

RANKS OF MODULES RELATIVE TO A TORSION THEORY

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ABSTRACT. Relative to a hereditary torsion theory τ we introduce a dimension for a module M , called τ -rank of M , which coincides with the reduced rank of M whenever τ is the Goldie torsion theory. It is shown that the τ -rank of M is measured by the length of certain decompositions of the τ -injective hull of M . Moreover, some relations between the τ -rank of M and complements to τ -torsionfree submodules of M are obtained.

1. Introduction

Throughout the paper, rings will have unit elements and modules will be unitary right modules. The category of all right R -modules is denoted by $\text{Mod-}R$, and the notation \leq_e will denote an essential submodule. In the paper $\tau = (\mathcal{T}, \mathcal{F})$ will denote a fixed hereditary torsion theory on $\text{Mod-}R$. Then, $\tau(M) = \sum\{N : N \leq M, N \in \mathcal{T}\}$ is the τ -torsion submodule of $M \in \text{Mod-}R$. The module M is called τ -torsion, if $M \in \mathcal{T}$, and τ -torsionfree, if $M \in \mathcal{F}$. In fact, M is τ -torsion, if $\tau(M) = M$, and τ -torsionfree, if $\tau(M) = 0$. A submodule A of M is called τ -dense, if M/A is τ -torsion and we denote this by $A \leq_{\tau-d} M$. It is clear that

$$\tau(M) = \{m \in M : \text{ann}(m) \leq_{\tau-d} R_R\}.$$

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More information on torsion-theoretic concepts can be found in [5]. In this paper we prove that whenever the uniform dimension of a complement to $\tau(M)$ is a finite number n then the uniform dimension of every complement to $\tau(M)$ is n . We call this integer n , the τ -rank of M , and if no such integer exists we say that M is not of finite τ -rank. We shall prove that the hereditary torsion τ is stable, if and only if the τ -rank of M coincides with the uniform dimension of $M/\tau(M)$ for every $M \in \text{Mod-}R$.

In Goldie's theory of uniform dimension, within a module M , one seeks for a submodule $A = A_1 \oplus \cdots \oplus A_l$ with the largest possible l such that each A_i is non-zero, and if such an A exists then A is essential in M . In fact, M is measured by the largest possible direct sum of non-zero submodules which it can contain. In section 2, we introduce the notion of *pseudo τ -essential submodule* which has a role in the subject of τ -rank similar to that of an essential submodule in Goldie's theory of uniform dimension. We say that a submodule A of M is pseudo τ -essential, if for every submodule B of M , $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. This concept is a generalization of the notion of *τ -essential submodule*. A submodule A of M is called τ -essential, if A is τ -dense and essential in M . Such submodules appear in many concepts such as (s-) τ -CS modules and the τ -injective hulls of modules which are torsion-theoretic analogues of CS modules and the injective hulls of modules; see [1] and [2]. Some properties of τ -essential submodules can be found in [1, Proposition 3.1], and in Propositions 2.3 and 2.6 we show that most of these properties hold generally for pseudo τ -essential submodules.

In section 3 we deal with the theory of τ -rank. Then, we will prove that to find the τ -rank of a module M , one should look for a submodule $A = A_1 \oplus \cdots \oplus A_l \oplus B$ with the largest possible l such that each A_i is non-zero τ -torsionfree and B is quasi- τ -torsion (that is $\tau(B) \leq_e B$), and if such an A exists then A is pseudo τ -essential in M . As the uniform dimension of a module M relates to the decomposition length of the injective hull of M , we show that the τ -rank of M is measured by the length of certain decompositions of the τ -injective hull of M . Indeed, in Proposition 3.11 we show that the τ -rank of M is a finite number n , if and only if the τ -injective hull of M is a direct sum of n pseudo τ -uniform modules and a quasi τ -torsion module. Finally section 4 is devoted to some relations between τ -ranks and certain complements analogous to the well known relations between uniform dimensions and complements.

In fact, the τ -rank of M is the supremum of the set of nonnegative integers k for which M contains a chain of length k of complements to τ -torsionfree submodules of M .

2. Pseudo τ -essential submodules

In this section we introduce the notion of a pseudo τ -essential submodule and give some properties of such submodules for later use. We say that a submodule A of a module M is *pseudo τ -essential in M* and write $A \leq_{p,\tau,e} M$, if for every submodule B of M , $A \cap B \leq \tau(M)$ implies that $B \leq \tau(M)$. Clearly every τ -torsionfree essential submodule of M is pseudo τ -essential. Moreover, if A is a submodule of a τ -torsionfree module M , then A is pseudo τ -essential in M if and only if A is essential in M .

The first result shows that the notion of pseudo τ -essential is a generalization of the notion of τ -dense. In particular, every τ -essential submodule is pseudo τ -essential.

Proposition 2.1. *Every τ -dense submodule of M is pseudo τ -essential in M .*

Proof. Assume that A is a τ -dense submodule of M . Let $A \cap B \leq \tau(M)$ for some submodule B of M , and $b \in B$. As $\tau(M/A) = M/A$, there exists a τ -dense right ideal I of R such that $bI \leq A$. Then, $bI \leq A \cap B \leq \tau(M)$, hence $b + \tau(M) \in \tau(M/\tau(M)) = 0$ and so $b \in \tau(M)$. \square

Proposition 2.2. *Let A be a τ -torsionfree submodule of M and B a submodule of M which is maximal with respect to the property $A \cap B = 0$. Then, $A \oplus B \leq_{p,\tau,e} M$ and $(A \oplus B)/B \leq_{p,\tau,e} M/B$.*

Proof. Assume that C is a submodule of M such that $(A \oplus B) \cap C \leq \tau(M)$. Let $a \in A$; if $a = b + c$, for some $b \in B$ and $c \in C$, then $a - b \in (A \oplus B) \cap C$ and so $(a - b)I = 0$ for some dense right ideal I of R . Thus, $aI = bI \leq A \cap B = 0$, hence $a \in \tau(M)$ and so $a = 0$. This implies that $A \cap (B + C) = 0$, hence by hypothesis $C \leq B$ and so $C \leq \tau(M)$. This shows that $A \oplus B \leq_{p,\tau,e} M$. The pseudo τ -essentiality of $(A \oplus B)/B$

is clear as it is a τ -torsionfree essential submodule of M/B . \square

A hereditary torsion theory is called *stable* if the torsion class is closed under injective envelopes; equivalently, $\tau(M)$ is (essentially) closed, for every module M .

Proposition 2.3. *The following statements are equivalent for a submodule A of M .*

- (1) $A \leq_{p.\tau.e} M$.
- (2) $(A + \tau(M))/\tau(M) \leq_e M/\tau(M)$.
- (3) For all $m \in M \setminus \tau(M)$, there exists $r \in R$ such that $mr \in A \setminus \tau(A)$.

If the hereditary torsion theory τ is stable, then the above statements are equivalent to

- (4) $A + \tau(M) \leq_e M$.
- (5) $A \oplus B \leq_e M$, for some τ -torsion submodule B of M .
- (6) $A \cap B \neq 0$, for every non-zero τ -torsionfree submodule B of M .

Proof. Clearly (1) \Rightarrow (3) and (3) \Rightarrow (2).

(2) \Rightarrow (1). Let B be a submodule of M , for which $A \cap B \leq \tau(M)$. If $a \in A$, $b \in B$ with $a + \tau(M) = b + \tau(M)$ then there exists a τ -dense right ideal I such that $(a - b)I = 0$. Then, $aI = bI \leq A \cap B \leq \tau(M)$, hence $a + \tau(M) \in \tau(M/\tau(M)) = 0$. This implies that $(A + \tau(M))/\tau(M) \cap (B + \tau(M))/\tau(M) = 0$, thus by hypothesis $B \leq \tau(M)$.

Now, assume that the hereditary torsion theory τ is stable.

(1) \Rightarrow (6). Let B be a non-zero τ -torsionfree submodule of M . Then, by (1), $A \cap B$ is non- τ -torsion, hence $A \cap B \neq 0$.

(6) \Rightarrow (5). There exists a submodule B of M such that $A \oplus B \leq_e M$. Also, there exists a submodule B' of B such that $B' \oplus \tau(B) \leq_e B$. Then, (6) implies that $B' = 0$, hence $\tau(B) \leq_e B$. Since $\tau(B)$ is closed in B we conclude that $B = \tau(B)$.

(5) \Rightarrow (4). This is obvious.

(4) \Rightarrow (2). This is clear as $A + \tau(M) \leq_e M$ and $\tau(M)$ is a closed submodule of M . \square

Corollary 2.4. *Every essential submodule of M is pseudo τ -essential in M if and only if $\tau(M)$ is (essentially) closed. Consequently, for every*

module M the set of essential submodules in M is a subset of pseudo τ -essential submodules in M if and only if the hereditary torsion theory τ is stable.

Proof. This follows by Proposition 2.3-(2) and [4, Proposition 1.27-((1) \Leftrightarrow (3))]. \square

Corollary 2.5. *The following statements are equivalent for a module M .*

- (1) *Every submodule of M is pseudo τ -essential.*
- (2) *Every submodule of M is τ -dense.*
- (3) *Every submodule of M is τ -torsion.*
- (4) *M has a τ -torsion pseudo τ -essential submodule.*
- (5) *$\tau(M)$ is pseudo τ -essential in M .*
- (6) *M is τ -torsion.*

Proposition 2.6. (1) *Suppose $A \leq B \leq C$ are modules. Then, $A \leq_{p,\tau,e} C$ if and only if $A \leq_{p,\tau,e} B$ and $B \leq_{p,\tau,e} C$.*

- (2) *Let A_1, A_2, B_1 and B_2 be modules such that $A_1 \leq_{p,\tau,e} B_1$ and $A_2 \leq_{p,\tau,e} B_2$. Then, $A_1 \cap A_2 \leq_{p,\tau,e} B_1 \cap B_2$.*
- (3) *Assume that $f : B \rightarrow C$ is a homomorphism of modules, and $A \leq_{p,\tau,e} C$. Then, $f^{-1}(A) \leq_{p,\tau,e} B$.*
- (4) *Let A_λ be a submodule of B_λ , for all λ in a set Λ . Then, $\bigoplus_\Lambda A_\lambda \leq_{p,\tau,e} \bigoplus_\Lambda B_\lambda$ if and only if $A_\lambda \leq_{p,\tau,e} B_\lambda$, for all $\lambda \in \Lambda$.*

Proof. (1) and (2) follow easily from the definition.

(3). Let $b \in B \setminus \tau(B)$. By Proposition 2.3-(3) it suffices to show that there exists $r \in R$ such that $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. If $f(b) \notin \tau(C)$, as $A \leq_{p,\tau,e} C$, there exists $r \in R$ such that $f(b)r \in A \setminus \tau(A)$. Hence, $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$. Now, assume that $f(b) \in \tau(C)$. There exists a τ -dense right ideal I of R such that $f(b)I = 0$. If $bI \leq \tau(B)$ then $b + \tau(B) \in \tau(B/\tau(B)) = 0$ which is a contradiction. Thus, $bI \not\leq \tau(B)$ and so $bI \not\leq \tau(f^{-1}(A))$. Hence, there exists $x \in I$ such that $bx \in f^{-1}(A) \setminus \tau(f^{-1}(A))$.

(4). The implication (\Rightarrow) follows from Proposition 2.3-(3). For the converse implication (\Leftarrow), by Proposition 2.3-(3) it is enough to check the case of a finite direct sum, and by induction it suffices to check the

case $\Lambda = \{1, 2\}$. The latter follows from (2) and (3) for projections $B_1 \oplus B_2 \rightarrow B_\lambda$ ($\lambda = 1, 2$). \square

Let us call a module M *quasi- τ -torsion* if $\tau(M) \leq_e M$. Clearly every submodule of a quasi- τ -torsion module is quasi- τ -torsion, and every direct sum of quasi- τ -torsion modules is quasi- τ -torsion. Every τ -torsion module is quasi- τ -torsion, however the next result shows that the class of τ -torsion modules coincides with the class of quasi- τ -torsion modules precisely when the hereditary torsion theory τ is stable.

Proposition 2.7. *The class of τ -torsion modules is equal to the class of quasi- τ -torsion modules if and only if the hereditary torsion theory τ is stable.*

Proof. For (\Rightarrow) , it suffices to show that $\tau(M)$ is (essentially) closed in M , for any module M . Then, assume that $\tau(M) \leq_e K \leq M$. Clearly $\tau(K) = \tau(M)$, hence K is quasi- τ -torsion and so it is τ -torsion by hypothesis. Thus, $K = \tau(M)$. The converse implication (\Leftarrow) is clear. \square

Proposition 2.8. *A module M is quasi- τ -torsion if and only if M has a pseudo τ -essential submodule which is quasi- τ -torsion.*

Proof. The implication (\Rightarrow) is clear. For (\Leftarrow) , let M have a pseudo τ -essential submodule A which is quasi- τ -torsion. If $\tau(M)$ is not essential in M then there exists a non-zero submodule K of M such that $\tau(M) \cap K = 0$. Since $A \leq_{p,\tau,e} M$ we conclude that $A \cap K \neq 0$. Thus, $\tau(A) \leq_e A$ implies that $\tau(A) \cap K \neq 0$ which is a contradiction. \square

3. τ -Ranks

Let M be a non-quasi- τ -torsion module. We say that M is *pseudo τ -uniform* if every non-quasi- τ -torsion submodule of M is pseudo τ -essential. Equivalently, a *pseudo τ -uniform module* is a non-quasi- τ -torsion module M such that for every $A, B \leq M$ if $A \cap B$ is τ -torsion then A is quasi- τ -torsion or B is τ -torsion. For a τ -torsionfree module the properties of uniform and pseudo τ -uniform are equivalent. By

Proposition 2.3, if M is a non-quasi- τ -torsion module for which $M/\tau(M)$ is uniform then M is pseudo τ -uniform. Moreover, by Proposition 2.1, if M is a non-quasi- τ -torsion module which is τ -uniform (see [1, Definition 3.18]) then M is pseudo τ -uniform.

Theorem 3.1. *Let $A_1 \oplus \cdots \oplus A_m \oplus K$ and $B_1 \oplus \cdots \oplus B_n \oplus L$ be pseudo τ -essential submodules of a module M such that each A_i and each B_j is pseudo τ -uniform and K and L are quasi- τ -torsion. Then, $m = n$.*

Proof. Clearly a complement to $\tau(A_i)$ in A_i is a non-zero τ -torsionfree submodule and so it is pseudo τ -essential in A_i . Thus, by Proposition 2.6-(4), we can assume that each A_i is τ -torsionfree and uniform. Similarly we can assume that each B_j is τ -torsionfree and uniform. Now, let $m \leq n$ and set $A = A_2 \oplus \cdots \oplus A_m$. If $A \cap B_j \neq 0$, for all j , then $A \cap B_j \leq_{p,\tau,e} B_j$ and so

$$(A \cap B_1) \oplus \cdots \oplus (A \cap B_n) \oplus L \leq_{p,\tau,e} B_1 \oplus \cdots \oplus B_n \oplus L,$$

hence $(A \cap (B_1 \oplus \cdots \oplus B_n)) \oplus L \leq_{p,\tau,e} B_1 \oplus \cdots \oplus B_n \oplus L \leq_{p,\tau,e} M$. Thus, by Proposition 2.6-(1), $A \oplus L \leq_{p,\tau,e} M$; note that $A \cap L$ is a τ -torsionfree submodule of the quasi- τ -torsion module L and so it is zero. However $(A_1 \oplus \cdots \oplus A_m) \cap L = 0$ and so $(A \oplus L) \cap A_1 = 0$ which is impossible since A_1 is non- τ -torsion. Hence, $A \cap B_j = 0$, for some j , say $j = 1$ and set $B = A \oplus B_1$. If $A_1 \cap B = 0$ then $A_1 + A + B_1$ would be a direct sum and so it is τ -torsionfree, hence $(A_1 \oplus A \oplus B_1) \cap K = 0$. Thus, $(A_1 \oplus A \oplus K) \cap B_1 = 0$ which is impossible as $A_1 \oplus A \oplus K \leq_{p,\tau,e} M$. Therefore, $A_1 \cap B \neq 0$ and so

$$(A_1 \cap B) \oplus A_2 \oplus \cdots \oplus A_m \oplus K \leq_{p,\tau,e} A_1 \oplus \cdots \oplus A_m \oplus K \leq_{p,\tau,e} M,$$

hence by Proposition 2.6-(1), $B \oplus K \leq_{p,\tau,e} M$. This shows that we can replace the summand A_1 of $A_1 \oplus \cdots \oplus A_m \oplus K$ by B_1 . By repeating this process we obtain

$$B_1 \oplus B_2 \oplus A_3 \oplus \cdots \oplus A_m \oplus K \leq_{p,\tau,e} M,$$

and after m steps we will arrive at $B_1 \oplus \cdots \oplus B_m \oplus K \leq_{p,\tau,e} M$ which is impossible if $m < n$, since $(B_1 \oplus \cdots \oplus B_m \oplus K) \cap B_{m+1} = 0$ and B_{m+1} is non- τ -torsion. Thus, $m = n$ as desired. \square

Proposition 3.2. *Let $M_1 \oplus \cdots \oplus M_n \oplus N$ be a pseudo τ -essential submodule of a module M such that each M_i is pseudo τ -uniform and N is quasi- τ -torsion. Then, M does not contain any direct sum of $n + 1$ non-quasi- τ -torsion submodules.*

Proof. If $n = 0$ then M is quasi- τ -torsion by Proposition 2.8 and so the conclusion is clear. Now, let $n > 0$ and assume that the statement holds for $n - 1$. Let M contain a direct sum $A_1 \oplus \cdots \oplus A_{n+1}$ of $n + 1$ non-quasi- τ -torsion submodules. As every non-quasi- τ -torsion module has a nonzero τ -torsionfree submodule, we can assume that A_1, \dots, A_{n+1} are non-zero τ -torsionfree. Moreover, $B_i = (M_1 \oplus \cdots \oplus M_n \oplus N) \cap A_i$ is non-quasi- τ -torsion since $M_1 \oplus \cdots \oplus M_n \oplus N$ is pseudo τ -essential, and clearly $B_1 \oplus \cdots \oplus B_{n+1} \leq M_1 \oplus \cdots \oplus M_n \oplus N$. Hence, we may assume that $M = M_1 \oplus \cdots \oplus M_n \oplus N$. Now, set $A = A_1 \oplus \cdots \oplus A_n$. If $A \cap M_1$ is quasi- τ -torsion then $A \cap M_1 = 0$ since A is τ -torsionfree. Then, we can embed A in $M_2 \oplus \cdots \oplus M_n \oplus N$ by using the natural projection $M \rightarrow M_2 \oplus \cdots \oplus M_n \oplus N$. Thus, $M_2 \oplus \cdots \oplus M_n \oplus N$ contains a direct sum of n non-quasi- τ -torsion submodules, contradicting the induction hypothesis. Therefore, $A \cap M_1$ is non-quasi- τ -torsion and similarly so is $A \cap M_i$, for all i . Thus, $A \cap M_i \leq_{p,\tau,e} M_i$ and so

$$(A \cap M_1) \oplus \cdots \oplus (A \cap M_n) \oplus N \leq_{p,\tau,e} M_1 \oplus \cdots \oplus M_n \oplus N \leq_{p,\tau,e} M.$$

Consequently $A \oplus N \leq_{p,\tau,e} M$. However $(A_1 \oplus \cdots \oplus A_{n+1}) \cap N$ is a τ -torsionfree submodule of the quasi- τ -torsion module N , hence it is zero and so $(A \oplus N) \cap A_{n+1} = 0$ which is impossible as A_{n+1} is non- τ -torsion. Hence, M does not contain a direct sum of $n + 1$ non-quasi- τ -torsion submodules. \square

Corollary 3.3. *For any module M , the uniform dimensions of all complements to $\tau(M)$ (in M) are equal.*

Proof. Assume that there exists a complement C to $\tau(M)$ of finite uniform dimension n . Then, C contains an essential submodule $C_1 \oplus \cdots \oplus C_n$ such that each C_i is uniform. By Proposition 2.2, there exists a submodule D such that $C \oplus D \leq_{p,\tau,e} M$, hence by Proposition 2.6-(4), (1), $C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p,\tau,e} M$. If D is non-quasi- τ -torsion then it contains a non-zero τ -torsionfree submodule B . Thus, $(B \oplus C) \cap \tau(M) = 0$ which is impossible, hence D is quasi- τ -torsion. Therefore, by Proposition 3.2,

if a complement to $\tau(M)$ is of finite uniform dimension then every complement to $\tau(M)$ is of finite uniform dimension and by Theorem 3.1, the uniform dimensions of all complements to $\tau(M)$ are equal. \square

As Corollary 3.3 shows, for any module M either all complements to $\tau(M)$ are not of finite uniform dimension or all complements to $\tau(M)$ are of finite uniform dimension n . Let us call this integer n , the τ -rank of M and denote this by $\mathbf{r}_\tau(M)$. Note that $\mathbf{r}_\tau(M) = 0$ if and only if M is quasi- τ -torsion. If a complement (hence, every complement) to $\tau(M)$ is not of finite uniform dimension, we say that M is not of finite τ -rank and write $\mathbf{r}_\tau(M) = \infty$. Let $\text{u.dim}(M)$ denote the uniform dimension of M . Clearly $\text{u.dim}(M) = \mathbf{r}_\tau(M) + \text{u.dim}(\tau(M))$, hence $\mathbf{r}_\tau(M) = \text{u.dim}(M)$ if M is τ -torsionfree and the converse holds if M is of finite uniform dimension.

Proposition 3.4. *The following statements are equivalent for a module M .*

- (1) M has finite τ -rank n .
- (2) M has a pseudo τ -essential submodule which is a finite direct sum of n τ -torsionfree uniform submodules and a quasi- τ -torsion submodule.
- (3) M has a pseudo τ -essential submodule which is a finite direct sum of n pseudo τ -uniform submodules and a quasi- τ -torsion submodule.
- (4) M contains a direct sum of n non-quasi- τ -torsion submodules, but no direct sum of $n + 1$ non-quasi- τ -torsion submodules.
- (5) M contains a direct sum of n non-zero τ -torsionfree submodules, but no direct sum of $n + 1$ non-zero τ -torsionfree submodules.

Proof. (1) \Rightarrow (2). Assume that C is a complement to $\tau(M)$. As the proof of Corollary 3.3 shows there exist some τ -torsionfree uniform submodules C_1, \dots, C_n of C and a quasi- τ -torsion submodule D of M such that $C_1 \oplus \dots \oplus C_n \oplus D \leq_{p.\tau.e} M$.

(2) \Rightarrow (3). This implication is clear.

(3) \Rightarrow (4). This follows by Proposition 3.2.

(4) \Rightarrow (5). This implication is clear as every non-quasi- τ -torsion submodule has a non-zero τ -torsionfree submodule and every non-zero

τ -torsionfree submodule is non-quasi- τ -torsion.

(5) \Rightarrow (1). By hypothesis there exists a direct sum of n non-zero τ -torsionfree submodules $K_1 \oplus \cdots \oplus K_n$. This direct sum can be enlarged into a complement C of $\tau(M)$. Then, C contains a direct sum of n non-zero τ -torsionfree submodules, but no direct sum of $n + 1$ non-zero τ -torsionfree submodules and so $\text{u.dim}(C) = n$. \square

Corollary 3.5. *The following statements are equivalent for a module M .*

- (1) M is of finite τ -rank.
- (2) M contains no infinite direct sum of non-quasi- τ -torsion submodules.
- (3) M contains no infinite direct sum of non-zero τ -torsionfree submodules.

Proof. The implication (1) \Rightarrow (2) follows by Proposition 3.4, and (2) \Rightarrow (3) is clear.

(3) \Rightarrow (1). Let C be a complement to $\tau(M)$. By hypothesis C contains no infinite direct sum of non-zero submodules, hence C is of finite uniform dimension. \square

Corollary 3.6. *For any module M ,*

$$\mathbf{r}_\tau(M) = \sup\{k : M \text{ contains a direct sum of } k \text{ non-quasi-}\tau\text{-torsion submodules}\} = \sup\{k : M \text{ contains a direct sum of } k \text{ non-zero } \tau\text{-torsionfree submodules}\}.$$

Corollary 3.7. $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(M/N) \leq \text{u.dim}(M/N)$, for every τ -torsion submodule N of M . In particular, $\mathbf{r}_\tau(M) \leq \text{u.dim}(M/\tau(M))$. Moreover, $\mathbf{r}_\tau(M) = \text{u.dim}(M/\tau(M))$ for every module M , if and only if the hereditary torsion theory τ is stable.

Proof. Assume that M contains a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ of non-zero τ -torsionfree submodules. If N is a τ -torsion submodule of M then M/N has a direct sum of non-zero τ -torsionfree submodules $(A_1 + N)/N \oplus (A_2 + N)/N \oplus \cdots \oplus (A_k + N)/N$. Thus, $\mathbf{r}_\tau(M) \leq$

$\mathbf{r}_\tau(M/N)$ by Corollary 3.6. If τ is stable and $A_1/\tau(M) \oplus \cdots \oplus A_k/\tau(M)$ is a direct sum of non-zero submodules of $M/\tau(M)$, then $B_1 \oplus \cdots \oplus B_k$ is a direct sum of non-zero τ -torsionfree submodules of M , where B_i is a complement to $\tau(M)$ in A_i . Thus, $\text{u.dim}(M/\tau(M)) \leq \mathbf{r}_\tau(M)$ and so $\mathbf{r}_\tau(M) = \text{u.dim}(M/\tau(M))$. Now, let $\mathbf{r}_\tau(M) = \text{u.dim}(M/\tau(M))$, for every module M . Then, $\text{u.dim}(M/\tau(M)) = 0$, for every quasi- τ -torsion module M . Hence, every quasi- τ -torsion module is τ -torsion and so the hereditary torsion theory is stable by Proposition 2.7. \square

Corollary 3.8. *Let A be a submodule of M .*

- (1) $\mathbf{r}_\tau(A) \leq \mathbf{r}_\tau(M)$.
- (2) $\mathbf{r}_\tau(A) = \mathbf{r}_\tau(M)$ if and only if $\mathbf{r}_\tau(A) = \infty$ or if $\mathbf{r}_\tau(A) = k < \infty$ then every complement in M of each direct sum of k non-zero τ -torsionfree submodules of A is quasi- τ -torsion.
- (3) $\mathbf{r}_\tau(A) = \mathbf{r}_\tau(M)$ if $\mathbf{r}_\tau(A) = \infty$ or $A \leq_{p,\tau,e} M$. The converse holds if the hereditary torsion theory τ is stable.

Proof. Clearly (1) follows by Corollary 3.6.

(2). (\Leftarrow). By Corollary 3.5, if $\mathbf{r}_\tau(A) = \infty$ then $\mathbf{r}_\tau(M) = \infty$. Now, assume that $\mathbf{r}_\tau(A) = k < \infty$. By Proposition 3.4, A contains a direct sum of k non-zero τ -torsionfree submodules $A_1 \oplus \cdots \oplus A_k$ and so by Proposition 2.2, $A_1 \oplus \cdots \oplus A_k \oplus B \leq_{p,\tau,e} M$ for a submodule B of M which is maximal with respect to the property $(A_1 \oplus \cdots \oplus A_k) \cap B = 0$. By hypothesis B is quasi- τ -torsion, hence $\mathbf{r}_\tau(M) = k$.

(\Rightarrow). Let $\mathbf{r}_\tau(A) = k < \infty$ and $A_1 \oplus \cdots \oplus A_k$ be a direct sum of k non-zero τ -torsionfree submodules of A . If a complement B in M of $A_1 \oplus \cdots \oplus A_k$ is non-quasi- τ -torsion, then there exists a non-zero τ -torsionfree submodule C of B and so M contains the direct sum $A_1 \oplus \cdots \oplus A_k \oplus C$ of $k + 1$ non-zero τ -torsionfree submodules which is impossible as $\mathbf{r}_\tau(M) = k$.

(3). The first statement is clear by Proposition 3.4 and Corollary 3.5. Now, assume that τ is stable and $\mathbf{r}_\tau(A) = k < \infty$, moreover A is not pseudo τ -essential in M . Then, A contains a direct sum $A_1 \oplus \cdots \oplus A_k$ of k non-zero τ -torsionfree submodules. Since A is not pseudo τ -essential in M , there exists a non- τ -torsion submodule B such that $A \oplus B \leq_e M$ by Proposition 2.3-(5). Thus, M contains the direct sum $A_1 \oplus \cdots \oplus A_k \oplus B$ of non- τ -torsion submodules. Hence, $\mathbf{r}_\tau(M) \geq k + 1$ as the notions of

non- τ -torsion and non-quasi- τ -torsion are the same whenever a hereditary torsion theory is stable. \square

Note that by Corollaries 3.7 and 3.8, $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A)$ if A is a τ -torsion submodule or a pseudo τ -essential submodule of M . The next corollary shows that the inequality holds for some other submodules of M . Recall that a submodule A of M is called τ -pure (or τ -closed) if M/A is τ -torsionfree.

Corollary 3.9. *Let A be a τ -torsionfree submodule or a τ -pure submodule of M . Then,*

$$\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A).$$

Proof. Let A be a τ -torsionfree submodule. There exists a submodule B such that $A \oplus B \leq_{p.\tau.e} M$. Then, $\mathbf{r}_\tau(M) = \mathbf{r}_\tau(A \oplus B) = \mathbf{r}_\tau(A) + \mathbf{r}_\tau(B)$. But, $B \cong (A \oplus B)/A \leq M/A$, hence $\mathbf{r}_\tau(B) \leq \mathbf{r}_\tau(M/A)$. Now, assume that A is τ -pure, moreover C is a complement to $\tau(M)$ in M . Clearly $C \cap A$ can be enlarged to a complement D to $\tau(A)$ in A . Then,

$$\begin{aligned} \text{u.dim}(C) &\leq \text{u.dim}(C \cap A) + \text{u.dim}(C/(C \cap A)) \\ &\leq \text{u.dim}(D) + \text{u.dim}(M/A). \end{aligned}$$

Since A is τ -pure, $\mathbf{r}_\tau(M/A) = \text{u.dim}(M/A)$ and so $\mathbf{r}_\tau(M) \leq \mathbf{r}_\tau(A) + \mathbf{r}_\tau(M/A)$. \square

A module M is called τ -injective if for any τ -dense submodule A of B , any homomorphism $A \rightarrow M$ extends to a homomorphism $B \rightarrow M$. If $E_\tau(M)$ is a τ -injective τ -essential extension of M , then $E_\tau(M)$ is the smallest τ -injective module containing M . Moreover, it is unique up to isomorphism. $E_\tau(M)$ is called the τ -injective hull of M . More properties of the τ -injective hull of a module can be found in [2, § 3]. Note that by Proposition 2.1, $M \leq_{p.\tau.e} E_\tau(M)$. Proposition 3.11 below interprets the finiteness of the τ -rank of M via a certain decomposition length of $E_\tau(M)$. The following lemma is helpful.

Lemma 3.10. *A module M is pseudo τ -uniform if and only if $E_\tau(M)$ is pseudo τ -uniform.*

Proof. Clearly if M is non-quasi- τ -torsion then $E_\tau(M)$ is non-quasi- τ -torsion and the converse holds by Proposition 2.8. For (\Rightarrow) , assume that $A \cap B \leq \tau(E_\tau(M))$. Then, $(A \cap M) \cap (B \cap M) \leq \tau(M)$, hence by hypothesis $A \cap M$ is quasi- τ -torsion or $B \cap M$ is τ -torsion. However $M \leq_{p.\tau.e} E_\tau(M)$ and so $A \cap M \leq_{p.\tau.e} A$, therefore A is quasi- τ -torsion by Proposition 2.8 or B is τ -torsion. The converse implication (\Leftarrow) is clear. \square

Proposition 3.11. $\mathbf{r}_\tau(M) = n < \infty$ if and only if $E_\tau(M)$ is a direct sum of n pseudo τ -uniform modules and a quasi- τ -torsion module.

Proof. (\Rightarrow) . By hypothesis M contains pseudo τ -uniform submodules A_1, \dots, A_n and a quasi- τ -torsion submodule B such that

$$A_1 \oplus \cdots \oplus A_n \oplus B \leq_{p.\tau.e} M.$$

There exists a submodule C of M for which $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_e M$. Then, C is τ -torsion and by Proposition 2.6-(1),

$$A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_{p.\tau.e} M.$$

Thus, $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C$ is essential and pseudo τ -essential in $E_\tau(M)$. Thus,

$$\begin{aligned} E_\tau(M) &= E_\tau(A_1 \oplus \cdots \oplus A_n \oplus B \oplus C) \\ &= E_\tau(A_1) \oplus \cdots \oplus E_\tau(A_n) \oplus E_\tau(B) \oplus E_\tau(C), \end{aligned}$$

where, each $E_\tau(A_i)$ is pseudo τ -uniform by Lemma 3.10 and $E_\tau(B)$ and $E_\tau(C)$ are quasi- τ -torsion by Proposition 2.8.

(\Leftarrow) . By Proposition 3.4, $\mathbf{r}_\tau(E_\tau(M)) = n$ and so by Corollary 3.8-(3), $\mathbf{r}_\tau(M) = \mathbf{r}_\tau(E_\tau(M)) = n$. \square

Corollary 3.12. $\mathbf{r}_\tau(\bigoplus_{i=1}^k M_i) = \sum_{i=1}^k \mathbf{r}_\tau(M_i)$.

4. Complements and τ -ranks

Recall that for a module M , if $\text{u.dim}(M) = n < \infty$, then any chain of complements has length $\leq n$. In addition, $\text{u.dim}(M) = \infty$ if and only if there exists an infinite strictly ascending chain of complements

in M if and only if there exists an infinite strictly descending chain of complements (See [3, Propositions (6.29) and (6.30)]). In this section we obtain similar relations for τ -rank of a module M in terms of certain complement submodules.

Proposition 4.1. *Let M be a module and $\mathbf{r}_\tau(M) = n < \infty$. Then, in M any chain of complements to τ -torsionfree submodules has length $\leq n$.*

Proof. Let $C_0 < C_1 < \cdots < C_k$, where each C_{i-1} is a complement to some τ -torsionfree submodule T_i of M . Then, each C_{i-1} is a complement to the τ -torsionfree submodule $T_i \cap C_i$ of C_i . Set $S_i = T_i \cap C_i$, for all $i = 1, \dots, k$. Since $C_{i-1} \neq C_i$, we have $S_i \neq 0$. Then, $S_1 \oplus \cdots \oplus S_k$ is a direct sum of k non-zero τ -torsionfree submodules of M , hence $k \leq n$ by Corollary 3.6. \square

Theorem 4.2. *The following statements are equivalent for a module M .*

- (1) $\mathbf{r}_\tau(M) = \infty$.
- (2) *There exists an infinite strictly ascending chain of complements to τ -torsionfree submodules in M .*
- (3) *There exists an infinite strictly descending chain of complements to τ -torsionfree submodules in M .*

Proof. (1) \Rightarrow (2). By Corollary 3.5, M contains an infinite direct sum $T_1 \oplus T_2 \oplus \cdots$, where T_i is a non-zero τ -torsionfree submodule. Enlarge T_1 into a complement to $T_2 \oplus T_3 \oplus \cdots$, say C_1 . Then, enlarge $C_1 \oplus T_2$ into a complement to $T_3 \oplus T_4 \oplus \cdots$, say C_2 . In this way, we get an ascending chain $C_1 \leq C_2 \leq \cdots$, where each C_i is a complement to a τ -torsionfree submodule in M . Since $T_i \leq C_i$ and $T_i \cap C_{i-1} = 0$, we have $C_{i-1} \neq C_i$, for all i .

(2) \Rightarrow (3). Assume that $C_0 < C_1 < \cdots$, where each C_i is a complement to a τ -torsionfree submodule in M . If C_k is τ -torsion then $C_k = \tau(M)$ and so only C_0 can be τ -torsion. Moreover, similar to the proof of Proposition 4.1, C_{i-1} is a complement to some non-zero τ -torsionfree submodule S_i in C_i . Enlarge $S_2 \oplus S_3 \oplus \cdots$ into a complement to S_1 , let L_1 be this complement. Then, enlarge $S_3 \oplus S_4 \oplus \cdots$

into a complement to S_2 in L_1 , say L_2 . Clearly L_2 is a complement to the non-zero τ -torsionfree submodule $S_1 \oplus S_2$ in M . Moreover, $L_2 < L_1$ since $S_2 \leq L_1$ and $L_2 \cap S_2 = 0$. By this process we get a strictly descending chain of complements to τ -torsionfree submodules in M , i.e., $L_1 > L_2 > \dots$.

(3) \Rightarrow (1) is clear by Proposition 4.1. □

Recall that $\text{u.dim}(M) = \sup\{k : M \text{ contains a chain of complements of length } k\}$. A similar result holds for the τ -rank of M .

Corollary 4.3. *For any module M ,*

$\mathbf{r}_\tau(M) = \sup\{k : M \text{ contains a chain of length } k \text{ of complements to } \tau\text{-torsionfree submodules}\}$.

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