Bulletin of the Iranian Mathematical Society Vol. 37 *No.* 2 (*2011*), *pp 273-290.*

Conference Paper

RIGID DUALIZING COMPLEXES

A. NEEMAN

Communicated by Siamak Yassemi

Abstract. Let X be a sufficiently nice scheme. We survey some recent progress on dualizing complexes. It turns out that a complex in $\mathbf{K}(\mathrm{Inj}/X)$ is dualizing if and only if tensor product with it induces an equivalence of categories from Murfet's new category $\mathbf{K}_m(\mathrm{Proj}/X)$ to the category $\mathbf{K}(\mathrm{Inj}/X)$. In these terms, it becomes interesting to wonder how to glue such equivalences.

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1. Introduction

Nowadays, dualizing complexes must count as a venerable, old part of algebraic geometry. They date back to the early 1960's, to Grothendieck's work on duality. In the past few years, we have come to a remarkable new perspective on them; as is always the case with recent progress, our understanding is still quite patchy.

In this survey, I would like to present a novel approach to dualizing complexes, and then raise one of the many puzzling questions. The way I have structured the article is that it begins with a review of classical, relatively well-understood facts. The next section summarizes the dualizing complexes, how they can be rigidified (Van den Bergh's style), and how one goes about proving their existence. Section 3 offers a short review of some developments in the past few years, and the final section

MSC(2010): Primary: 14F05; Secondary: 14A22, 18E30.
Keywords: Dualizing complex, Grothendieck duality.
Received: 9 March 2010, Accepted: 11 November 2010.
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consists of some problems, addressing an aspect of the theory which is understand very little.

2. **Old stuff**

Let X be a noetherian, separated scheme. Let $\mathbf{D}^b(\mathrm{Coh}/X)$ be the bounded derived category of coherent sheaves on X. We define a dualizing complex to be an object $\mathcal{I} \in \mathbf{D}^b(\mathrm{Coh}/X)$ so that the natural functor

$$\mathbf{R}\mathcal{H}om(-,\mathcal{I}) \ : \ \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} \longrightarrow \mathbf{D}^b(\mathrm{Coh}/X)$$

is an equivalence of categories.[1] Historically, dualizing complexes formed the core of Grothendieck's theory of duality; the main technical lemmas in the subject can be viewed as controlling the behavior of dualizing complexes under morphisms of schemes, and the traditional approach, in which the subject was originally developed, with built on these lemmas. Of course, the Grothendieck duality contains theorems which make no explicit mention of dualizing complexes, but the proofs, at least the old ones that go back to Grothendieck, all depended crucially on dualizing complexes.

Nowadays, we know that much of the theory can be set up without any reference to dualizing complexes. Furthermore, the modern approach is smoother than the original clunky version. It might help if we sketch for the reader the current status of the theory.

Notation 2.1. Let $f : X \longrightarrow Y$ be a morphism of quasicompact, separated schemes. Then, $\mathbf{R}f_* : \mathbf{D}(\mathrm{Qcoh}/X) \longrightarrow \mathbf{D}(\mathrm{Qcoh}/Y)$ stands for the derived pushforward map, from the (unbounded) derived category of quasicoherent sheaves on X to the derived category of quasicoherent sheaves on Y. We let $\mathbf{L}f^*$ denote its left adjoint. If f is flat, then there is no need to take left derived functors, and we feel free to write f^* for $\mathbf{L}f^*$. The first result is given next.

[1]It is traditional to impose on $\mathcal{I}$ a finiteness condition, concerning the injective dimension. The theory works fine without the restriction; the reader can find this in [11]. In the traditional terminology, what we call here "dualizing complexes" would go by the name "pointwise dualizing complexes", and one consequence of the results of [11] is that pointwise dualizing complexes share practically all the good properties of dualizing complexes.

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Theorem 2.2. *With the notation as above, the functor $\mathbf{R}f_*$ has a right adjoint $f^\# : \mathbf{D}(\mathrm{Qcoh}/Y) \longrightarrow \mathbf{D}(\mathrm{Qcoh}/X)$. This right adjoint satisfies the following properties.*

(*i*) *$f^\#$ takes $\mathbf{D}^+(\mathrm{Qcoh}/Y) \subset \mathbf{D}(\mathrm{Qcoh}/Y)$ into $\mathbf{D}^+(\mathrm{Qcoh}/X) \subset \mathbf{D}(\mathrm{Qcoh}/X)$.*

(*ii*) *Suppose X and Y are noetherian and f is proper. Assume that we are given a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with g a flat morphism of noetherian schemes. Then, for every object $\mathcal{V} \in \mathbf{D}^+(\mathrm{Qcoh}/Y)$, there is a canonical, natural base-change isomorphism

$$(g')^* f^\# \mathcal{V} \longrightarrow (f')^\# g^* \mathcal{V}.$$

Proof. A modern proof for the existence of $f^\#$ can be found in [9, Example 4.2]. For (i), see the first two paragraphs in the proof of [9, Proposition 6.3], while (ii) is found in [13, Theorem 2] or [7, Corollary 4.4.3]. It should be noted that the hypothesis in (ii), that X and Y both be noetherian, can be relaxed somewhat; the reader can find a more general statement in the second of the references. In this survey, I am not trying to give the sharpest known results; I am happy to sacrifice generality in the interest of transparency. If the strongest known statement is at all technical, which is in the case of (ii), then I choose simplicity over generality. □

Notation 2.3. Assume $f : X \longrightarrow Y$ is a finite type morphism of noetherian, separated schemes. By a theorem of Nagata's [2, Theorem 4.1], we may factor $f : X \longrightarrow Y$ as $X \xrightarrow{g} \overline{X} \xrightarrow{h} Y$, where, g is an open immersion and h is proper. We will write $f^!$ for the composite

$$\mathbf{D}^+(\mathrm{Qcoh}/Y) \xrightarrow{h^\#} \mathbf{D}^+(\mathrm{Qcoh}/\overline{X}) \xrightarrow{g^* = \mathbf{L}g^*} \mathbf{D}^+(\mathrm{Qcoh}/X).$$

The first important fact is that $f^!$ is well defined. Even though the proof is standard[2], we will give it in some detail, to underline why it only works

[2]See, for example, [13, Corollary 1]. A more extensive discussion may be found in [7, Section 4.8].

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for $\mathbf{D}^+$; note that we were careful to only define $f^!$ on bounded-below complexes.

Lemma 2.4. *Up to canonical isomorphism, the functor* $f^!\colon \mathbf{D}^+(\mathrm{Qcoh}/Y) \longrightarrow \mathbf{D}^+(\mathrm{Qcoh}/X)$ *is independent of the factorization* $f = hg$.

Proof. Suppose we are given another factorization $X \xrightarrow{g'} \widehat{X} \xrightarrow{h'} Y$. We can factor g and g' through the pullback $\overline{X} \times_Y \widehat{X}$. Let $\widetilde{X} \subset \overline{X} \times_Y \widehat{X}$ be the scheme-theoretic closure of X in the pullback. We obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & \widetilde{X} & \xrightarrow{\beta} & \overline{X} \\ & & {\scriptstyle\beta'}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ & & \widehat{X} & \xrightarrow[h']{} & Y \end{array}$$

where, $g = \beta\alpha$, $g' = \beta'\alpha$. The map α is an open immersion, and β, β' are proper maps which are isomorphisms on the image of α. Given any object $\mathcal{V} \in \mathbf{D}^+(\mathrm{Qcoh}/Y)$, by Theorem 2.2(i), we have that $h^{\#}\mathcal{V}$ (respectively $(h')^{\#}\mathcal{V}$) lies in $\mathbf{D}^+(\mathrm{Qcoh}/\overline{X})$ (respectively $\mathbf{D}^+(\mathrm{Qcoh}/\widehat{X})$). We also have two cartesian squares

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \widetilde{X} \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ X & \xrightarrow[g]{} & \overline{X} \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\alpha} & \widetilde{X} \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle\beta'} \\ X & \xrightarrow[g']{} & \widehat{X} \end{array} \qquad (*)$$

with β and β' being proper, while $g = \beta\alpha$ and $g' = \beta'\alpha$ are open immersions (hence flat). We have nautral isomorphisms

$$\begin{aligned} g^* h^{\#}\mathcal{V} &\cong \alpha^*\beta^{\#}h^{\#}\mathcal{V} && \begin{cases} \text{Theorem 2.2(ii) applied to left square of } (*) \\ \text{and to } h^{\#}\mathcal{V} \in \mathbf{D}^+(\mathrm{Qcoh}/\overline{X}), \end{cases} \\ &\cong \alpha^*(\beta')^{\#}(h')^{\#}\mathcal{V} && \text{because } h\beta = h'\beta', \\ &\cong (g')^*(h')^{\#}\mathcal{V} && \begin{cases} \text{Theorem 2.2(ii) applied to right square of } (*) \\ \text{and to } (h')^{\#}\mathcal{V} \in \mathbf{D}^+(\mathrm{Qcoh}/\widehat{X}). \end{cases} \end{aligned}$$

Note that we appealed to Theorem 2.2(ii) twice, which requires the complexes to be bounded below; see [9, Example 6.5] for what dreadful things can happen with unbounded complexes. □

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The proof of the next lemma is similar, and is left to the reader.

Lemma 2.5. *Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are composable morphisms, both of finite type, of noetherian, separated schemes. Then, up to canonical isomorphism,* $f^! g^! \cong (gf)^!$.

The relevance of all this, to dualizing complexes, comes from the following result.

Proposition 2.6. *Let* $f : X \longrightarrow Y$ *be a finite type morphism of noetherian, separated schemes. If* $\mathcal{J}$ *is a dualizing complex on* Y*, then* $f^!\mathcal{J}$ *is a dualizing complex on* X*.*

Proof. The question is local in X; see [11, Theorem 3.12]. Let $U \subset X$ be an affine open subset, whose image under f is contained in an affine open subset $V \subset Y$. Let $g : U \longrightarrow X$ be the inclusion; it suffices to show that $g^* f^! \mathcal{J}$ is dualizing on U. By Lemma 2.5, we have that $g^* f^! = (fg)^!$; replacing f by fg, we may assume that $X = U$ is affine. But, now the map $f : X \longrightarrow Y$ factors through the open affine $V \subset Y$. In the factorization $X \xrightarrow{\alpha} V \xrightarrow{\beta} Y$, the map β is an open immersion, and hence $\beta^! \mathcal{J} = \beta^* \mathcal{J}$ is dualizing. We are therefore reduced to the case where both X and Y are affine. The map $f : X \longrightarrow Y$ may be factored as

$$X \xrightarrow{\ \alpha\ } X' \xrightarrow{\ \beta\ } X'' \xrightarrow{\ \gamma\ } Y$$

where, α is an open immersion, β is a closed immersion, and γ is a projection $\mathbb{P}^n \times Y \longrightarrow Y$. Now, each of $\alpha^!$, $\beta^!$ and $\gamma^!$ takes dualizing complexes to dualizing complexes; for $\alpha^! = \alpha^*$, this comes from [11, Theorem 3.12], for $\beta^! = \beta^{\#}$, we use [11, Lemma 3.18 and Theorem 3.14], and for $\gamma^! = \gamma^{\#}$, this can be proved many ways, including by modifying slightly the argument of [11, Section 4].[3] □

[3] We could appeal to [11, Fact 0.3(ii)]. But, then we would need to assume that Y satisfies the technical condition $(*)$ of [11, Conjecture 4.16]. The modification I am suggesting, which quite easily handles the case of the projection $\gamma : \mathbb{P}^n_Y = \mathbb{P}^n \times Y \longrightarrow Y$, is the following. In [11, Lemma 4.4], we learn that the functor $\left[\gamma^! \mathcal{O}_Y\right] \overset{\mathbf{L}}{\otimes} (-)$ commutes with all products of complexes of the form $\gamma^* \mathcal{L}$, with $\mathcal{L} \in \mathbf{D}(\mathrm{Qcoh}/Y)$. It certainly follows that $\left[\gamma^! \mathcal{O}_Y\right] \overset{\mathbf{L}}{\otimes} (-)$ commutes with the much more restricted class of products, where we only allow $\mathcal{L}$ to be of the form $\Sigma^n \mathcal{O}_Y$; this is formalized in [11, Definition 4.7], and [11, Sections 4 and 5] studies what one can deduce about $\gamma^! \mathcal{O}_Y$, from the weaker hypothesis. In the case of the projection $\gamma : \mathbb{P}^n_Y \longrightarrow Y$, it is helpful to use all the complexes $\gamma^* \mathcal{L}$, since it is easy to express any complex on $\mathbb{P}^n_Y$ in terms of $\gamma^* \mathcal{L}$'s plus a finite number of the $\mathcal{O}_{\mathbb{P}^n_Y}(m)$.

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And now we come to the punchline.[4]

Theorem 2.7. *Let $f : X \longrightarrow Y$ be a finite type morphism of noetherian, separated schemes. Suppose Y is Gorenstein, that is, the structure sheaf $\mathcal{O}_Y$ is a dualizing complex on Y. Then, the complex $\mathcal{I} = f^!\mathcal{O}_Y$ is a dualizing complex on X.*

If we assume furthermore that f is flat, then there is a canonical isomorphism $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$. Here, $\Delta : X \longrightarrow X \times_Y X$ is the diagonal map, and $\mathcal{I} \boxtimes \mathcal{I} = \mathcal{I} \boxtimes_Y \mathcal{I}$ is the external tensor product.

Proof. The fact that $\mathcal{I} = f^!\mathcal{O}_Y$ is a dualizing complex comes from Proposition 2.6. Assume f to be flat; we need to produce the isomorphism $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$. For this, consider the composite

$$X \xrightarrow{\ \Delta\ } X \times_Y X \xrightarrow{\ f\times f\ } Y \times_Y Y \xrightarrow{\ e\ } Y,$$

where, Δ is the diagonal map, and e is the obvious identification. The composite is equal to f. Lemma 2.5 therefore gives us a canonical isomorphism

$$f^!\mathcal{O}_Y \quad \cong \quad \Delta^!(f \times f)^! e^!\mathcal{O}_Y \,. \qquad (**)$$

Now, $e^!\mathcal{O}_Y$ is just the pullback of $\mathcal{O}_Y$ by the isomorphism $Y \times_Y Y = Y$, and hence identifies with the external tensor product $\mathcal{O}_Y \boxtimes \mathcal{O}_Y$. This makes

$$(f \times f)^! e^!\mathcal{O}_Y \quad \cong \quad (f \times f)^!\big[\mathcal{O}_Y \boxtimes \mathcal{O}_Y\big] \quad \cong \quad f^!\mathcal{O}_Y \boxtimes f^!\mathcal{O}_Y \quad = \quad \mathcal{I} \boxtimes \mathcal{I}\,,$$

and from $(**)$ we deduce our isomorphism $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$.

Perhaps we should explain where the hypothesis of flatness plays a role. It is in the proof that $(f \times f)^!\big[\mathcal{O}_Y \boxtimes \mathcal{O}_Y\big] \cong f^!\mathcal{O}_Y \boxtimes f^!\mathcal{O}_Y$. Observe that

$$\begin{aligned}(f \times f)^!\big[\mathcal{O}_Y \boxtimes \mathcal{O}_Y\big] \;&\cong\; (f \times 1)^!(1 \times f)^!\big[\mathcal{O}_Y \boxtimes \mathcal{O}_Y\big] \\ &\cong\; (f \times 1)^!\big[\mathcal{O}_Y \boxtimes f^!\mathcal{O}_Y\big] \\ &\cong\; f^!\mathcal{O}_Y \boxtimes f^!\mathcal{O}_Y \,,\end{aligned}$$

where, the last isomorphism appeals to the two facts

[4] A version of the second part of Theorem 2.7 first appeared in Van den Bergh [12]. In the form given in Theorem 2.7 it is due to Lipman, and may be found in [14, Remark 6.20].

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(i) If $g : Z \longrightarrow Z'$ is of finite tor-dimension, then $g^!\mathcal{F} \cong g^!\mathcal{O}_{Z'} \overset{\mathbf{L}}{\otimes} g^*\mathcal{F}$.[5] In our case, g is the map

$$f \times 1 \ : \ X \times_Y X \ \longrightarrow \ Y \times_Y X, \qquad (*)$$

while $\mathcal{F} = \mathcal{O}_Y \boxtimes f^!\mathcal{O}_Y$.

(ii) We apply Theorem 2.2(ii), slightly generalized to work for an $f^!$ in place of an $f^{\#}$, to the pullback square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{1\times f} & X \times_Y Y \\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle f\times 1} \\ Y \times_Y X & \xrightarrow[1\times f]{} & Y \times_Y Y \end{array}$$

and to the object $\mathcal{O}_Y \boxtimes \mathcal{O}_Y \in \mathbf{D}^+(\text{Qcoh}/Y \times_Y Y)$. This is where we absolutely need flatness.

□

Remark 2.8. By the first half of Theorem 2.7, every noetherian scheme X, of finite type over a Gorenstein scheme Y, automatically has a dualizing complex. Dualizing complexes are cheap and widely available.

Remark 2.9. If $Y = \text{Spec}(k)$, where k is a field, then we know that

(i) any morphism $f : X \longrightarrow Y$ is flat, and
(ii) Y is Gorenstein.

Both parts of Theorem 2.7 therefore apply to any morphism of finite type $f : X \longrightarrow \text{Spec}(k)$. In Remark 2.8, we observed that any scheme, of finite type over a field k, has a dualizing complex $\mathcal{I}$; the flatness part of Theorem 2.7 guarantees that $\mathcal{I}$ may be chosen to have a canonical rigidifying isomorphism $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$. The reader is encouraged to look at Van den Bergh's [12], where a version of this was originally observed; for applications, see also the later articles by Yekutieli and Zhang [14, 15, 16].

It should also be mentioned that rigid dualizing complexes, that is dualizing complexes $\mathcal{I}$ together with a rigidifying isomorphisms $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$, are useful in non-commutative algebraic geometry. One should not scoff and say that we already know all we really need to know about the existence of dualizing complexes. This might be true in commutative

[5] If g is proper, in which case $g^! = g^{\#}$, then the proof may be found in [9, Example 5.2 and Theorem 5.4]. For general g, we first reduce to the case where Z and Z' are both affine, and then use the standard factorization.

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algebraic geometry, but we are very far from such a satisfactory state of affairs in the non-commutative case.

We note in passing: the definition of rigid dualizing complexes, in Van den Bergh's sense, assumes we have a morphism $X \longrightarrow Y$ so that we can speak of an isomorphism $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes_Y \mathcal{I}]$. In order to prove existence, using Theorem 2.7, we need to assume further that Y is Gorenstein and X is flat over it; but for all we know, these hypotheses might not be the best possible.

Remark 2.10. Dualizing complexes are not unique: if $\mathcal{J}$ is a dualizing complex on X, if $\mathcal{L}$ is a line bundle on X and if n is any integer, then $\Sigma^n\mathcal{L} \otimes \mathcal{J}$ is also a dualizing complex. The only ambiguity corresponds to the case when X is connected; see [11, Lemma 3.9].

The useful fact is that rigid dualizing complexes are unique; if X is a scheme of finite type over a noetherian, separated Y, and if we have, on X, two dualizing complexes $\mathcal{I}$ and $\mathcal{J}$ and isomorphisms

$$\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes_Y \mathcal{I}] \ , \qquad\qquad \mathcal{J} \longrightarrow \Delta^![\mathcal{J} \boxtimes_Y \mathcal{J}] \ ,$$

then $\mathcal{I}$ and $\mathcal{J}$ must be isomorphic; the isomorphism between them is even unique (that is, canonical) if we require it to be compatible with rigidifications, and no ambiguity up to suspension or twisting by a line bundle. If X and Y are smooth and X is flat over Y, then $\mathcal{I}$ is a shift of the relative canonical bundle by the relative dimension.

Remark 2.11. The argument we have given so far shows that one can use the functor $f^!$ to deduce the existence of dualizing complexes on many schemes, and in some cases even the existence of rigid dualizing complexes. This is the reverse of the historical approach to the subject; in Grothendieck's original development, dualizing complexes were used to prove the existence of $f^!$. This means that the old guys had to construct dualizing complexes without the help of Proposition 2.6.

The classical way people proceeded was by gluing together dualizing complexes on affine bits of a scheme. Given a scheme X, the idea was to find an open cover $X = \cup U_i$, where the U_i are affine and for each i we have a dualizing complex $\mathcal{I}_i$ on $U_i \subset X$. The problem was to glue to a global dualizing complex. Without getting bogged down in technicalities, let me remind the reader that gluing objects in derived categories is not for the faint-hearted; derived categories are singularly ill-suited for patching objects together, and the objects involved have to be very special for it to have any chance of success. The way Grothendieck set about this was by looking at the minimal injective resolutions of the

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$\mathcal{I}_i$ and their remarkable properties; the reader can look up [3, 1] or [8, Section 9] to see how convoluted the arguments become. It seems to me that, at a deep, fundamental level, we really do not yet understand why dualizing complexes glue.

One could take the cheap way out; we know, by virtue of Remark 2.8, that global dualizing complexes exist in great generality. Once we know the global existence, it becomes clear that local data must be glueable. This seems to me a cheat; there are many hard theorems in the literature informing us how to glue dualizing complexes, and they beg for a clearer understanding.

Remark 2.12. Until now, the most inspired idea was to use the rigidifying isomorphisms $\mathcal{I} \longrightarrow \Delta^![\mathcal{I} \boxtimes \mathcal{I}]$ of Van den Bergh; see Yekutieli and Zhang [14] for more. Again let me not go into detail; suffice it to say that the technicalities involved in gluing rigid dualizing complexes are much more pleasant than those that come up in the more classical, non-rigid context. In view of Remark 2.10, this is hardly surprising; unlike their unrigidified cousins, rigid dualizing complexes cannot be perturbed by tensoring with line bundles and shifting.

The drawback of the approach is that, in order to produce rigidifying isomorphisms, Theorem 2.7 would require all the affine schemes being glued to be flat over a fixed Gorenstein scheme. This is OK for schemes over a field k, but the flatness hypothesis becomes a problem as soon as we deal with more general base schemes. The older gluing results were much more general (albeit infinitely uglier to prove). The question, which I will raise in Section 4, is whether there is some alternative rigidifying structure.

3. Some very new results

In the past few years, we have come to have a new perspective on dualizing complexes; the approach is so novel, and so beset with obvious questions which beg to be answered, that it is hard to know where to start; it is unclear which of the many gaps in our understanding to mention first. In this section, we will briefly survey the developments of the recent past and then, in Section 4, we will raise only one of the multitude of open problems, the one we hinted at towards the end of Section 2.

But, first we need to review the recent progress. To state the new results, let me remind the reader of the general formalism of compact objects in a [TR5] triangulated category. We recall the definitions.

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Definition 3.1. *A triangulated category satisfies [TR5] if it has arbitrary small coproducts.*

Example 3.2. The following are examples:

(i) Let X be a noetherian scheme, and let $\mathbf{K}(\mathrm{Inj}/X)$ be the category whose objects are chain complexes of injective objects in the category of quasicoherent sheaves on X, and whose morphisms are the homotopy equivalence classes of chain maps. Then, $\mathbf{K}(\mathrm{Inj}/X)$ satisfies [TR5]. The point is that, over a noetherian scheme, direct sums of injective quasicoherent sheaves are injective.

(ii) Let R be a commutative ring, and let $X = \mathrm{Spec}(R)$ be its spectrum. The category $\mathbf{K}(\mathrm{Proj}/X)$ has for its objects the chain complexes of projective objects in the category of quasicoherent sheaves, and the morphisms are the homotopy equivalence classes of chain maps. The category $\mathbf{K}(\mathrm{Proj}/X)$, which is obviously equivalent to $\mathbf{K}(R\text{–Proj})$, also satisfies [TR5].

Definition 3.3. *Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. An object $k \in \mathcal{T}$ is* compact *if any map*

$$k \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$$

factors as

$$\begin{array}{ccc} k & \longrightarrow & \coprod_{\lambda\in\Lambda} X_\lambda \\ & \searrow & \nearrow \\ & \coprod_{i=1}^{n} X_{\lambda_i} & \end{array}$$

that is, any map, from k into an arbitrary coproduct, factors through a finite coproduct.

Definition 3.4. *Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. The full subcategory of all compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$.*

Definition 3.5. *Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. We say that $\mathcal{T}$ is* compactly generated *if*

(i) *$\mathcal{T}^c$ is essentially small, and*

(ii) *$\mathcal{T}^c$ generates $\mathcal{T}$. This means that one of the following two equivalent conditions holds:*

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(a) *If* $\mathcal{S}$ *is a triangulated subcategory of* $\mathcal{T}$*, closed under coproducts and containing* $\mathcal{T}^c$*, then* $\mathcal{S} = \mathcal{T}$.

(b) *If* $X \in \mathcal{T}$ *is any non-zero object, then there is a non-zero map* $k \longrightarrow X$ *with* $k \in \mathcal{T}^c$.

This reminds us of the basic definitions in the theory of compactly generated triangulated categories. The relation of dualizing complexes, with compactly generated triangulated categories, comes from two recent theorems, which appeared in two lovely articles in 2005.

Theorem 3.6. (Krause [6]**)** *Suppose that* X *is a noetherian, separated scheme.*[6] *Then, the category* $\mathbf{K}(\text{Inj}/X)$ *is compactly generated. Furthermore, there is a natural equivalence*

$$\mathbf{K}(\text{Inj}/X)^c \quad \cong \quad \mathbf{D}^b(\text{Coh}/X).$$

Theorem 3.7. (Jørgensen [5]**)** *Let* R *be a noetherian, commutative ring,*[7] *and put* $X = \text{Spec}(R)$*. Then, the category* $\mathbf{K}(\text{Proj}/X) \cong \mathbf{K}(R\text{–Proj})$ *is compactly generated. Furthermore, there is a natural equivalence*

$$\mathbf{K}(\text{Proj}/X)^c \quad \cong \quad \mathbf{D}^b(\text{Coh}/X)^{\text{op}} .$$

To be precise means the following. Let X be a noetherian, separated scheme. In theorems 3.7 and 3.6, Jørgensen and Krause respectively, exhibit two functors

$$\mathbf{D}^b(\text{Coh}/X)^{\text{op}} \xrightarrow{\ \Phi\ } \mathbf{K}(\text{Proj}/X)\ , \quad \mathbf{D}^b(\text{Coh}/X) \xrightarrow{\ \Psi\ } \mathbf{K}(\text{Inj}/X).$$

Jørgensen's functor Φ was defined only if X was affine. When they exist, these functors are fully faithful, and in each case the essential image is the subcategory of compact objects. The functor Ψ is simple to describe; it takes an object in $\mathbf{D}^b(\text{Coh}/X)$ to its injective resolution. The functor Φ is slightly more subtle and I would rather not go into details.

Suppose X is a noetherian scheme, and let $\mathcal{I}$ be any bounded-below complex of quasicoherent sheaves, that is, $\mathcal{I} \in \mathbf{D}^+(\text{Qcoh}/X)$. For any object $\mathcal{F}$ of $\mathbf{D}^b(\text{Coh}/X)$, the Hom-complex $\mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{I})$ is bounded below;

[6] Krause's result is more general than what we state here. We only care about the scheme version.

[7] Jørgensen's theorem [5, Theorem 2.4] is more general in that the ring is not assumed commutative, and needs only be right and left coherent, which is less restrictive than noetherian. But then, Jørgensen's result imposes a further hypothesis on R, a condition which turns out to be unnecessary; see [10, Facts 2.8(iii)] for an assertion which covers Theorem 3.7, and [10, Remark 2.10] for a comparison with Jørgensen's result.

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it belongs to $\mathbf{D}^+(\mathrm{Qcoh}/X)$. The functor $\mathbf{R}\mathcal{H}om(-,\mathcal{I})$ takes $\mathbf{D}^b(\mathrm{Coh}/X)$ into $\mathbf{D}^+(\mathrm{Qcoh}/X)$. Noting that the natural map $\mathbf{K}^+(\mathrm{Inj}/X) \longrightarrow \mathbf{D}^+(\mathrm{Qcoh}/X)$ is an equivalence, we may view $\mathbf{R}\mathcal{H}om(-,\mathcal{I})$ as a functor

$$\mathbf{R}\mathcal{H}om(-,\mathcal{I}) \;:\; \mathbf{D}^b(\mathrm{Coh}/X) \longrightarrow \mathbf{K}^+(\mathrm{Inj}/X)\ .$$

With this notation, we now state a lemma.

Lemma 3.8. *Let X be a noetherian, affine scheme, and let $\mathcal{I}$ be a bounded-below complex of injective quasicoherent sheaves, that is, $\mathcal{I} \in \mathbf{K}^+(\mathrm{Inj}/X)$. Then, the following diagram of functors commutes up to canonical natural isomorphism*

$$\begin{array}{ccc}
\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\ \mathbf{R}\mathcal{H}om(-,\mathcal{I})\ } & \mathbf{K}^+(\mathrm{Inj}/X) \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle I} \\
\mathbf{K}(\mathrm{Proj}/X) & \xrightarrow[\ \mathcal{I}\otimes-\ ]{} & \mathbf{K}(\mathrm{Inj}/X)
\end{array} \qquad (\dagger)$$

In the diagram ($\dagger$), the map Φ is Jørgensen's functor, while I is the natural inclusion. The functor $\mathcal{I}\otimes -$ takes $\mathcal{F}$ to $\mathcal{I}\otimes\mathcal{F}$.

Since I have not disclosed to the reader what is the functor Φ, I must ask her to accept Lemma 3.8 on faith; the fact that this square commutes was first observed by Iyengar and Krause [4], who then proceeded to cleverly use it. Much of the argument below follows their footsteps.

Remark 3.9. Let the notation be as in Lemma 3.8. For a general $\mathcal{I} \in \mathbf{K}^+(\mathrm{Inj}/X)$ and a general $\mathcal{F} \in \mathbf{D}^b(\mathrm{Coh}/X)$, all we know is that $\mathbf{R}\mathcal{H}om(\mathcal{F},\mathcal{I})$ belongs to $\mathbf{D}^+(\mathrm{Qcoh}/X) \cong \mathbf{K}^+(\mathrm{Inj}/X)$. If $\mathcal{I}$ is carefully chosen, then it may just happen that for every $\mathcal{F} \in \mathbf{D}^b(\mathrm{Coh}/X)$, the complex $\mathbf{R}\mathcal{H}om(\mathcal{F},\mathcal{I})$ actually lies in the subcategory $\mathbf{D}^b(\mathrm{Coh}/X)$. For such a wisely selected $\mathcal{I}$, the following square clearly commutes, once again up to canonical natural isomorphism:

$$\begin{array}{ccc}
\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\ \mathbf{R}\mathcal{H}om(-,\mathcal{I})\ } & \mathbf{D}^b(\mathrm{Coh}/X) \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\
\mathbf{K}(\mathrm{Proj}/X) & \xrightarrow[\ \mathcal{I}\otimes-\ ]{} & \mathbf{K}(\mathrm{Inj}/X)
\end{array} \qquad (\dagger\dagger)$$

This comes from the commutativity of ($\dagger$) coupled with the definition of the map Ψ; we remind the reader that the map Ψ, given to us in Krause's Theorem 3.6, is nothing other than taking injective resolutions.

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Dualizing complexes are examples of wisely chosen complexes, as in Remark 3.9. We now deduce the following result.

Corollary 3.10. *Let X be a noetherian, affine scheme, and let $\mathcal{I}$ be (the injective resolution of) a dualizing complex. Then, the functor*

$$\mathcal{I} \otimes - \ : \ \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$$

is an equivalence of categories.

Proof. By Remark 3.9, the diagram (††) commutes, and because $\mathcal{I}$ is a dualizing complex, the top row in (††) is an equivalence. This means that the functor $\mathcal{I} \otimes - : \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ is a coproduct-preserving exact functor of compactly generated categories, inducing an equivalence on the subcategories of compact objects. Formal nonsense about compactly generated triangulated categories permit us to deduce that $\mathcal{I} \otimes - : \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ is an equivalence. □

The converse also holds. We prove the following.

Corollary 3.11. *Let X be a noetherian, affine scheme, and let $\mathcal{I}$ be any complex of injectives. Assume that the functor*

$$\mathcal{I} \otimes - \ : \ \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$$

is an equivalence of categories. Then, $\mathcal{I}$ is a dualizing complex.

Proof. Any equivalence must take compact objects to compact objects. It follows that the image of the compact object $\mathcal{O}_X \in \mathbf{K}(\mathrm{Proj}/X)$ must be compact in $\mathbf{K}(\mathrm{Inj}/X)$; but the image is nothing other than $\mathcal{I} = \mathcal{I} \otimes \mathcal{O}_X$. Its compactness forces it to lie in the essential image of Krause's functor Ψ; thus, $\mathcal{I}$ is isomorphic in $\mathbf{K}(\mathrm{Inj}/X)$ to a bounded below complex of injectives. Replace it by such an isomorph. Then, Lemma 3.8 applies, and the square (†) commutes. The fact that $\mathcal{I} \otimes -$ takes compacts to compacts, applied to the top row of (†), forces the functor $\mathbf{R}\mathcal{H}om(-, \mathcal{I})$ to take $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ into $\mathbf{D}^b(\mathrm{Coh}/X) \subset \mathbf{K}^+(\mathrm{Inj}/X)$. The square (††) of Remark 3.9 therefore also commutes. Since $\mathcal{I} \otimes -$ is an equivalence on the large categories, it must restrict to an equivalence between the subcategories of compact objects. We conclude that $\mathcal{I}$ must be a dualizing complex. □

Putting together Corollaries 3.10 and 3.11, we obtain the following.

Summary 3.12. Let X be a noetherian, affine scheme. An object $\mathcal{I} \in \mathbf{K}(\mathrm{Inj}/X)$ is a dualizing complex if and only if the functor $\mathcal{I} \otimes - : \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ is an equivalence of categories.

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The first puzzle, now solved, was to try to find a global version of this affine result. Perhaps we should explain why this was a puzzle: the reader should note that the category $\mathbf{K}(\mathrm{Proj}/X)$, of projective quasi-coherent sheaves, is nearly worthless except when X is affine; it often consists only of the zero object. The problem was to find a suitable replacement.

In [10, Theorem 1.2], we proved that, when X is a noetherian, affine scheme, the category $\mathbf{K}(\mathrm{Proj}/X)$ can also be described as a Verdier quotient $\mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}$. Here, $\mathbf{K}(\mathrm{Flat}/X)$ is the homotopy category of chain complexes of flat, quasicoherent sheaves on X, and the subcategory $\mathcal{E} \subset \mathbf{K}(\mathrm{Flat}/X)$ has many characterizations. The one we want to generalize here, to all noetherian, separated schemes, is the following.

Definition 3.13. *Let X be a noetherian, separated scheme. Let $\mathbf{K}(\mathrm{Flat}/X)$ be the homotopy category of chain complexes of flat, quasicoherent sheaves. We define the full subcategory $\mathcal{E} = \mathcal{E}(X) \subset \mathbf{K}(\mathrm{Flat}/X)$ as having objects $\mathcal{F}^*$ which are complexes*

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\partial^{i-1}} \mathcal{F}^{i} \xrightarrow{\partial^{i}} \mathcal{F}^{i+1} \longrightarrow \cdots$$

such that

(i) *the complex $\mathcal{F}^*$ is acyclic, and*
(ii) *for each $i \in \mathbb{Z}$, the image of the map $\partial^i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ is flat.*

As explained above, when X is affine, the quotient category $\mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}$ is equivalent to $\mathbf{K}(\mathrm{Proj}/X)$; more concretely, the natural composite

$$\mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Flat}/X) \xrightarrow{\pi} \frac{\mathbf{K}(\mathrm{Flat}/X)}{\mathcal{E}}$$

is an equivalence of categories. The Verdier quotient map π makes sense for any noetherian, separated scheme. We make this a definition.

Definition 3.14. *Let X be a noetherian, separated scheme. Define the functor $\pi : \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)$ to be the Verdier quotient map*

$$\pi \ : \ \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \frac{\mathbf{K}(\mathrm{Flat}/X)}{\mathcal{E}} \ ;$$

this means, in particular, that $\mathbf{K}_m(\mathrm{Proj}/X)$ is defined to be the Verdier quotient $\mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}$.

Remark 3.15. This definition is the starting point of Daniel Murfet's Ph.D. thesis. The logic behind the terminology was that, except in the special case where X is affine, $\mathbf{K}_m(\mathrm{Proj}/X)$ can be thought of as a fake substitute for $\mathbf{K}(\mathrm{Proj}/X)$. The subscript m in $\mathbf{K}_m(\mathrm{Proj}/X)$ stands for

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the *mock* homotopy category of projectives. But Murfet probably grew tired of being teased that m also happens to be the first letter of his last name;[8] he has recently begun changing his notation.

The results in Murfet's Ph.D. thesis inform us as followings.

Theorem 3.16. *Let X be a noetherian, separated scheme. Then, the category $\mathbf{K}_m(\mathrm{Proj}/X)$ is compactly generated. Furthermore, there is a fully faithful functor*

$$\Phi \;:\; \mathbf{D}^b(\mathrm{Coh}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X),$$

whose essential image is the subcategory of compact objects $\mathbf{K}_m(\mathrm{Proj}/X)^c$.

Once again, I would prefer not to give the definition of the functor Φ. I ask the reader to believe that Lemma 3.8 generalizes; we have

Lemma 3.17. *Let X be a noetherian, separated scheme, and let $\mathcal{I}$ be a bounded-below complex of injective quasicoherent sheaves. Then, the following diagram of functors commutes up to canonical natural isomorphism*

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\mathbf{R}\mathcal{H}om(-,\mathcal{I})} & \mathbf{K}^+(\mathrm{Inj}/X) \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle I} \\ \mathbf{K}_m(\mathrm{Proj}/X) & \xrightarrow[\mathcal{I}\otimes -]{} & \mathbf{K}(\mathrm{Inj}/X). \end{array} \qquad (\dagger)$$

In this diagram, Φ is Murfet's functor, while I is the natural inclusion.

Remark 3.18. The careful reader will object about the functor $\mathcal{I}\otimes -$. Clearly, there is a well-defined functor

$$\mathcal{I}\otimes - \;:\; \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X).$$

In order to define the tensor product functor on the Verdier quotient $\mathbf{K}_m(\mathrm{Proj}/X) = \mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}$, we need to know that, for every object $\mathcal{F} \in \mathcal{E}$, the tensor product $\mathcal{I}\otimes\mathcal{F}$ is contractible. This is true; for X affine, it is in [10, Corollary 9.7(ii)], and the general case may be found in Murfet's thesis.

Once you know that the diagram $(\dagger)$ commutes, for any noetherian, separated scheme X, then the argument we gave, in proceeding from Remark 3.9 through to Summary 3.12, generalizes formally. We conclude as follows.

[8] Including, I must admit, by his Ph.D. advisor.

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Corollary 3.19. *Let X be a noetherian, separated scheme. An object $\mathcal{I} \in \mathbf{K}(\mathrm{Inj}/X)$ is a dualizing complex if and only if the functor*

$$\mathcal{I} \otimes - \;:\; \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$$

is an equivalence of categories.

4. Open problems

Let me begin with the one question to which I know the answer. The first suggestion that might pop into the mind of the unwary is that, perhaps, every equivalence of categories $\mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ takes $\mathcal{O}_X \in \mathbf{K}_m(\mathrm{Proj}/X)$ to a dualizing complex. This is false.

Example 4.1. Let X be a principally polarized abelian variety, with principal polarization $\mathcal{P}$. Because X is smooth, the categories $\mathbf{K}_m(\mathrm{Proj}/X)$, $\mathbf{D}(\mathrm{Qcoh}/X)$ and $\mathbf{K}(\mathrm{Inj}/X)$ all agree. We need to produce an autoequivalence of $\mathbf{D}(\mathrm{Qcoh}/X)$ which takes $\mathcal{O}_X$ to something which is not a dualizing complex.

The idea is to use the Fourier–Mukai correpondence; it is the functor which takes an object $\mathcal{F} \in \mathbf{D}(\mathrm{Qcoh}/X)$, pulls it back to $\pi_1^*\mathcal{F}$ in $\mathbf{D}(\mathrm{Qcoh}/X \times X)$, forms the tensor product $\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_1^*\mathcal{F}$ in $\mathbf{D}(\mathrm{Qcoh}/X \times X)$, and then projects to $\mathbf{R}\{\pi_2\}_*\big[\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_1^*\mathcal{F}\big]$ in $\mathbf{D}(\mathrm{Qcoh}/X)$. This Fourier–Mukai correspondence takes $\mathcal{O}_X$ to a skyscraper sheaf supported at $0 \in X$, and the skyscraper sheaf is not a dualizing complex.

What we learn from Example 4.1 is that equivalences of categories $E : \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ are not all equal. And this is the point of the questions I want to ask here.

Some equivalences, such as tensoring with a dualizing complex, take $\mathcal{O}_X$ to a dualizing complex. Other equivalences do not. Can one single out, in some concrete fashion, the equivalences which take $\mathcal{O}_X$ to dualizing complexes with restrictions, for example to rigid dualizing complexes? What is the right notion of a rigid equivalence of categories $E : \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$? How can one glue them?

Acknowledgments

This article was written while the author was visiting the CRM in Barcelona for six months. Many thanks go to the CRM for its hospitality and congenial working environment. The visit to the CRM was partly supported by a sabbatical grant from the Spanish Ministry of Education (grant number SAB2006-0135). The research was also partly

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supported by the Australian Research Council. The author also thanks an anonymous referee for the helpful suggestions leading to an improvement presentation of the manuscript.

References

[1] B. Conrad, *Grothendieck Duality and Base Change*, Lecture Notes in Mathematics **1750**, Springer-Verlag, Berlin, 2000.

[2] B. Conrad, Deligne's notes on Nagata compactifications, *J. Ramanujan Math. Soc.* **22** (2007) 205-257.

[3] R. Hartshorne, *Residues and Duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics (20), Springer-Verlag, Berlin, New York, 1966.

[4] S. Iyengar and H. Krause, Acyclicity versus total acyclicity for complexes over Noetherian rings, *Doc. Math.* **11** (2006) 207-240.

[5] P. Jørgensen, The homotopy category of complexes of projective modules, *Adv. Math.* **193** (2005) 223-232.

[6] H. Krause, The stable derived category of a Noetherian scheme, *Compos. Math.* **141** (2005) 1128-1162.

[7] J. Lipman, Notes on Derived Functors and Grothendieck Duality, *Foundations of Grothendieck duality for diagrams of schemes*, Lecture Notes in Math. **1960**, Springer, Berlin, 2009, pp. 1-259.

[8] J. Lipman, S. Nayak, and P. Sastry, Pseudofunctorial behavior of Cousin complexes on formal schemes, *Variance and Duality for Cousin Complexes on Formal Schemes*, Contemp. Math. **375**, *Amer. Math. Soc., Providence, RI*, 2005, pp. 3-133.

[9] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, *J. Amer. Math. Soc.* **9** (1996) 205-236.

[10] A. Neeman, The homotopy category of flat modules, and Grothendieck duality, *Invent. Math.* **174** (2008) 255-308.

[11] A. Neeman, Derived categories and Grothendieck duality, *Triangulated Categories*, London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, Cambridge, 2010, pp. 290-350.

[12] M. van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, *J. Algebra* **195** (1997) 662-679.

[13] J.-L. Verdier, Base change for twisted inverse images of coherent sheaves, *1969 Algebraic Geometry* (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), *Oxford Univ. Press*, London, pp. 393-408.

[14] A. Yekutieli and J. J. Zhang, *Rigid Dualizing Complexes on Schemes*, arXiv:math/0405570v3.

[15] A. Yekutieli and J. J. Zhang, Dualizing complexes and perverse modules over differential algebras, *Compos. Math.* **141** (2005) 620-654.

[16] A. Yekutieli and J. J. Zhang, Dualizing complexes and perverse sheaves on non-commutative ringed schemes, *Selecta Math. (N.S.)* **12** (2006) 137-177.

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Amnon Neeman
Centre for Mathematics and its Applications, Mathematical Sciences Institute, John Dedman Building, The Australian National University Canberra, ACT 0200, Australia
Email: Amnon.Neeman@anu.edu.au

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