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ALGEBRAS WITH CYCLE-FINITE STRONGLY SIMPLY CONNECTED GALOIS COVERINGS

J. A. DE LA PEÑA

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Abstract. Let A be a finite dimensional k-algebra and R be a locally bounded category such that $R \to R/G = A$ is a Galois covering defined by the action of a torsion-free group of automorphisms of R. Following [30], we provide criteria on the convex subcategories of a strongly simply connected category R in order to be a cycle-finite category and describe the module category of A. We provide criteria for A to be of polynomial growth.

1. Introduction

Throughout the article, algebras are finite dimensional associative k-algebras with identity over a fixed algebraically closed field k. By a module over an algebra A we mean a left A-module of finite dimension over k, if not specified otherwise.

From Drozd's Tame and Wild Theorem [19] (see also [13]), the class of algebras are divided into two disjoint classes. On the one hand, we have tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. On the other hand, we have wild algebras whose

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representation theory includes the representation theories of all finite dimensional algebras over k (see [35, Chapter XIX]). A well understood class of tame algebras is formed by the algebras of finite type which accept only finitely many isoclasses of indecomposable modules (see [4, 5, 8, 9]). In the more general situation, the representation theory of tame algebras is slowly emerging. Tame tilted algebras [24], domestic and tubular extensions of tame concealed algebras [33], coil algebras [3] and more generally, (tame) algebras of polynomial growth [37], for which there exists an integer m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d, by d^m, are among the type of algebras studied in the past years.

The methods of the representation theory of algebras work best for triangular algebras $A = kQ_A/I$, where the Gabriel quiver Q_A has no oriented cycles (see [1, 33, 34, 35]). To deal with arbitrary algebras, covering techniques were developed (see [8, 16, 18, 20, 28]). In many situations, an algebra A admits a Galois covering $R \to R/G = A$, where R is a triangular locally bounded category and G is a torsion-free group acting freely on the objects of R, which allows to study the representation theory of A by the consideration of finite dimensional algebras inside R. For instance, assume that R is a strongly simply connected category (see [38]). Then, tameness of A implies tameness of R, which happens exactly when R does not accept convex subcategories which are hypercritical [11]. The converse is expected to hold. Moreover, under these assumptions, A is of polynomial growth if and only if R does not accept convex subcategories which are hypercritical or pg-critical; see [37].

An important role in the representation theory of algebras is played by cycles of modules. A cycle in the category mod A of finite dimensional modules over an algebra A is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

of non-zero non-isomorphisms between indecomposable modules in mod A, and the cycle is said to be finite if the homomorphisms $f_1, \dots, f_s$ do not belong to the infinite Jacobson radical of mod A. An algebra A is said to be *cycle-finite* if all cycles in mod A are finite [2]. Representation-finite algebras, tame tilted algebras, tame generalized multicoil algebras [26] are examples of cycle-finite algebras. In general, a cycle-finite algebra A is of polynomial growth, while the converse holds if A is a strongly simply connected algebra [40]. Recently, it was shown in [30] that every algebra

A, which admits a Galois covering $R \to R/G = A$ with R a cycle-finite locally bounded category and G a torsion-free group, is tame and the indecomposable finite dimensional A-modules were described. Moreover, for such a Galois covering, the algebra A is of polynomial growth if and only if the number of G-orbits of isoclasses of indecomposable locally finite dimensional R-modules with non-trivial stabilizers is finite.

Here, we recall the main results and related techniques of the context discussed so far. Namely, we consider algebra A and Galois covering $R \to R/G = A$ where R is a "nice" locally bounded category and G is a torsion-free group of automorphisms of R. The nicest situation corresponds to R being a strongly simply connected and cycle-finite category. Assuming that R is a strongly simply connected category, we show that R is cycle-finite if and only if R does not accept convex subcategories which are hypercritical, $pg-$critical or of type $(2, 2, \infty)$. Here, we say that a category is of type $(2, 2, \infty)$ if it is the direct limit of domestic extensions of type $(2, 2, n)$, for $0 \leq n \in \mathbb{N}$, of a fixed tame concealed algebra of type $(2, 2, s)$. These conditions are satisfied when there is a set of representatives $\mathcal{S}_0$ of the G-orbits in a separating family $\mathcal{S}$ of convex subcategories of R with respect to G which is formed by lines ${}_{\infty}\mathbb{A}_{\infty}$; see [30]. Moreover, if $R \to R/G = A$ is a covering in the nicest situation and $\mathcal{S}_0$ is not empty, then A is of polynomial growth exactly when $G = \mathbb{Z}$.

The remainder of the paper is organized as follows. In Section 1, we recall basic facts on Galois coverings of algebras essential for further considerations. Section 2 contains results on cycle-finite strongly simply connected categories. Section 3 is devoted to the proof of the main result and its immediate consequences. In Section 4, we establish a criterion for polynomial growth. In the final Section 5, we exhibit a couple of examples illustrating our results.

For basic background on the representation theory of algebras, refer to the books [1, 33, 34, 35].

2. Galois coverings of algebras

Following [8], by a *locally bounded category* we mean a k-category R which is isomorphic to a factor category kQ_R/I, where Q_R is a locally finite quiver and I is an admissible ideal of the path category kQ_R of Q_R. An algebra A will be considered as a *finite category*, that is, a locally bounded category given by a finite quiver. A full subcategory C

of a locally bounded category R is said to be *convex* if any path in Q_R with source and target in Q_C lies entirely in Q_C.

Throughout this section, we denote by R a fixed locally bounded category (over k). By an R-*module*, we mean a covariant functor M from R to the category $\mathrm{MOD}\, k$ of all vector spaces over k [8]. An R-module M is called *finite dimensional* (respectively, *locally finite dimensional*) if $\dim M = \sum_{x\in R} \dim_k M(x) < \infty$ (respectively, $\dim_k M(x) < \infty$ for any object x of R). We denote by $\mathrm{MOD}\, R$, (respectively, $\mathrm{Mod}\, R$ or $\mathrm{mod}\, R$) the category of all (respectively, all locally finite dimensional or all finite dimensional) R-modules, and by $\mathrm{Ind}\, R$, (respectively, $\mathrm{ind}\, R$) the full subcategory of $\mathrm{Mod}\, R$ (respectively, $\mathrm{mod}\, R$) formed by all indecomposable modules. The *support* $\mathrm{supp}\, M$ of an R-module M is the full subcategory of R given by all objects x such that $M(x) \neq 0$.

Let G be a group of k-linear automorphisms of R acting freely on the objects of R. Then, following [20], we may consider the orbit category R/G with objects being the G-orbits of the objects of R, and, for any two objects a and b of R/G, the morphism k-space $(R/G)(a, b)$ is defined as

$$(R/G)(a,b) = \left\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} R(x,y) \,\Big|\, g f_{y,x} = f_{gy,gx} \underset{g\in G, x\in a, y\in b}{\forall} \right\}$$

with the natural composition. Then, we have a canonical *Galois covering functor*

$$F : R \longrightarrow R/G$$

which assigns to any object x of R its G-orbit Gx and maps a morphism $f \in R(x, y)$ onto the family $F(f) \in (R/G)(Gx, Gy)$ such that $F(f)_{hy,gx} = gf$ or 0 in accordance with $h = g$ or $h \neq g$. Moreover, F induces the k-linear isomorphisms

$$\bigoplus_{F(y)=a} R(x,y) \xrightarrow{\sim} (R/G)(F(x), a), \qquad \bigoplus_{F(y)=a} R(y,x) \xrightarrow{\sim} (R/G)(a, F(x)),$$

for all objects x of R and a of R/G. For a full subcategory D of R, we denote by $g(D)$ the full subcategory of R formed by the objects $g(x)$, $x \in D$, and its stabilizer $G_D = \{g \in G \,|\, g(D) = D\}$. Then, we may consider the locally bounded category D/G_D. The group G acts on $\mathrm{Mod}\, R$ by the translations $(-)^g$ which assign to each R-module M the R-module $M^g = M \circ g$. For each R-module M, we denote by G_M the stabilizer $\{g \in G \,|\, M^g \cong M\}$ of M. Following [18], a module Y in $\mathrm{Ind}\, R$

is said to be *weakly G-periodic* if $\operatorname{supp} Y$ is infinite and $(\operatorname{supp} Y)/G_Y$ is a finite category. Observe that in such a case, G_Y is infinite.

Assume now that G is a group of k-linear automorphisms of R acting freely on the isoclasses of modules in $\operatorname{ind} R$. Clearly, then G acts freely on the objects of R, since G acts freely on the isoclasses of indecomposable projective R-modules $R(x,-)$, $x \in R$. Consider the associated Galois covering functor $F : R \to R/G$. We denote by $F_\bullet : \operatorname{MOD} R/G \to \operatorname{MOD} R$ the *pull-up functor*, which assigns to an R/G-module M the R-module $M \circ F$, and by $F_\lambda : \operatorname{MOD} R \to \operatorname{MOD} R/G$ the *push-down functor*, left adjoint to $F_\bullet$ (see [8, (3.2)]). Since G acts freely on the isoclasses in $\operatorname{ind} R$, F_λ induces an injection from the set $(\operatorname{ind} R/\cong)/G$ of G-orbits of isoclasses in $\operatorname{ind} R$ into the set $(\operatorname{ind} R/G)/\cong$ of isoclasses in $\operatorname{ind} R/G$ [20, (3.5)]. We denote by $\operatorname{mod}_1 R/G$ the full subcategory of $\operatorname{mod} R/G$ consisting of all modules isomorphic to $F_\lambda(M)$ for some module M in $\operatorname{mod} R$, and by $\operatorname{mod}_2 R/G$ the full subcategory of $\operatorname{mod} R/G$ formed by all modules without nonzero direct summands from $\operatorname{mod}_1 R/G$. It was shown in [18, (2.2) and (2.3)] that a module X from $\operatorname{mod} R/G$ belongs to $\operatorname{mod}_1 R/G$ (respectively, $\operatorname{mod}_2 R/G$) if and only if $F_\bullet(X)$ is a direct sum of finite dimensional R-modules (respectively, weakly G-periodic R-modules). We denote by $\operatorname{ind}_1 R/G$ (respectively, $\operatorname{ind}_2 R/G$) the full subcategory of $\operatorname{mod}_1 R/G$ (respectively, $\operatorname{mod}_2 R/G$) formed by the indecomposable modules. Following [18], the modules from $\operatorname{ind}_1 R/G$ (respectively, $\operatorname{ind}_2 R/G$) are called *indecomposable modules of the first kind* (respectively, *indecomposable modules of the second kind*). The category R is said to be *G-exhaustive* if $\operatorname{mod} R/G = \operatorname{mod}_1 R/G$ [18].

Assume that R is not G-exhaustive. Following [18, (3.1)], a family $\mathcal{S}$ of full subcategories of R is called *separating* (with respect to G) if $\mathcal{S}$ satisfies the following conditions:

(i) for each $L \in \mathcal{S}$ and $g \in G$, $gL \in \mathcal{S}$;
(ii) for each $L \in \mathcal{S}$ and each G-orbit $\mathcal{O}$ of R, $\mathcal{O} \cap L$ is contained in finitely many G_L-orbits;
(iii) for any two different $L, L' \in \mathcal{S}$, $L \cap L'$ is locally support-finite;
(iv) for each weakly G-periodic R-module Y, there exists an $L \in \mathcal{S}$ such that $\operatorname{supp} Y \subseteq L$.

The following theorem is the main result in [18, Theorem 3.1].

Theorem 2.1. *Let R be a locally bounded k-category and G be a group of k-linear automorphisms of R acting freely on the isoclasses in* $\operatorname{ind} R$.

Let $\mathcal{S}$ be a separating family of convex subcategories of R with respect to G and $\mathcal{S}_0$ be a fixed set of representatives of G-orbits in $\mathcal{S}$. There are natural embedding functors $E_\lambda^L : \operatorname{mod} L/G_L \to \operatorname{mod} R/G$, $L \in \mathcal{S}_0$ which induce a natural k-linear equivalence of categories

$$E : \coprod_{L\in\mathcal{S}_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L] \longrightarrow (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

In particular, the Auslander-Reiten quiver $\Gamma_{R/G}$ of R/G is the disjoint union of the translation quivers

$$\Gamma_{R/G} = (\Gamma_R/G) \sqcup \left(\coprod_{L\in\mathcal{S}_0} \left(\Gamma_{L/G_L}\right)_2 \right),$$

where $\left(\Gamma_{L/G_L}\right)_2$ is the union of all connected components of Γ_{L/G_L} formed by the indecomposable L/G_L-modules of the second kind.

For a convex subcategory L of a locally bounded category R, the canonical embedding $E^L : \operatorname{MOD} L \to \operatorname{MOD} R$ is defined for a module N in $\operatorname{MOD} L$, $E^L(N)$ as an R-module such that $E^L(N)(x) = N(x)$ for any object x of L, $E^L(N)(f) = N(f)$ for any morphism f in L, and $E^L(N)(y) = 0$ for any object y of R which is not in L. Moreover, we have a commutative diagram of functors

$$\begin{array}{ccc} \operatorname{MOD} L & \xrightarrow{E^L} & \operatorname{MOD} R \\ \downarrow F_\lambda^L & & \downarrow F_\lambda \\ \operatorname{MOD} L/G_L & \xrightarrow{E_\lambda^L} & \operatorname{MOD} R/G \end{array}$$

where F_λ^L is the push-down functor associated to the Galois covering $F^L : L \to L/G_L$, F_λ is the push-down functor associated to the Galois covering $F : R \to R/G$, and E_λ^L assigns to a module X in $\operatorname{MOD} L/G_L$ the module $E_\lambda^L(X)$ in $\operatorname{MOD} R/G$ such that $F_\bullet E_\lambda^L(X) = \bigoplus_{g\in U_L} F_\bullet^L(X)^g$, where $F_\bullet : \operatorname{MOD} R/G \to \operatorname{MOD} R$ and $F_\bullet^L : \operatorname{MOD} L/G_L \to \operatorname{MOD} L$ are the pull-up functors associated to F and F^L, and U_L is a fixed set of representatives of the cosets of G modulo G_L (see [18], (2.4) and (3.2)).

The following is an important special case of the last Theorem; see [18].

Proposition 2.2. *Let R be a tame locally bounded k-category, G be a group of k-linear automorphisms of R acting freely on the objects of R, and Y be a weakly G-periodic R-module. Then, the followings hold:*

(1) *the stabilizer* G_Y *is an infinite cyclic group;*
(2) *the push-down module* $F_\lambda(Y)$ *carries a canonical structure of a* kG_Y*-*R/G*-bimodule which is a free module of finite rank as left module over the group algebra* kG_Y *of* G_Y*. In particular, we have a canonical functor*

$$\Phi^Y = - \otimes_{kG_Y} F_\lambda(Y) : \operatorname{mod} kG_Y \longrightarrow \operatorname{mod} R/G,$$

whose image is contained in $\operatorname{mod}_2 R/G$*.*

Let R be a locally bounded k-category and G be a group of k-linear automorphisms of R acting freely on the objects of R. A *line* in R is a convex subcategory L of R which is isomorphic to the path category kQ of a linear quiver Q of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or ${}_\infty A_\infty$. A line L in R is said to be G*-periodic* if its stabilizer G_L is nontrivial. Clearly, in this case, the quiver Q_L of L is of type

$${}_\infty\mathbb{A}_\infty : \quad \cdots \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \cdots$$

and has a G_L-periodic orientation. With each G-periodic line L of R we may associate a canonical weakly G-periodic R-module M_L by setting $M_L(x) = k$ for any vertex x of Q_L, $M_L(y) = 0$ for all vertices y of $Q_R \backslash Q_L$, and $M_L(\gamma) = \mathrm{id}_k$ for each arrow γ of Q_L. Since $G_{M_L} = G_L = \mathbb{Z}$, we then obtain a canonical functor

$$\Phi^L = - \otimes_{k[T,T^{-1}]} F_\lambda(M_L) : \operatorname{mod} k[T, T^{-1}] \longrightarrow \operatorname{mod} R/G$$

where $\operatorname{mod} k[T, T^{-1}]$ denotes the category of finite dimensional modules over $k[T, T^{-1}]$.

Proposition 2.3. *Let* R *be a cycle-finite strongly simply connected category and* $F : R \to A$ *be a Galois covering functor of a finite dimensional algebra* A *defined by the action of a torsion-free group* G*. Let* $\mathcal{S}_0$ *be a set of representatives of the* G*-orbits in a separating family* $\mathcal{S}$ *of convex subcategories of* R *with respect to* G*. The followings hold:*

(i) *each* $L \in \mathcal{S}$ *is a convex subcategory of* R *which is a line* L *in* R*, that is, the quiver* Q_L *of* L *is of type*

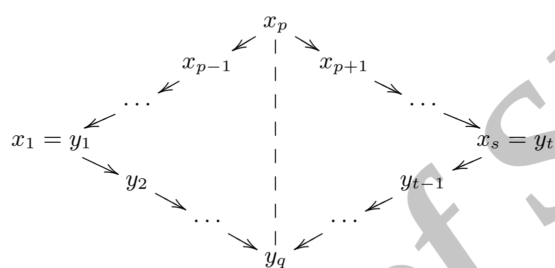
$${}_\infty\mathbb{A}_\infty : \quad \cdots \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \cdots$$

and $L = kQ_L$*;*

(ii) *for any two different* $L, L' \in \mathcal{S}$*, the intersection* $L \cap L'$ *is a connected finite linear quiver.*

Proof. (i): In [30] (3.1), without assuming that R is strongly simply connected, it was shown that L is a convex subcategory of R admitting a simply connected Galois covering $F' : \tilde{L} \to \tilde{L}/H = L$ determined by the action of a torsion free group H and $\tilde{L}$ is a line of type ${}_\infty\mathbb{A}_\infty$. Assuming that R is strongly simply connected, then $\tilde{L} = L$ as desired.

Assume (ii) fails. Since $L \cap L'$ is locally suport finite, then $L \cap L'$ is formed by at least two disconnected finite intervals of the line L. Thus, we get a convex segment x_1 —— x_2 —— $\cdots x_{s-1}$ —— x_s in L with $x_1, x_s \in L \cap L'$ and $x_i \notin L'$, $2 \le i \le s-1$. Then, the convex closure of x_1, x_s in R is of the shape



where all $y_i \in L'$ but $y_j \notin L$, for $2 \le j \le t-1$, and there is a commutativity relation from x_p to y_q. Since the stabilizer G_L acts on L, we get a weakly G-periodic convex subcategory of R which is not a line, contradicting (i). □

3. Cycle-finite strongly simply connected categories

By a *tame concealed algebra*, we mean a tilted algebra $C = \text{End}_H(T)$, where H is the path algebra kQ of a quiver Q of Euclidean type $\widetilde{\mathbb{A}}_m (m \ge 1)$, $\widetilde{\mathbb{D}}_n (n \ge 4)$, or $\widetilde{\mathbb{E}}_p (6 \le p \le 8)$, and T is a (multiplicity-free) preprojective tilting H-module. Recall that the Auslander-Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{I}^C,$$

where $\mathcal{P}^C$ is a preprojective component containing all indecomposable projective C-modules, $\mathcal{I}^C$ is a preinjective component containing all indecomposable injective C-modules, and $\mathcal{T}^C$ is a $\mathbb{P}_1(k)$-family $\mathcal{T}_\lambda^C, \lambda \in \mathbb{P}_1(k)$, of pairwise orthogonal standard stable tubes, all but finite number of them of rank one (see [33, Chapter 4]) and [34]).

By a *tubular algebra*, we mean a tubular extension (equivalently, tubular coextension) of a tame concealed algebra of tubular type $(2, 2, 2, 2)$,

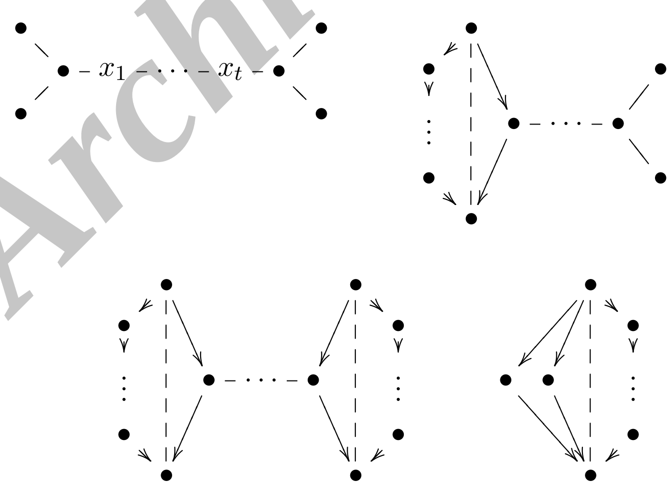
$(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, as defined in [33]. Recall that a tubular algebra B admits two different tame concealed convex subcategories C_0 and C_∞ such that the Auslander-Reiten quiver Γ_B of B is of the form

$$\Gamma_B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \vee \mathcal{T}_\infty^B \vee I_\infty^B,$$

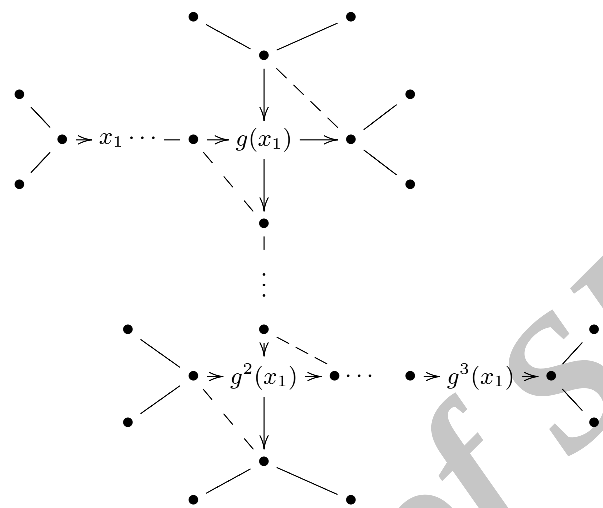
where $\mathcal{P}_0^B$ is the preprojective component $\mathcal{P}^{C_0}$ of Γ_{C_0}, $\mathcal{T}_0^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard ray tubes, obtained from the stable tubes of $\mathcal{T}^{C_0}$ by ray insertions, I_∞^B is the preinjective component I^{C_∞} of Γ_{C_∞}, $\mathcal{T}_\infty^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard coray tubes, obtained from the stable tubes of $\mathcal{T}^{C_\infty}$ by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers), $\mathcal{T}_q^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard stable tubes; see [33].

Lemma 3.1. *Let R be a tame strongly simply connected locally bounded category and G be a group acting freely on R. Let C be a tame concealed algebra of type $\tilde{\mathbb{D}}_n$ which is a convex subcategory of R. Assume x_1 is a vertex of C in a convex line $y - x_1 - x_2 - \cdots - x_t - y'$ such that each x_i has exactly two neighbors in the quiver of C and $x_t = g(x_1)$, for some $g \in G$. Then, for every number s there are indecomposable R-modules Y_s containing at least s convex tame concealed subcategories in the support supp Y_s.*

Proof. Tame concealed algebras of type $\tilde{\mathbb{D}}_s$ are given by the following frames:



with all commutativity relations. For the sake of simplicity, we assume that x_i, for $1 \leq i \leq t$, are given as in the first frame. Then, in R we get a convex subcategory B_3 of the shape



up to change of some arrow orientations. Clearly, B_3 accepts an indecomposable sincere module Y_3 whose support contains 6 tame concealed convex subcategories. Similarly, we may construct the desired indecomposable R-modules Y_s, for $s \geq 4$. □

Let B be an algebra, $\mathcal{C}$ be a standard component of Γ_B and X be an indecomposable module in $\mathcal{C}$. In [3], three *admissible operations* (ad 1), (ad 2) and (ad 3) were defined depending on the shape of the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ in order to obtain a new algebra B'.

(ad 1) If the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

then we set $B' = (B \times D)[X \oplus Y_1]$, where D is the full $t \times t$ lower triangular matrix algebra and Y_1 is the indecomposable projective-injective D-module.

(ad 2) If the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

with $t \geq 1$, so that X is injective, then we set $B' = B[X]$.

(ad 3) If the support of $\operatorname{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$\begin{array}{ccccccccccccc} & & Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & \rightarrow & Y_t & & & & \\ & & \uparrow & & \uparrow & & & & \uparrow & & & & \\ X= & & X_0 & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_{t-1} & \rightarrow & X_t & \rightarrow & \cdots \end{array}$$

with $t \geq 2$, so that X_{t-1} is injective, then we set $B' = B[X]$.

In each case, the module X and the integer t are called, respectively, the *pivot* and the *parameter of the admissible operation*. The dual operations are denoted by (ad 1*), (ad 2*) and (ad 3*).

Following [3], an algebra A is a *coil enlargement* of the critical algebra C if there is a sequence of algebras $C = A_0, A_1, \ldots, A_m = A$ such that for $0 \leq i < m$, A_{i+1} is obtained from A_i by an admissible operation with pivot in a stable tube of Γ_C or in a component (coil) of Γ_{A_i} obtained from a stable tube of Γ_C by means of the admissible operations done so far. When A is tame, then we call A a *coil algebra*.

If A is a coil enlargement of a critical algebra C, then there is a maximal branch coextension A^- of C inside A which is full and convex in A, and such that A is obtained from A^- by a sequence of admissible operations of types (ad 1), (ad 2) and (ad 3). Dually, there is a maximal branch extension A^+ of C inside A which is full and convex in A, and such that A is obtained from A^+ by a sequence of admissible operations of types (ad 1*), (ad 2*) and (ad 3*).

For a coil enlargement A of a critical algebra C, we consider the type $r(A)$ of A as follows: Let $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ be the separating tubular family of $\operatorname{mod} C$. For each $\lambda \in \mathbb{P}_1(k)$, let n_λ be the rank of $\mathcal{T}_\lambda$ and $r_\lambda^+ - n_\lambda$ (respectively, $r_\lambda^- - n_\lambda$) be the number of rays (respectively, corays) inserted in $\mathcal{T}_\lambda$ by the sequence of admissible operations that leads from C to A. Finally, let $r(A) = (r_\lambda^+, r_\lambda^-)_{\lambda\in\mathbb{P}_1(k)}$, where we write down only those numbers greater or equal to 1.

Proposition 3.2. *Let B be a coil enlargement of a tame concealed algebra C. The following conditions are equivalent.*

(*a*) *B is tame.*
(*b*) *B^+ and B^- are tame.*
(*c*) *Every cycle*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_s} X_s = X$$

of non-zero non-isomorphisms between indecomposable modules in $\operatorname{mod} B$*. belongs to a standard coil in* Γ_B*.*

(*d*) *B is of polynomial growth.*
(*e*) *B is of linear growth.*
(*f*) *B is cycle-finite.*
(*g*) *Each of* $r^+(B)$ *and* $r^-(B)$ *is one of the following:* (p, q) *where* $1 \leq p \leq q$*,* $(2, 2, r)$ *with* $r \geq 2$*,* $(2, 3, 3), (2, 3, 4), (2, 3, 5), (3, 3, 3), (2, 4, 4), (2, 3, 6), (2, 2, 2, 2)$.

Essential for our considerations is the following theorem which is the main result of [42].

Theorem 3.3. *Let A be a strongly simply connected algebra. The following conditions are equivalent.*

(*a*) *A is of polynomial growth.*
(*b*) *A is of linear growth.*
(*c*) *A is cycle-finite.*
(*d*) *A does not contain a convex subcategory which is pg-critical or hypercritical.*
(*e*) $\operatorname{rad}^\infty(\operatorname{mod} A)$ *is locally nilpotent.*
(*f*) *The component quiver* $C(A)$*, whose vertices are components of the Auslander-Reiten quiver* Γ_A *and arrows* $\mathcal{C} \to \mathcal{C}'$ *are set when there are modules* $X \in \mathcal{C}$ *and* $X' \in \mathcal{C}'$ *with* $\operatorname{rad}^\infty(X, X') \neq 0$*, has no oriented cycles.*
(*g*) *Every connected component of* Γ_A *is standard.*

A special situation of the above Theorem is the following.

Lemma 3.4. *Let B be a strongly simply connected cycle-finite algebra and M be an indecomposable B-module. Assume that*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

is a cycle of non-zero non-isomorphisms between pairwise different indecomposable modules in $\operatorname{mod} B$*, such that* $6 \leq s$ *and* f_1 *factorizes non-trivially in* $\operatorname{mod} B[M]$*. Then, one of the following two situations occur:*

(*i*) *B contains a convex subcategory* B' *which is a coil extension such that one of the two* $r^+(B')$ *or* $r^-(B')$ *is* $(2, 2, s)$.
(*ii*) $B[M]$ *is of wild type.*

Proof. Indeed, by [42] (2.3), the algebra B is multicoil and the given cycle belongs to a standard coil $\mathcal{T}$ of a multicoil of Γ_B. Let C be a tame concealed algebra such that B' is a convex subcategory of B and coil extension of a tame concealed algebra C. Assume (i) does not hold, that is, B' is of type (r_1, r_2, r_3), with $r_1 \leq r_2 \leq r_3$ and $3 \leq r_2$, or of type $(2, 2, 2, 2)$.

Let $\mathcal{T}'$ be the component of $\Gamma_{B[M]}$ where X belongs. Observe that $\mathrm{Hom}_B(M, \mathcal{T}) \neq 0$ and since f_1 is factorized there is a cycle of non-zero non-isomorphisms between $6 < s+1$ pairwise different indecomposable modules in $\mathrm{mod}\, B[M]$. If M belongs to $\mathcal{T}'$, then either M is not a pivot module or the extension type of $B'[M]$ is not tame. In the latter case, $B[M]$ is wild. Moreover, if M is not a pivot module, according to [29], the one-point extension $B'[M]$ is tame only when B' is of type $(2, 2, s)$. Since this is forbidden, then $B[M]$ is wild.

If M does not belong to $\mathcal{T}'$, then there is a regular C-module Y such that $\mathrm{Hom}_B(M, Y) \neq 0$, and $B[M]$ contains a convex subcategory of the form $C[N]$ for a preprojective C-module N. The extension $C[N]$ being wild implies that $B[M]$ is wild. □

The following theorem is the main result of [30].

Theorem 3.5. *Let R be a connected cycle-finite locally bounded k-category over an algebraically closed field k, G be a torsion-free admissible group of k-linear automorphisms of R, and $A = R/G$. Let $\mathcal{S}$ be a separating family of convex subcategories of R with respect to G and $\mathcal{S}_0$ be a fixed set of representatives of G-orbits in $\mathcal{S}$. Then, the functors $\Phi^Y = F_\lambda(Y) \otimes_{k[T,T^{-1}]} - : \mathrm{mod}\, k[T, T^{-1}] \to \mathrm{mod}\, A$, $Y \in \mathcal{S}_0$, induce a k-linear equivalence of categories*

$$\Phi : \coprod_{\mathcal{S}_0} \mathrm{mod}\, k[T, T^{-1}] \xrightarrow{\sim} \mathrm{mod}\, A/[\mathrm{mod}_1\, A].$$

Moreover, the following statements hold.

(*i*) *A is tame.*

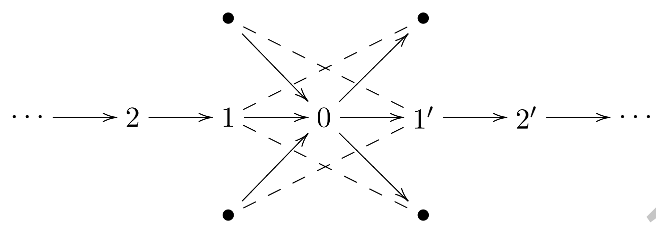
(*ii*) *Every indecomposable finite dimensional A-module X is isomorphic either to $F_\lambda(M)$ for some indecomposable finite dimensional R-module M or to $\Phi^Y(V)$ for some $Y \in \mathcal{S}_0$ and some indecomposable finite dimensional $k[T, T^{-1}]$-module V.*

(*iii*) *The Auslander-Reiten quiver Γ_A of A has the disjoint union decomposition*

$$\Gamma_A = (\Gamma_R/G) \sqcup \left(\coprod_{\mathcal{S}_0} \Gamma_{k[T,T^{-1}]} \right)$$

where $\Gamma_{k[T,T^{-1}]}$ is the Auslander-Reiten quiver of the category of finite dimensional $k[T, T^{-1}]$-modules.

There are strongly simply connected categories R of polynomial growth which are not cycle-finite, as the following example shows. Consider the category R given by the following quiver with relations as indicated by the dotted edges:



Since R has tame coil enlargements R_s of a hereditary algebra C of Euclidean type $\tilde{\mathbb{D}}_4$ of type $(2,2,s)$, for arbitrary $s \geq 1$, then $\operatorname{mod} R$ accepts cycles of non-zero morphisms between indecomposable R-modules of arbitrary length. We may build a non-trivial infinite cycle in $\operatorname{mod} R'$, where R' is the quotient of R obtained by adding a zero-relation from 1 to $1'$, of the form

$$S_0 \to P_1 \to P_s \to \begin{smallmatrix} 0 & & 1 \\ & 1 & \\ 0 & & 1 \end{smallmatrix} \to \begin{smallmatrix} 1 & & 1 \\ & 2 & \\ 1 & & 1 \end{smallmatrix} \to \begin{smallmatrix} 1 & & 0 \\ & 1 & \\ 1 & & 0 \end{smallmatrix} \to I_{s'} \to I_{1'} \to S_0$$

where S_0 is the simple module at 0, P_j (respectively, I_j) is the indecomposable projective cover (respectively, injective envelope) of S_j in $\operatorname{mod} R'$ and the dimension vectors correspond to indecomposable C-modules X_i, $i = 1,2,3$. Observe that the composition of maps $S_0 \to X_1$ is non-zero in $\operatorname{rad}^{\infty}(\operatorname{mod} R)$.

We say that the category R is of **type** $(2,2,\infty)$ if for every m it contains a convex subcategory B_m which is a coil enlargement of type $(2,2,m)$, B_m is a subcategory of B_{m+1} and $R = \bigcup_m B_m$.

The next result is preparatory for the main theorem of our work.

Lemma 3.6. *Let R be a strongly simply connected cycle-finite category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra A defined by the action of a torsion-free group G. Assume that R is of polynomial growth. Then, the followings hold:*

(*i*) *there is a number s_0 such that, for any finite convex subcategory B of R, any periodic B-module has period at most s_0;*

(*ii*) *for any cycle*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

of length $s \geq s_0$, there is a convex subcategory B of R and a coil $\mathcal{T}$ in $\operatorname{mod} B$ containing all modules X_i, $1 \leq i \leq s$, and at least $s - s_0$ indecomposable projective modules.

Proof. (i) : Let $s_0 = 2n + 4$, where n is the number of vertices in the quiver of A. Consider a convex subcategory B of R with a periodic module X of period $p > s_0$. Since B is a multicoil algebra, then X lies in a stable tube. By [3], the support of X is a tame concealed or a tubular algebra. Without lost of generality, we may asume that B is tame concealed or a tubular algebra.

Since $p > 6$, then B is tame concealed of type $\tilde{\mathbb{D}}_{p-2}$. From the structure of the frames of the tame concealed algebras, we get a linear convex subcategory of B of the shape $y - x_1 - x_2 - \cdots - x_t - y'$ such that each x_i has exactly two neighbors in the quiver of B and $x_t = g(x_1)$, for some $g \in G$. By Lemma 3.1, there is an indecomposable R-module whose support contains at least 4 convex tame concealed subcategories. This contadicts with the result in [25].

(ii) : is a consequence of (i) and the structure of multicoil components of the Auslander-Reiten quiver of multicoil algebras. □

4. **The main results**

Theorem 4.1. *Let R be a strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra A defined by the action of a torsion free group G. The followings are equivalent.*

- (a) *R is of polynomial growth and does not contain a convex subcategory of type $(2, 2, \infty)$.*
- (b) *R is of linear growth and does not contain a convex subcategory of type $(2, 2, \infty)$.*
- (c) *R is cycle-finite.*
- (d) *R does not contain a convex subcategory which is of type $(2, 2, \infty)$, pg-critical or hypercritical.*
- (e) *R does not contain a convex subcategory which is pg-critical or hypercritical and there exists a set of representatives $\mathcal{S}_0$ of the G-orbits in a separating family $\mathcal{S}$ of convex subcategories of R with respect to G formed by lines.*

  Moreover, if any of the above holds, then the following holds:
- (f) $\operatorname{rad}^{\infty}(\operatorname{mod}_1 A)$ *is locally nilpotent.*

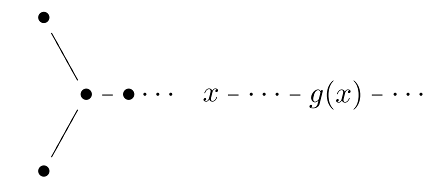
Proof. The equivalence of (a), (b) and (d) follows obviously from Theorem (4.1) in [42]. If (c) is satisfied, then clearly (a) is satisfied. Assume that (a) holds, that is, R is of polynomial growth not accepting convex subcategories of type $(2, 2, \infty)$. We shall show that there is a number s such that the maximal length of a cycle in $\mathrm{mod}\, R$ is s and therefore R is cycle finite.

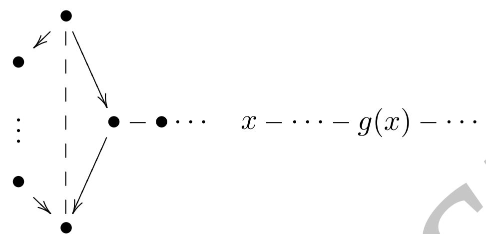
Suppose, to get a contradiction, that for every number s there is a cycle

$$\eta_s : X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{t(s)}} X_{t(s)} = X$$

of length $t(s) \geq s$. As in Lemma 3.6, there is a number s_0 such that, for any finite convex subcategory B of R, any periodic B-module has period at most s_0. In particular, any tame concealed convex subcategory C of R is of Euclidean type $(2, 3, 3), (2, 3, 4), (2, 3, 5)$ or $(2, 2, r)$ with $2 \leq r \leq s_0$. Moreover, each cycle η_s lies in a coil $\mathcal{T}_s$ in $\mathrm{mod}\, B_s$ containing at least $t(s) - s_0$ indecomposable projective modules, where B_s is a convex subcategory of R which is a coil extension of a tame concealed algebra C_s. Moreover, without lost of generality, we may assume that $B_s = B'_s[M_s]$ is a one-point extension of a coil algebra B'_s by a module in $\mathcal{T}_s$. Since there are only finitely many orbits of the action of G on R, there is a finite set F of numbers such that for every number s there is an element $g_s \in G$ such that $g_s(C_s) = C_{f(s)}$, for some $f(s) \in F$. Replacing η_s by $g_s(\eta_s)$ and choosing some $s' \in F$ with an infinite preimage $f^{-1}(s')$, we may assume, without lost of generality, that every B_s is a coil extension of the tame concealed algebra C. By Lemma 3.6 and $s \geq 7$, C is of type $(2, 2, t_0)$ with $t_0 \leq s_0$ and therefore, for $t_0 \leq s$, the cycle η_s lies in a coil $\mathcal{T}_s$ with at least $t(s) - t_0$ projective modules. Moreover, $\mathcal{T}_{s'}$ is a coil extension of the coil $\mathcal{T}_s$, for any $s' \geq s$. Clearly, R contains a convex subcategory of type $(2, 2, \infty)$, a contradiction showing (c).

(c) is equivalent to (e): we already observed that weakly G-periodic subcategories of a strongly simply connected cycle-finite category R are lines. For the converse, assume that (e) is satisfied. By theorem 3.3, every finite convex subcategory of R is of polynomial growth, that is, R is of polynomial growth. Assume, to get a contradiction, that B is a convex subcategory of R of type $(2, 2, \infty)$; in particular, there is a convex subcategory D of R tilted of type $\mathbb{D}_s$ with $s > n + 2$ for n, the number of vertices of the quiver Q_A, given by a quiver with relations corresponding to one of the following frames of categories:





for some $x \in Q_D$ and some $g \in G$. Clearly, this yields a convex subcategory D' of R which is tame concealed of type $\tilde{\mathbb{D}}_t$ and a convex line $x - x_1 - x_2 - \cdots - x_t - g(x)$ such that each x_i has exactly two neighbors in the quiver of C. Applying Lemma 3.1, we get indecomposable R-modules Y whose support contain at least 4 different tame concealed algebras. This contradicts the main result in [25].

(c) implies (f): Assume (c) holds. Consider M an indecomposable A-module of the first kind and a linear map $f : M \to M$ in $\mathrm{rad}^{\infty}(\mathrm{mod}_1 A)$. Suppose that $F_\lambda(X) = M$, for some indecomposable R-module X. Then, there are maps $f_g \in \mathrm{Hom}_R(X, X^g)$, almost all $f_g = 0$, such that $\sum\limits_{g\in G} F(f_g)$
$= f$. Since $f \in \mathrm{rad}^s(\mathrm{mod}_1 A)$ then, $f^g \in \mathrm{rad}^s(\mathrm{mod}\, R)$, for any $s \geq 1$. Suppose $0 \neq f = f_1 \cdots f_r$, for some $f_i \in \mathrm{rad}^{\infty}(M, M)$, there exist maps $f_{(i,g)} \in \mathrm{rad}^{\infty}(X, X^g)$, for $1 \leq i \leq r$, with almost all $f_{(i,g)} = 0$, such that $\sum\limits_{g\in G} F(f_{(i,g)}) = f_i$. We get

$$f_g = \sum_{g=g_r\cdots g_1} f_{(r,g_r)}^{g_{r-1}\cdots g_1} \cdots f_{(2,g_2)}^{g_1} f_{(1,g_1)}.$$

Call $X_0 = X, X_1 = X^{g_1}, X_2 = X^{g_2 g_1}, \cdots, X_r = X^{g_r \cdots g_2 g_1}$ and consider a non-zero composition of maps $0 \neq f_r \cdots f_2 f_1$ with $f_i \in \mathrm{rad}^{\infty}(X_{i-1}, X_i)$, $1 \leq i \leq r$. Since R is cycle-finite and therefore $\mathrm{rad}^{\infty}(Y, Y) = 0$ for any indecomposable R-module Y, then the modules X_i, $0 \leq i \leq r$, are pairwise non-isomorphic indecomposable R-modules with the same dimension $d = \dim_k M$. The Harada-Sai lemma yields a contradiction, in case $r \geq 2^d$. This shows that $\mathrm{rad}^\infty(\mathrm{mod}_1 A)$ is locally nilpotent. □

Given a Galois covering $R \to R/G = A$ of a finite dimensional k-algebra A, we observe that a component $\mathcal{C}'$ of Γ_A is either of the *first kind*, that is formed by the modules $F_\lambda(X)$, for $X \in \mathcal{C}$ for a component $\mathcal{C}$ in Γ_R, or of the *second kind*, that is formed by the modules $\Phi^Y(V)$, for Y a fixed weakly G-periodic module and V an indecomposable $k[T, T^{-1}]$-module. The following consequence for the structure of components of the Auslander-Reiten quiver Γ_A is obtained.

Proposition 4.2. *Let R be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra A defined by the action of a torsion free group G. Let $\mathcal{C}$ be a component of the Auslander-Reiten quiver Γ_R. The followings hold:*

- (a) *the set of vertices a such that $X(a) \neq 0$, for some indecomposable $X \in \mathcal{C}$, form a convex subcategory $B(\mathcal{C})$ of R;*
- (b) *the stabilizer $G' = G_{\mathcal{C}}$ of $\mathcal{C}$ is a normal subgroup of G;*
- (c) *the category $B(\mathcal{C})$ is strongly simply connected and cycle-finite, the induced functor $F' : B(\mathcal{C}) \to A'$ is a Galois covering defined by the action of a torsion free group G', and $\mathcal{C}$ is a component of $\Gamma_{B(\mathcal{C})}$ with stabilizer $G'_{\mathcal{C}} = G'$;*
- (d) *every component of the Auslander-Reiten quiver $\Gamma(\mathrm{mod}_1 A)$ is generalized standard.*

Proof. (a) : Assume $a_1 \to a_2 \to \cdots \to a_r$ is a path in the quiver Q_R such that $X(a_1) \neq 0 \neq Y(a_r)$, for indecomposable modules $X, Y \in \mathcal{C}$ and $Z(a_i) = 0$, for $2 \leq i \leq r-1$, and all $Z \in \mathcal{C}$. We shall construct a cycle in the componental quiver $C(R)$. This contradicts [42](4.1).

Indeed, consider the quotient R' of R obtained by adding relations $a_1 \to a_2 \to b$ and $c \to a_{r-1} \to a_r$, for all arrows $a_2 \to b$ and $c \to a_{r-1}$. Consider I_x to be the injective envelope and P_x to be the projective cover of the simple module S_x corresponding to a vertex x in the category $\mathrm{mod}\, R'$. We get a path of morphisms in $\mathrm{mod}\, R$ to be

$$Y \to I_{a_r} \to S_{a_{r-1}} \to F(a_{r-1}, a_{r-2}) \to S_{a_{r-2}} \to \cdots \to F(a_3, a_2) \to S_{a_2} \to P_{a_1} \to X$$

where for any arrow $y \to x$ in Q_R, the R-module $F(x, y)$ is the unique indecomposable whose composition factors are S_x and S_y. Since S_{a_i} does not belong to $\mathcal{C}$, for $2 \leq i \leq r-1$, we get a cycle through $\mathcal{C}$ in the componental quiver $C(R)$.

(b) and (c) are obvious.

(d): Let M and N be two modules in $\mathcal{C}'$ and $0 \neq f \in \operatorname{rad}_A^\infty(M,N)$. Assuming that $\mathcal{C}'$ is of the first kind implies that there exists a component $\mathcal{C}$ in Γ_R and indecomposable R-modules $X, Y \in \mathcal{C}$ such that $F_\lambda(X) = M$ and $F_\lambda(Y) = N$. Lifting the morphism f provides morphisms $f_g \in \operatorname{rad}_R^\infty(X, Y^g)$, for $g \in G'$, almost all zero, such that $\sum_{g\in G'} F_\lambda(f_g) = f$. We remark that, for the algebra A', we have $\operatorname{rad}_{A'}^\infty(M,N) = 0$. Indeed, a morphism $f' \in \operatorname{rad}_{A'}^\infty(M,N)$ yields the existence of morphisms $f'_g \in \operatorname{rad}_R^\infty(X, Y^g)$, for $g \in G'$, almost all zero, such that $\sum_{g\in G'} F_\lambda(f'_g) = f'$. Since for $g \in G'$ the module $Y^g \in \mathcal{C}$, then [42](4.1) implies that $\operatorname{rad}_R^\infty(X, Y^g) = 0$. Hence $f'_g = 0$ and $f' = 0$.

Since $\operatorname{rad}_{A'}^\infty(M,N) = 0$ then, for every $g \in G$ such that $f_g \neq 0$, we have $\operatorname{rad}_{B(\mathcal{C})}^\infty(X, Y^g) = 0$ and there is a chain of irreducible maps connecting X and Y^g, that is, $Y^g \in \mathcal{C}$ and $g \in G_{\mathcal{C}}$. Up to a change of orientation, we may assume that there is an indecomposable projective R-module $P_a \notin \mathcal{C}$ such that $\operatorname{rad} P_a = L$ and the one-point extension category $B' = B(\mathcal{C})[L]$ is convex in R. Moreover, $f_g \in \operatorname{rad}_{B'}^\infty(X, Y^g)$ factorizes through a module $Z \in \operatorname{mod} B'$ satisfying $Z(a) \neq 0$. Therefore, there is a direct summand Z' of Z satisfying $Z' \notin \mathcal{C}$ and there is a cycle in the componental quiver $C(R)$ of the form $\mathcal{C} = [X] \to [Z'] \to [Y^g] = \mathcal{C}$, where $[Z']$ denotes the component in Γ_R containing Z'. $\square$

5. **Criteria for polynomial growth**

The aim of this section is to establish a criterion for an algebra with a cycle-finite Galois covering to be of polynomial growth (respectively, domestic type). We start by recalling a criterion in [30].

Theorem 5.1. *Let R be a connected cycle-finite locally bounded k-category, G be a torsion-free admissible group of k-linear automorphisms of R, and $A = R/G$. Then the followings hold.*

- (i) *A is of polynomial growth if and only if the number of G-orbits of isoclasses of weakly G-periodic R-modules is finite.*
- (ii) *A is domestic if and only if R does not contain a convex subcategory which is tubular and the number of G-orbits of isoclasses of weakly G-periodic R-modules is finite.*

Part of the following result is explicit in [30].

Theorem 5.2. *Let R be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra A defined by the action of a torsion free group G. Let $\mathcal{S}_0$ be a set of representatives of the G-orbits in a separating family $\mathcal{S}$ of convex subcategories of R with respect to G. Then the followings hold.*

(1) *The category* $\text{mod}_1 A$ *of modules of the first type is of polynomial growth.*
(2) *The category* $\text{mod}_2 A$ *of modules of the second kind is of polynomial growth if and only if $\mathcal{S}_0$ is a finite set.*
(3) *The algebra A is of polynomial growth if and only if the cardinality of $\mathcal{S}_0$ is bounded by the number n of vertices in Q_A.*

Proof. (1): Since every convex subcategory of R is of polynomial growth, by [16], Lemma 3, the category of modules of the first kind $\text{mod}_1 A$ is of polynomial type.

(2): This results from Theorem 4.1 in [30].

(3): Assume there are different lines $L_1, \dots, L_s \in \mathcal{S}_0$ for any $s > n$. Obviously, not all sets of vertices $F(L_i)$ are disjoint. We may suppose x is a vertex in $L_1 \cap L_2$. Let $1 \neq g \in G_{L_1}$ and observe that $g(x) \notin L_2$, since otherwise, by Proposition 2.3, we would have $L_1 = L_2$. Consider the line L'_s, for $s \in \mathbb{N}$ formed by the vertices

$$\cdots - y_{-2} - y_{-1} - x - x_1 - \cdots - x_t = g(x) \quad \cdots - g^2(x) - \cdots - g^s(x) -$$
$$- g(y_1) - g(y_2) - \cdots$$

where $x - x_1 - \cdots x_t = g(x) - \cdots - x_{2t} = g^2(x) - \cdots - x_{st} = g^s(x)$ is the convex segment of L_1 connecting x and $g^s(x)$, and $\cdots - y_{-2} - y_{-1} - x - y_1 - y_2 - \cdots$

is the line L_2. We may assume that y_{-1} and $g(y_1)$ are not in the line L_1. We claim that the lines L'_s determine pairwise different elements in $\mathcal{S}_0$. Indeed, assume that $h(L'_p) = L'_q$, for some $p \leq q$, and $h \in G$. Then h sends infinite segments of L_2 to L_2, and hence $h \in G_{L_2}$. Moreover, L'_p contains exactly tp vertices of L_1, which yields $p = q$. □

The structure of G is sometimes a source of information on the families of second kind modules, and hence on the representation type of R/G. Namely, we show the following proposition.

Proposition 5.3. *Let R be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional* *algebra A defined by the action of a torsion-free group G. If G is cyclic, then A is of polynomial growth.*

Proof. Assume that $\mathcal{S}_0$ is not empty and assume that G is cyclic. Take lines $L_1, \ldots, L_s \in \mathcal{S}_0$, for any $s > n$, where n is the number of vertices in Q_A. Obviously, not all the sets of vertices $F(L_i)$ are disjoint. We may suppose x to be a vertex in $L_1 \cap L_2$. Since G_{L_1} and G_{L_2} are non-trivial cyclic subgroups of G, then $G/(G_{L_1} \cap G_{L_2})$ is a finite group. Let $1 \neq g \in G_{L_1} \cap G_{L_2}$ and observe that x and $g(x)$ belong to $L_1 \cap L_2$ which, according to Proposition 2.3, is formed by a unique connected segment. This yields $L_1 = L_2$, and the cardinality of $\mathcal{S}_0$ is at most n. □

6. Examples

Here we illustrate some results of our work in four parts.

(1) We start by giving an example (see [12]) of the relation between structural properties of the category R and the group G defining the Galois covering.

Theorem 6.1. *Let R be a strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional triangular algebra A defined by the action of a torsion free group G. Then, G is a free (non-abelian) group.*

Sketch of proof: (i) Assume $A = B[M]$ to be a one-point extension of an algebra B by a module M. Let a be a source vertex in Q_A such that $\operatorname{rad} P_a = M$ and x be a vertex in Q_R such that $F(x) = a$. Consider R' the convex subcategory formed by those vertices at the preimage $F^{-1}(B)$ and choose a connected component R^B of R'. The stabilizer G^B of R^B is a normal subgroup of G. Consider $F^B : R^B \to B$ to be the functor obtained as the restriction of F. We get that R_B is a strongly simply connected category and $F^B : R^B \to B$ is a Galois covering functor of a finite dimensional triangular algebra B defined by the action of a torsion free group G^B.

By induction on the $\dim_k A$, we may assume that G^B is a free group.

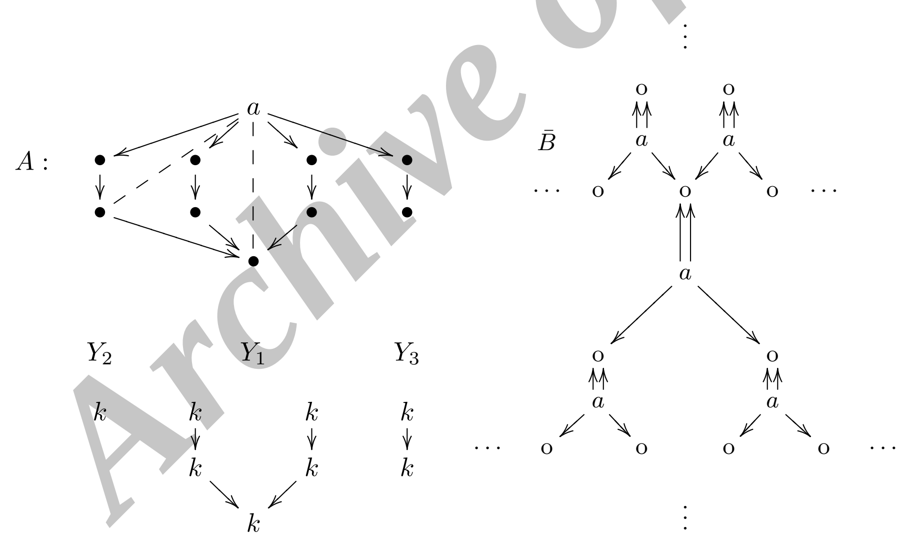
(ii) We show that F is a covering of the first kind, that is, if $\operatorname{rad} P_a = M_1 \oplus \cdots \oplus M_t$ is an indecomposable decomposition in $\operatorname{mod} B$, then there is an indecomposable decomposition $\operatorname{rad} P_x = Y_1 \oplus \cdots \oplus Y_s$ in $\operatorname{mod} R'$ such that $s = t$ and a permutation σ satisfying $F_\lambda(Y_i) = M_{\sigma(i)}$, for $1 \leq i \leq t$.

Indeed, since R is strongly simply connected, then the source x *separates* R, that is, there are connected components $R_1, \cdots, R_s$ of R' such that the support of Y_i is contained in R_i, for $1 \leq i \leq s$. Therefore, $\operatorname{rad} P_a = \operatorname{rad} F_\lambda(Px) = \bigoplus_{i=1}^{s} F_\lambda(Y_i)$ is an indecomposable decomposition and the claim follows.

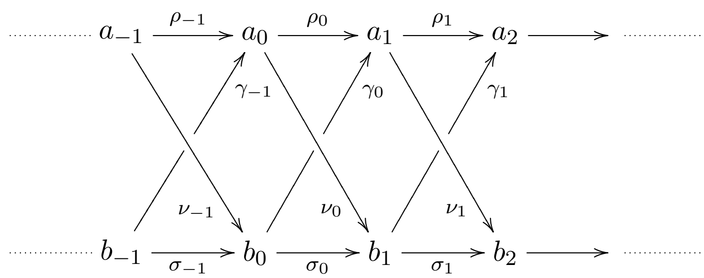
(iii) The group G/G^B is a free group $F(t-1)$ of rank $t-1$.

(iv) G is isomorphic to the free product $G^B * F(t-1)$ and it is therefore a free group.

To illustrate the idea of the proof, assume that $t = 3$. Construct the category $\bar{B}$ as a model for a covering $F' : \bar{B} \to A$ defined by the action of $F(t-1) = F(2)$. Sustitute each o in the diagram by the category R_B in such a way that, for every vertex a, the radical $\operatorname{rad} P_a = Y_1 \oplus Y_2 \oplus Y_3$ (in the representations Y_i the arrows stand for identity maps; observe that the vertical arrows in $\bar{B}$ contribute Y_1 to the radical of P_a). The functor $F : R \to A$ factorizes as $F = \bar{F}F'$ by a Galois covering functor $\bar{F} : R \to \bar{B}$ defined by the action of G^B (in the example $G^B = \mathbb{Z}$):

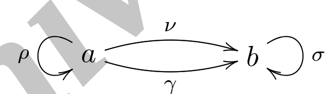


(2) As another series of examples, consider the categories $R_{\alpha,\beta}$ given by the quiver with relations



$$\left.\begin{aligned} \sigma_{i+1}\sigma_i &= \alpha\nu_{i+1}\gamma_i \\ \nu_{i+1}\rho_i &= \beta\sigma_{i+1}\nu_i \\ \gamma_{i+1}\nu_i &= \rho_{i+1}\rho_i \\ \rho_{i+1}\gamma_i &= \gamma_{i+1}\sigma_i \end{aligned}\right\}$$

and $(\alpha, \beta) \neq (1, 1)$, to be locally support finite; it is simply connected but not strongly simply connected. Moreover, the group $\mathbb{Z}$ generated by the action $(a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1})$ acts freely on $R_{\alpha,\beta}$ and on $\operatorname{ind}_{R_{\alpha,\beta}} / \cong$. Hence, the Galois covering $F\colon R_{\alpha,\beta} \to A_{\alpha,\beta}/\mathbb{Z}$ yields a bijection $F_\lambda\colon$ (ind $R_{\alpha,\beta}/\cong)/\mathbb{Z} \to (\operatorname{ind} A_{\alpha,\beta})/\cong$. The algebra $A_{\alpha,\beta}$ is given by the quiver with relations



$$\left.\begin{aligned} \sigma^2 &= \alpha\gamma\nu \\ \rho\nu &= \beta\nu\sigma \\ \nu\gamma &= \rho^2 \\ \gamma\rho &= \sigma\gamma \end{aligned}\right\}$$

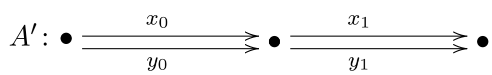
Since $R_{\alpha,\beta}$ is tame (respectively polynomial growth for $\alpha\beta \neq 1$), so is $A_{\alpha,\beta}$.

(3) Consider the Galois covering

$$F\colon A = A_{1,1}^{(2)} \to \bar{A} = A_{1,1}^{(2)}/\mathbb{Z}_2$$

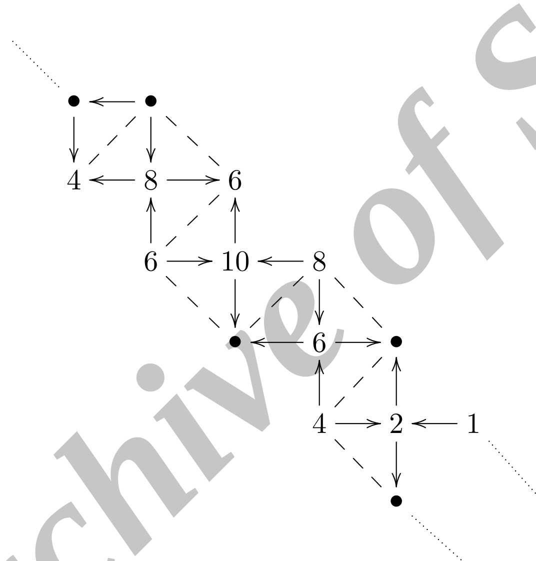
and assume that the characteristic of k is 2. As a tubular algebra, we know that A is tame. We show that $\bar{A}$ is a wild algebra.

Set $x_0 = \alpha_0 + \beta_0$, $y_0 = \beta_0$, $x_1 = \alpha_1 + \beta_1$, $y_1 = \beta_1$. Then, $\bar{A}$ is isomorphic to the algebra A' given by the quiver with relations



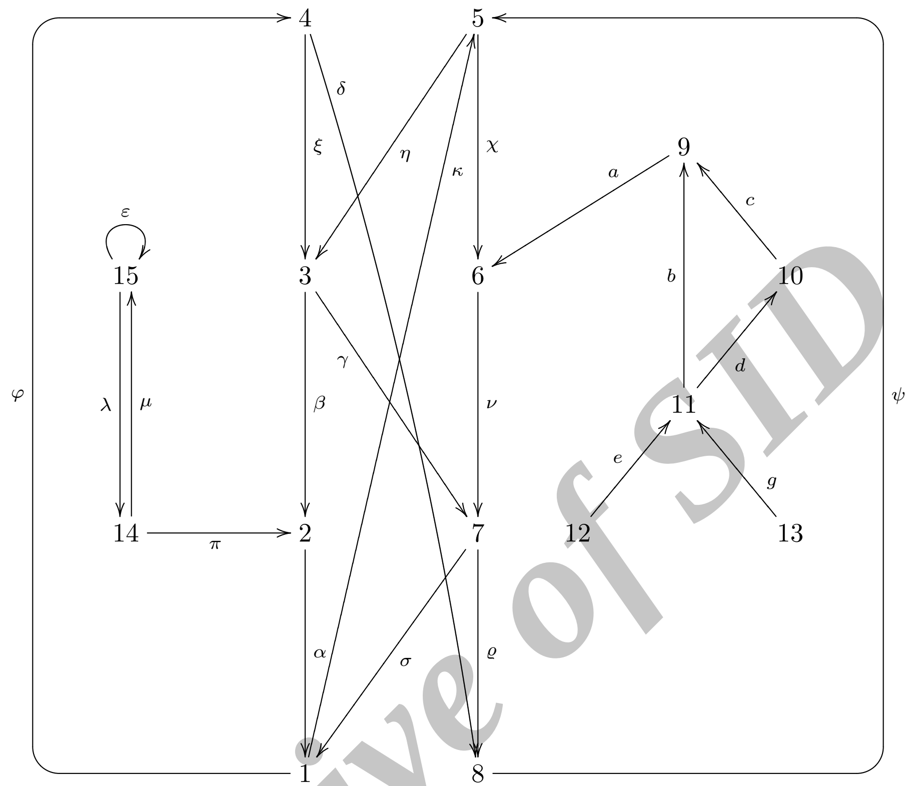
$$\left.\begin{array}{rcl} x_1x_0 & = & 0 \\ y_1x_0 & = & x_1y_0 \end{array}\right\}$$

Observe that A' accepts a Galois covering $R \to R/\mathbb{Z} = A'$, given by the category



which is strongly simply connected. Therefore, R is tame if and only if the Tits form q_R is weakly non-negative. Observe that the vector y marked on the vertices of R determines a convex subcategory B of R whose Tits form takes value $q_B(y) = -1$. Therefore, R is a wild category and A' a wild algebra.

(4) Our last example is similar to an example given in [30]. Let A be the bound quiver algebra kQ/I given by the quiver



and the ideal I of the path algebra kQ of Q be generated by the elements $\alpha\varphi$, $\alpha\kappa$, $\sigma\varphi$, $\sigma\kappa$, $\gamma\sigma$, $\gamma\varrho$, $\nu\varrho$, $\nu\sigma$, $\eta\gamma$, $\eta\beta$, $\xi\beta$, $\xi\gamma$, $\varphi\delta$, $\delta\psi$, $\psi\eta$, $\psi\chi$, $\kappa\chi$, $\kappa\eta$, $a\nu$, ba, dca, eb, $gb - gdc$, $\pi\alpha$, $\lambda\pi$, $\mu\lambda\mu$, $\varepsilon^2 - \lambda\mu$, $\mu\lambda - \mu\varepsilon\lambda$.

For k of characteristic 2, the convex subcategory B of A given by the objects 14 and 15 is a penny-farthing, and hence is a non-standard representation-finite algebra. Hence, for k of characteristic 2, the algebra A does not admit a simply connected (even triangular) Galois covering. For characteristic different from 2, A is isomorphic to kQ/I', where I' is obtained by sustituting the last given relation by $\mu\lambda$. For this presentation, the algebra A accepts a covering $R \to R/G = A$, where R is strongly simply connected and G is a torsion-free group.

The convex subcategory C given by the vertices $9, 10, 11, 12, 13$ determines a convex subcategory of R which contains subcategories of type $(2, 2, \infty)$; that is, R is not cycle-finite.

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José A. de la Peña*
Instituto de Matemáticas, UNAM, Ciudad Universitaria, D.F. México, MéxicoCentro de Investigación en Matemáticas, Guanajuato, México
Email: jap@matem.unam.mx