 8(\frac{\gamma\rho}{12})(\frac{4\gamma\rho}{3}) \sup_k E(u_k^n)^2 \\ &+ \sigma^2\Delta t \sup_k \ E(u_k^n)^2 \\ &= \left(1 + \frac{1}{9}(\gamma\rho)^2 + \frac{2}{3}(\gamma\rho) + \sigma^2\Delta t\right) \sup_k E(u_k^n)^2. \end{aligned}$$

It is enough to select λ such that $\frac{1}{9}(\gamma\rho)^2 + \frac{2}{3}(\gamma\rho) + \sigma^2\Delta t \leq \lambda^2\Delta t$ holds, for all k. Therefore,
(3.10)
$$\sup_k E(u_k^{n+1})^2 \leq (1+\lambda^2\Delta t) \sup_k E(u_k^n)^2 \leq \cdots \leq (1+\lambda^2\Delta t)^{n+1} \sup_k E(u_k^0)^2,$$
and by substituting Δt with $\frac{t}{n+1}$,

$$E\|u^{n+1}\|_\infty^2 \leq (1 + \frac{\lambda^2 t}{n+1})^{n+1} E\|u^0\|_\infty^2 \leq e^{\lambda^2 t} E\|u^0\|_\infty^2. \tag{3.11}$$

Hence, the scheme is conditionally stable with $\beta = \lambda^2$and $K = 1$.

3.3. Consistency condition of stochastic difference schemes.

3.3.1. Consistency of stochastic Saul'yev scheme.

Theorem 3.11. *The stochastic Saul'yev scheme is consistent in mean square in the sense of Definition 3.3.*

Proof. Let $\Phi(x,t)$ be a smooth function (at least continuously differentiable in x and continuous in t). Then, we have

$$L(\Phi)|_k^n = \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)$$

$$- \quad \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)ds - \sigma \int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)dW(s)$$

and

$$\begin{aligned} L_k^n(\Phi) &= \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ &- \gamma \frac{\Delta t}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t) \\ &- \Phi(k\Delta x, (n+1)\Delta t) + \Phi((k-1)\Delta x, (n+1)\Delta t)] \\ &- \sigma\Phi(k\Delta x, n\Delta t)(W((n+1)\Delta t) - W(n\Delta t)). \end{aligned}$$

Therefore, in mean square, we obtain:

$$\begin{aligned} &E|L(\Phi)|_k^n - L(\Phi)|_k^n|^2 \leq 2\gamma^2 E| \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) \\ &- \quad \frac{1}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t) - \Phi(k\Delta x, (n+1)\Delta t) \\ &+ \quad \Phi((k-1)\Delta x, (n+1)\Delta t)]\, ds|^2 + 2\sigma^2 E| \int_{n\Delta t}^{(n+1)\Delta t} (\Phi(k\Delta x, s) \\ &- \quad \Phi(k\Delta x, n\Delta t))dW(s)|^2 \leq 2\gamma^2 E| \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) \\ &- \quad \frac{1}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t) - \Phi(k\Delta x, (n+1)\Delta t) \\ &+ \quad \Phi((k-1)\Delta x, (n+1)\Delta t)]ds|^2 \\ &+ \quad 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} |\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)|^2 d(s). \end{aligned}$$

Since $\Phi(x, t)$ is only a deterministic function and $0 \leq \gamma\rho \leq 1$, we have

$$E|L(\Phi)|_k^n - L_k^n(\Phi)| \to 0 \tag{3.12}$$

when $n, k \to \infty$. This proves the consistency.

3.3.2. Consistency of stochastic higher order finite difference scheme.

Theorem 3.12. *The stochastic finite difference in the form (3.5) is consistent in the mean square sense.*

Proof. Assume that $\Phi(x, t)$ is a smooth function. Then

$$L(\Phi)|_k^n \quad = \quad \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)$$

$$- \quad \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s) ds - \sigma \int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s) dW(s)$$

and

$$\begin{aligned} L_k^n \Phi = & \ \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ & - \gamma\rho \left(-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) \right. \\ & - \left. \frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) - \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t) \right) \\ & - \sigma\Phi(k\Delta x, n\Delta t) \left(W((n+1)\Delta t) - W(n\Delta t) \right). \end{aligned}$$

In the mean square sense, we get

$$\begin{aligned} & E\left(|L(\Phi)|_k^n - L_k^n\Phi|^2 \right) \leq 2\gamma^2 E \left[\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, n\Delta t) \right. \\ & - \frac{1}{\Delta x^2} \left(-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) \right. \\ & \quad -\frac{5}{2}\Phi((k)\Delta x, n\Delta t) \\ & + \left. \left. \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) - \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t) \right) ds \right]^2 \\ & + 2\sigma^2 E \left[\int_{n\Delta t}^{(n+1)\Delta t} \left(\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t) \right) dW(s) \right]^2. \end{aligned}$$

Also, $\Phi(x,t)$ is deterministic, and thus we have

$$\begin{aligned} & E\left(|L(\Phi)|_k^n - L_k^n\Phi|^2 \right) \leq 2\gamma^2 \left[\int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{\partial^2}{\partial x^2}\Phi(k\Delta x, s) \right. \right. \\ & - \frac{1}{\Delta x^2} [-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) \\ & - \quad \frac{5}{2}\Phi((k)\Delta x, n\Delta t) \\ & + \left. \left. \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) - \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t)] \right) ds \right]^2 \\ & + 2\sigma^2 \left[\int_{n\Delta t}^{(n+1)\Delta t} |\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)|^2 \, ds \right]. \end{aligned}$$

Therefore, we have $E\left(|L(\Phi)|_k^n - L_k^n\Phi|^2\right) \to 0$, as $n, k \to \infty$.

3.4. **Convergence of Stochastic Difference Schemes.** By considering the theorems proved for stability and consistency of the Saul'yev and the higher order finite difference stochastic difference schemes, and according to the stochastic version of the Lax-Richtmyer theorem, both proposed methods are conditionally convergent for $\|.\|_\infty$ [4].

In spite of the above, in the following theorem, we investigate the convergency condition for the stochastic Saul'yev scheme directly.

Theorem 3.13. *Let $v \in H^3$. The stochastic Saul'yev scheme is convergent for the $\|.\|_\infty - norm$, for $0 \leq \gamma(\Delta t/\Delta x^2) =: \gamma\rho \leq 1$.*

Proof. The Saul'yev scheme is given by

$$u_k^{n+1} = u_k^n + \gamma\frac{\Delta t}{\Delta x^2}[u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}] + \sigma u_k^n[W((n+1)\Delta t) - W(n\Delta t)].$$

The exact solution v_k^{n+1} can be represented by the Taylor expansion $v_{xx}(x,s)$ with respect to the space variable as

$$\begin{aligned} v_k^{n+1} &= v_k^n + \gamma\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} ds + \sigma\int_{n\Delta t}^{(n+1)\Delta t} v(k\Delta x, s)dW(s) \\ &= \quad v_k^n + \gamma\int_{n\Delta t}^{(n+1)\Delta t}\Big(\frac{v_{k+1}^n - v_k^n - v_k^{n+1} + v_{k-1}^{n+1}}{(\Delta x)^2} \\ &\quad + \gamma v_{xxx}(k\Delta x + \nu\Delta x, s)(\frac{\Delta t}{\Delta x})\Big)\ ds + \sigma\int_{n\Delta t}^{(n+1)\Delta t} v(k\Delta x, s)dW(s), \end{aligned}$$

where $\nu \in (0,1)$.

Let $z_k^n = v_k^n - u_k^n$ and $0 \leq \gamma(\Delta t/\Delta x^2) =: \gamma\rho \leq 1$. We get

$$\begin{aligned} z_k^{n+1} &= v_{k+1}^n - u_k^{n+1} \\ &= v_k^n + \gamma\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x,s)|_{x=x_k} ds + \sigma\int_{n\Delta t}^{(n+1)\Delta t} v(k\Delta x, s)dW(s) \\ &- u_k^n - \gamma\frac{\Delta t}{\Delta x^2}[u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}] - \sigma u_k^n[W((n+1)\Delta t) \\ &\quad -W(n\Delta t)] \\ &= z_k^n + \gamma\int_{n\Delta t}^{(n+1)\Delta t}(v_{xx}(k\Delta x, s) - \frac{[u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}]}{\Delta x^2})ds \end{aligned}$$

$$+ \sigma \int_{n\Delta t}^{(n+1)\Delta t} (v(k\Delta x, s) - u_k^n) dW(s).$$

Then, we have

$$\begin{aligned} E|z_k^{n+1}|^2 \leq\ & 2E|z_k^n + \gamma \int_{n\Delta t}^{(n+1)\Delta t} (v_{xx}(k\Delta x, s) \\ & - \frac{[u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}]}{\Delta x^2}) ds|^2 \\ & + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v(k\Delta x, s) - u_k^n)|^2 \ ds \end{aligned}$$

or

$$\begin{aligned} E|z_k^{n+1}|^2 \leq\ & 2E|z_k^n + \gamma \int_{n\Delta t}^{(n+1)\Delta t} \frac{v_{k+1}^n - v_k^n - v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} \\ & - \frac{u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2} + \gamma\rho \ v_{xxx}((k+\nu)\Delta x, s)\Delta x ds|^2 \\ & + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v(k\Delta x, s) - u_k^n)|^2 \ ds. \end{aligned}$$

Assume $A_k = \gamma\rho \ v_{xxx}((k+\nu)\Delta x_k, s) < \infty$. Then, we continue the estimate

$$\begin{aligned} E|z_k^{n+1}|^2 \leq\ & 2E|z_k^n + \gamma \int_{n\Delta t}^{(n+1)\Delta t} (\frac{v_{k+1}^n - v_k^n - v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} \\ & - \frac{u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2} + A_k\Delta x) ds|^2 \\ & + 2\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v(k\Delta x, s) - u_k^n)|^2 ds \\ E|z_k^{n+1}|^2 \leq\ & 2E|z_k^n + \gamma\rho(z_{k+1}^n - z_k^n - z_k^{n+1} + z_{k-1}^{n+1}) + A_k\Delta x\Delta t|^2 \\ & + 4\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|(v(k\Delta x, s) - v_k^n)|^2 ds + 4\sigma^2 \int_{n\Delta t}^{(n+1)\Delta t} E|v_k^n - u_k^n| ds. \end{aligned}$$

We have the following estimation for the first integral term [3]:

$$\int_{n\Delta t}^{(n+1)\Delta t} E|(v(k\Delta x, s) - v_k^n)|^2 \ ds \leq D_2\Delta t.$$

Therefore,

$$E|z_k^{n+1}|^2 \leq 2E|z_k^n + \gamma\rho(z_{k+1}^n - z_k^n - z_k^{n+1} + z_{k-1}^{n+1}) + A_k\Delta x\Delta t|^2$$

$$
\begin{aligned}
& \qquad\qquad +4\sigma^2\Delta t E|z_k^n|^2 + D_2\Delta t \\
\leq\ & 4E|z_k^n + \gamma\rho(z_{k+1}^n - z_k^n - z_k^{n+1} + z_{k-1}^{n+1})|^2 + 4E|A_k\Delta x\Delta t|^2 \\
& \qquad\qquad +4\sigma^2\Delta t E|z_k^n|^2 + D_2\Delta t \\
\leq\ & 4E|z_k^n + \gamma\rho(z_{k+1}^n - z_k^n - z_k^{n+1} + z_{k-1}^{n+1})|^2 + 4\sigma^2\Delta t E|z_k^n|^2 \\
& \qquad\qquad +D_1\Delta x\Delta t + D_2\Delta t \\
\leq\ & 4E|(1-\gamma\rho)z_k^n + (\gamma\rho)z_{k+1}^n + \gamma\rho(z_{k-1}^{n+1} - z_k^{n+1})|^2 \\
& \qquad\qquad +4\sigma^2\Delta t E|z_k^n|^2 + D_3\Delta t \\
\leq\ & 8E|(1-\gamma\rho)z_k^n + (\gamma\rho)z_{k+1}^n|^2 + 8(\gamma\rho)^2 E|z_{k-1}^{n+1} - z_k^{n+1}|^2 \\
& \qquad\qquad +4\sigma^2\Delta t E|z_k^n|^2 + D_3\Delta t \\
\leq\ & 8[(|1-\gamma\rho| + |\gamma\rho|)^2 + \sigma^2\Delta t]\sup_k E|z_k^n|^2 \\
& \qquad\qquad +8(\gamma\rho)^2 E|z_{k-1}^{n+1} - z_k^{n+1}|^2 + D_3\Delta t.
\end{aligned}
$$

If we consider $0 \leq \gamma\rho \leq 1$, then

$$
\begin{aligned}
E|z_k^{n+1}|^2 &\leq 8(1+\sigma^2\Delta t)\sup_j E|z_j^n|^2 + 8(\gamma\rho)^2 E|z_{k-1}^{n+1} + z_k^{n+1}|^2 + D_3\Delta t \\
&\leq 8(1+\sigma^2\Delta t)E\|z^n\|_\infty^2 + 8(\gamma\rho)^2 E|z_{k-1}^{n+1} + z_k^{n+1}|^2 + D_3\Delta t \\
&\leq 8(1+\sigma^2\Delta t)E\|z^n\|_\infty^2 + 16(\gamma\rho)^2 \sup_k E|z_k^{n+1}|^2 + D_3\Delta t.
\end{aligned}
$$

This holds for all k, and so we have

$$
\sup_k E|z_k^{n+1}|^2 \leq 8(1+\sigma^2\Delta t)E\|z^n\|_\infty^2 + 16(\gamma\rho)^2 \sup_k E|z_k^{n+1}|^2 + D_3\Delta t
$$

$$
(1-16(\gamma\rho)^2)\sup_k E|z_k^{n+1}|^2 \leq 8(1+\sigma^2\Delta t)E\|z^n\|_\infty^2 + D_3\Delta t.
$$

Therefore, we have

$$
\begin{aligned}
E\|z^{n+1}\|_\infty^2 \leq\ & \frac{8}{1-16(\gamma\rho)^2}(1+\sigma^2\Delta t)\ E\|z^n\|_\infty^2 + \frac{8}{1-16(\gamma\rho)^2}D_3\Delta t \\
\leq\ & \frac{8}{1-16(\gamma\rho)^2}(1+\sigma^2\Delta t)\ E\|z^n\|_\infty^2 + D_4\Delta t \\
\leq\ & \sum_{j=1}^{n}(1+\frac{\sigma^2 t}{n+1})^j(\frac{8D_4\Delta t}{1-16(\gamma\rho)^2})^j + D_4\Delta t \\
\leq\ & (1+\frac{\sigma^2 t}{n+1})^{n+1}\sum_{j=1}^{n}(\frac{8D_4\Delta t}{1-16(\gamma\rho)^2})^j + D_4\Delta t.
\end{aligned}
$$

For a sufficiently small grid, that is, for a large enough n ($n\Delta t = t$), with $\frac{8D_4\Delta t}{1-16(\gamma\rho)^2} \leq 1$, we have

$$\begin{aligned} E\|z^{n+1}\|_\infty^2 \quad &\leq 8(n-1)(1+\frac{\sigma^2 t}{n+1})^{n+1}(\frac{8D_4\Delta t}{1-16(\gamma\rho)^2})^2 \\ &\qquad +8(1+(1+\frac{\sigma^2 t}{n+1})^{n+1})(\frac{D_4\Delta t}{1-16(\gamma\rho)^2})^2 \\ &\leq \quad \frac{8}{1-16(\gamma\rho)^2}(1+(1+t)e^{\sigma^2 t})D_4^2\Delta t \to 0, \end{aligned}$$

as $\Delta t \to 0$ or $n \to \infty$.

Theorem 3.14. *The stochastic scheme in the form (3.5), with the assumptions $\rho\gamma < \frac{2}{5}$ and $\gamma > 0$, is convergent in the $\|.\|_\infty$-norm.*

Proof. According to Theorem 3.10 and Theorem 3.12 and the stochastic version of Lax-Richtmyer theorem, the stochastic higher order finite difference scheme is conditionally convergent in $\|.\|_\infty$ [4].

Note that here we have only considered the left to right Saul'yev scheme for approximating the solution of the stochastic diffusion equation; the right to left case can be investigated similarly.

4. Numerical results

In this section, the performance of the numerical methods described in the previous sections are considered by their application to two test examples. Also, the convergency and stability of the Saul'yev and the higher finite difference schemes are numerically investigated. For computational purposes, it is useful to consider the discrete Brownian motion, where $W(t)$ is specified at discrete t values.

Example 1:

We examine the performance of the proposed left to right Saul'yev scheme for a linear stochastic parabolic equation. Consider the equation

$$u_t(x,t) = 0.01u_{xx}(x,t) + u(x,t)dW(t), \qquad t \in [0,1]\ ,\ x \in [0,1], \tag{4.1}$$

subject to the following initial condition

$$u(x,0) = x^2(1 - sin(\frac{\pi}{2}x)^2), \qquad x \in [0,1],$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0.$$

Therefore, the difference scheme for the Saul'yev method can be written as:

(4.2)
$$u_k^{n+1} = u_k^n + 0.01\rho(u_{k+1}^n - u_k^n - u_k^{n+1} + u_{k-1}^{n+1}) + \sigma u_k^n (W(n+1)\Delta t - W(n\Delta t)).$$

Let M and N be the total number of grid points for the space and

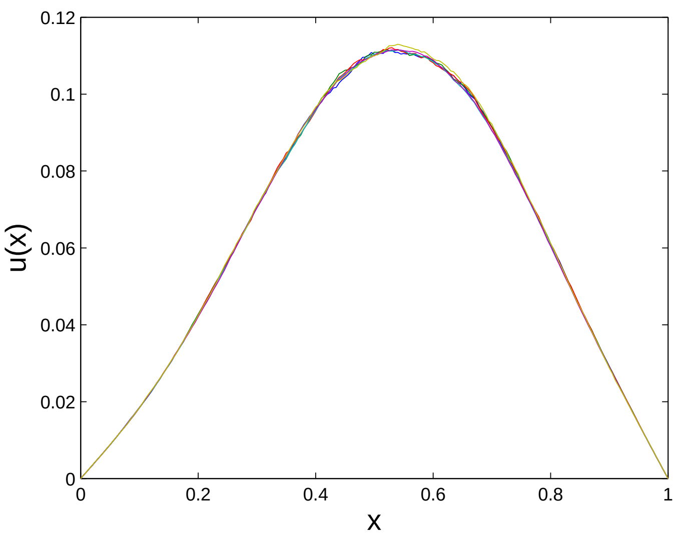

FIGURE 1. Numerical approximation with the left to right Saul'yev scheme using six different values

time discretizations, respectively. The convergence of the scheme at the end of time interval $t = 1$, for the fixed space grid points $M = 200$ and various time grid points $N = 350, 380, 400, 420, 450, 500$ is considered. In our experiments, we have adjusted the variable time step Δt and the space step Δx according to the conditions expressed in the theorems to ensure the stability and convergence of the numerical scheme.

It is clear from Figure 1 that the numerical solution obtained for the stochastic parabolic equation for the different time steps is convergent at time $t = 1$.

In Figure 2, we presented the stability of the stochastic scheme with Δx and Δt selected according to the stability condition. If we consider

$$E\|u^{n+1}\|^2 = Ke^{\beta t} E\|u^0\|^2,$$

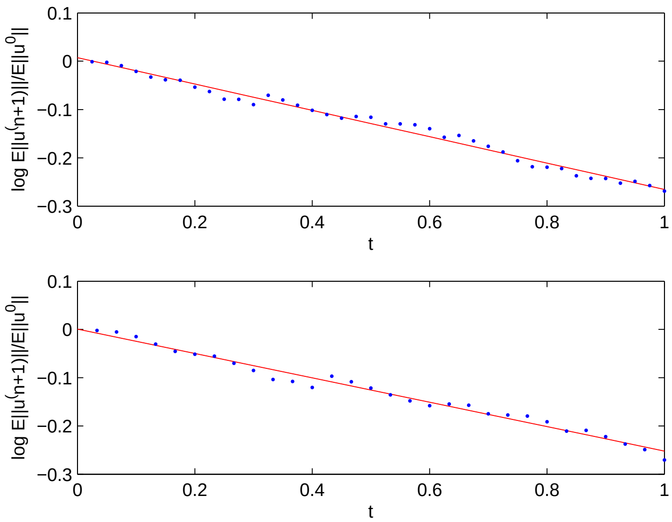

Figure 2. Stability of the left to right Saul'yev scheme and estimation of β and K, for $M = 200$ and $N = 400$ (top) and $N = 500$ (bottom) in (4.3)

Then we have

$$\ln \frac{E\|u^{n+1}\|^2}{E\|u^0\|^2} = \ln(K) + \beta t. \tag{4.3}$$

In these tests, we estimate β and K for different values of M and N. The top figure presents the approximation of $\beta = -.272208$ and $K = 1.002080$, for $M = 200$ and $N = 400$, and in the same way for the second case (the bottom figure), $\beta = -0.268733$ and $K = 1.005249$, for $M = 200$ and $N = 500$, are estimated.

To demonstrate the effect of randomness, we have considered the stochastic problem (4.1) subject to the following initial condition,

$$u(x,0) = x^2(1 - sin(\frac{\pi}{2}x))^2 \; x \in [0,1],$$

and compared the mean solution with its corresponding deterministic solution in Figure 3. In this problem and other test examples, by a deterministic solution we mean the numerical solution of the unperturbed problems, i.e, $\sigma = 0$ in (1.1). It is clear from Figure 3 that the numerical solution obtained for the stochastic and the deterministic parabolic equations are different, but in general the stochastic result is in a good agreement with the result obtained for the deterministic case.

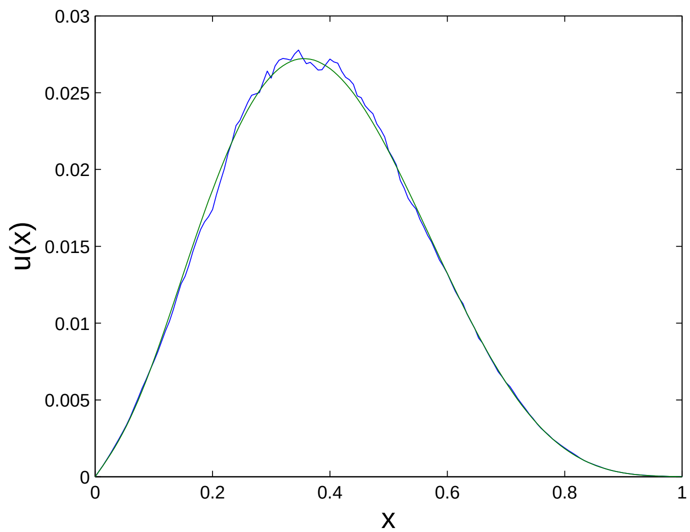

FIGURE 3. Comparison between deterministic and stochastic numerical solution of (4.1) using the Saul'yev scheme

Example 2:
Consider

$$u_t(x,t) = 0.001u_{xx}(x,t) - u(x,t)dW(t), \qquad t \in [0\ 1]\ ,\ x \in [0\ 1],$$

with $u_0(x) = x^2(1-x)^2$ as the initial condition and the boundary conditions $u(0,t) = u(1,t) = 0$. The discrete form with the higher order finite difference scheme is

$$u_k^{n+1} = u_k^n + \frac{\rho}{1000}\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right) \\ -u_k^n(W((n+1)\Delta t) - W(n\Delta t)),$$

where $\Delta t = \frac{1}{N}$ and $\Delta x = \frac{1}{M}$, for some positive integer N and M. The above form is conditionally stable with $\beta = \lambda^2$ and $K = 1$ and

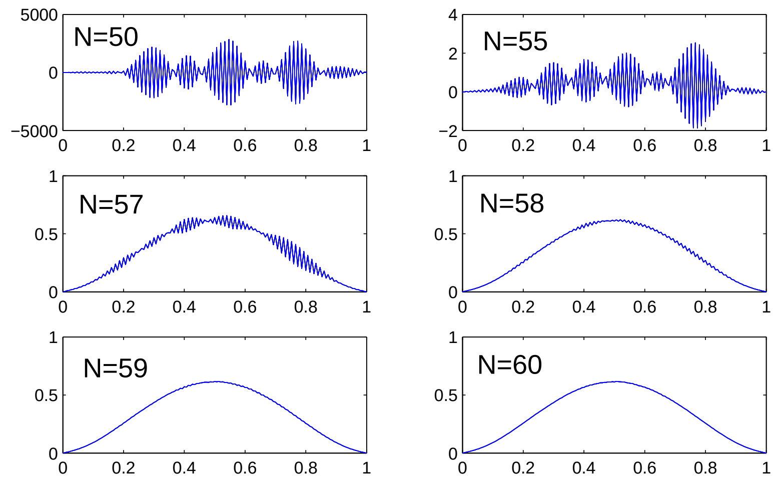

FIGURE 4. Representation of conditional convergence, $u(x, 1)$ for different values of N

for $\gamma\rho < \frac{2}{5}$, $\gamma \geq 0$. Therefore, if $M = 150$, then for the stability (or convergency) condition, we must have $\Delta t \leq \frac{1}{58}$ or $N \geq 58$. We have shown this in Figure 4. In the proof of Theorem 3.10, we assumed that $\frac{1}{9}(\gamma\rho)^2 + \frac{2}{3}(\gamma\rho) + \sigma^2\Delta t \leq \lambda^2\Delta t$, and for different values of N, we obtained the least value of λ^2 in Table 1.

TABLE 1. λ^2 for stability

N	50	58	60	120	250	500	1000
λ^2	17.1250	16.9698	16.9374	16.3750	16.2250	16.1124	16.0562

On the other hand, in (3.11) we had

(4.4)

$$E\|u^{n+1}\|_\infty^2 \leq e^{\lambda^2 t} E\|u^0\|_\infty^2 \Rightarrow y = \ln\left(\frac{E\|u^{n+1}\|_\infty^2}{E\|u^0\|_\infty^2}\right) \leq \lambda^2 t\ , \ (n+1)\Delta t = t.$$

According to (4.4) and Figure 5 (or Figure 4) and Table 1, the stability condition is satisfied for $N \geq 58$. In Figure 6, we investigate the con-

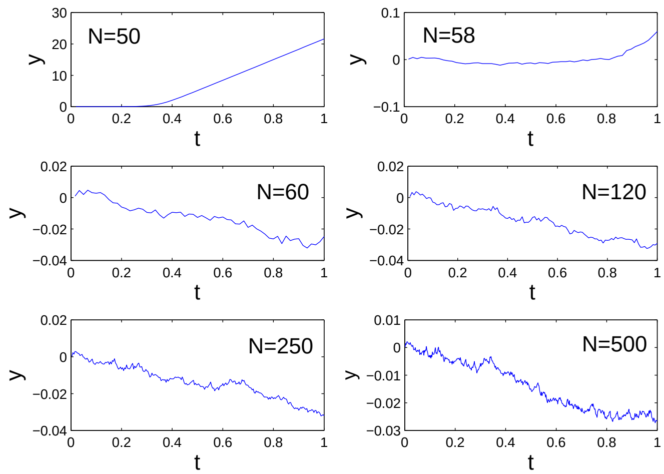

FIGURE 5. Figures of y in (4.4) against t

vergency of the solutions. We do not have the exact solution for this example, and so the numerical approximation at $t = 1$, for $N = 120$, is chosen as a basic fixed solution (Figure 6, left). The right hand side of Figure 6, gives the log-scale of the difference between the numerical approximations with $N = 60$ and $N = 500$ having the basic fixed solution at the mesh points.

5. Conclusion

Two numerical methods for approximating the solution of linear stochastic parabolic equations using the Saul'yev and a higher order finite difference schemes were provided. The Saul'yev scheme was unconditionally stable and explicit, and so it did not need to solve simultaneous equations in each time step in the comparison with other unconditionally stable implicit methods. On the other hand, the higher order finite difference scheme seemed to be advantageous because of its high accuracy. So, the main idea in this approach is to extend these explicit finite difference methods for approximating the solution of linear parabolic *SPDEs*. Stability conditions, convergence and consistency as most important properties of finite difference schemes were studied and proved for the stochastic cases. In the numerical results, the performance of

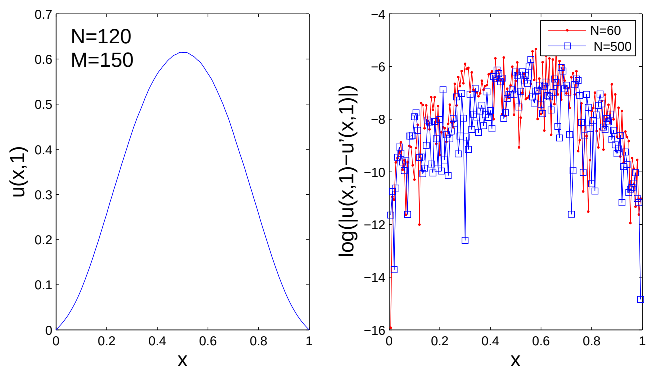

FIGURE 6. Log difference numerical approximation (right figure) for $N = 60$ and $N = 500$ with $N = 120$ (left figure) and common value $M = 150$

these two numerical schemes for stochastic parabolic equations were discussed.

An open question is how to extend other main methods of solving deterministic partial differential equations to the stochastic case. For example, it seems that method of lines or adaptive methods for stochastic initial-boundary value problems can be applied and the performance of these methods can be examined with test problems such as stochastic advection-diffusion problem or stochastic Burger's equation.

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References

[1] W. F. Ames, *Numerical Methods for Partial Differential Equations*, Third edition, Computer Science and Scientific Computing, Academic Press, Inc., Boston, MA, 1992.

[2] L. J. Campbell and B. Yin, On the stability of alternating-direction explicit methods for advection-diffusion equations, *Numer. Methods Partial Differential Equations* **23** (2007) 1429-1444.

[3] C. Roth, Difference methods for stochastic partial differential equations, 4th GAMM-Workshop "Stochastic Models and Control Theory" (Lutherstadt Wittenberg, 2001) *ZAMM. Z. Angew. Math. Mech.* **82** (2002) 821-830.

[4] C. Roth, *Approximations of Solutions of a First Order Stochastic Partial Differential Equation*, Report, Institut Optimierung und Stochastik, Martin-Luther-Universität Halle-Wittenberg, Halle, 1989.

[5] J. W. Thomas, *Numerical Partial Differential Equations, Finite Difference Methods*, Texts in Applied Mathematics, 22, Springer-Verlag, New York, 1995.

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