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# RANKS OF MODULES RELATIVE TO A TORSION THEORY

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ABSTRACT. Relative to a hereditary torsion theory  $\tau$  we introduce a dimension for a module M, called  $\tau$ -rank of M, which coincides with the reduced rank of M whenever  $\tau$  is the Goldie torsion theory. It is shown that the  $\tau$ -rank of M is measured by the length of certain decompositions of the  $\tau$ -injective hull of M. Moreover, some relations between the  $\tau$ -rank of M and complements to  $\tau$ -torsionfree submodules of M are obtained.

# 1. Introduction

Throughout the paper, rings will have unit elements and modules will be unitary right modules. The category of all right *R*-modules is denoted by Mod-*R*, and the notation  $\leq_e$  will denote an essential submodule. In the paper  $\tau = (\mathcal{T}, \mathcal{F})$  will denote a fixed hereditary torsion theory on Mod-*R*. Then,  $\tau(M) = \sum \{N : N \leq M, N \in \mathcal{T}\}$  is the  $\tau$ -torsion submodule of  $M \in \text{Mod-}R$ . The module *M* is called  $\tau$ -torsion, if  $M \in \mathcal{T}$ , and  $\tau$ -torsionfree, if  $M \in \mathcal{F}$ . In fact, *M* is  $\tau$ -torsion, if  $\tau(M) = M$ , and  $\tau$ -torsionfree, if  $\tau(M) = 0$ . A submodule *A* of *M* is called  $\tau$ -dense, if M/A is  $\tau$ -torsion and we denote this by  $A \leq_{\tau-d} M$ . It is clear that

$$\tau(M) = \{ m \in M : \operatorname{ann}(m) \leq_{\tau-d} R_R \}.$$

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More information on torsion-theoretic concepts can be found in [5]. In this paper we prove that whenever the uniform dimension of a complement to  $\tau(M)$  is a finite number n then the uniform dimension of every complement to  $\tau(M)$  is n. We call this integer n, the  $\tau$ -rank of M, and if no such integer exists we say that M is not of finite  $\tau$ -rank. We shall prove that the hereditary torsion  $\tau$  is stable, if and only if the  $\tau$ -rank of M coincides with the uniform dimension of  $M/\tau(M)$  for every  $M \in$ Mod-R.

In Goldie's theory of uniform dimension, within a module M, one seeks for a submodule  $A = A_1 \oplus \cdots \oplus A_l$  with the largest possible l such that each  $A_i$  is non-zero, and if such an A exists then A is essential in M. In fact, M is measured by the largest possible direct sum of nonzero submodules which it can contain. In section 2, we introduce the notion of *pseudo*  $\tau$ -essential submodule which has a role in the subject of  $\tau$ -rank similar to that of an essential submodule in Goldie's theory of uniform dimension. We say that a submodule A of M is pseudo  $\tau$ essential, if for every submodule B of M,  $A \cap B \leq \tau(M)$  implies that  $B \leq \tau(M)$ . This concept is a generalization of the notion of  $\tau$ -essential submodule. A submodule A of M is called  $\tau$ -essential, if A is  $\tau$ -dense and essential in M. Such submodules appear in many concepts such as  $(s)\tau$ -CS modules and the  $\tau$ -injective hulls of modules which are torsiontheoretic analogues of CS modules and the injective hulls of modules; see [1] and [2]. Some properties of  $\tau$ -essential submodules can be found in [1, Proposition 3.1], and in Propositions 2.3 and 2.6 we show that most of these properties hold more generally for pseudo  $\tau$ -essential submodules.

In section 3 we deal with the theory of  $\tau$ -rank. Then, we will prove that to find the  $\tau$ -rank of a module M, one should look for a submodule  $A = A_1 \oplus \cdots \oplus A_l \oplus B$  with the largest possible l such that each  $A_i$  is non-zero  $\tau$ -torsionfree and B is quasi- $\tau$ -torsion (that is  $\tau(B) \leq_e B$ ), and if such an A exists then A is pseudo  $\tau$ -essential in M. As the uniform dimension of a module M relates to the decomposition length of the injective hull of M, we show that the  $\tau$ -rank of M is measured by the length of certain decompositions of the  $\tau$ -injective hull of M. Indeed, in Proposition 3.11 we show that the  $\tau$ -rank of M is a finite number n, if and only if the  $\tau$ -injective hull of M is a direct sum of n pseudo  $\tau$ -uniform modules and a quasi  $\tau$ -torsion module. Finally section 4 is devoted to some relations between  $\tau$ -ranks and certain complements analogous to the well known relations between uniform dimensions and complements. In fact, the  $\tau$ -rank of M is the supremum of the set of nonnegative integers k for which M contains a chain of length k of complements to  $\tau$ -torsionfree submodules of M.

#### 2. Pseudo $\tau$ -essential submodules

In this section we introduce the notion of a pseudo  $\tau$ -essential submodule and give some properties of such submodules for later use. We say that a submodule A of a module M is pseudo  $\tau$ -essential in M and write  $A \leq_{p.\tau.e} M$ , if for every submodule B of M,  $A \cap B \leq \tau(M)$  implies that  $B \leq \tau(M)$ . Clearly every  $\tau$ -torsionfree essential submodule of Mis pseudo  $\tau$ -essential. Moreover, if A is a submodule of a  $\tau$ -torsionfree module M, then A is pseudo  $\tau$ -essential in M if and only if A is essential in M.

The first result shows that the notion of pseudo  $\tau$ -essential is a generalization of the notion of  $\tau$ -dense. In particular, every  $\tau$ -essential submodule is pseudo  $\tau$ -essential.

**Proposition 2.1.** Every  $\tau$ -dense submodule of M is pseudo  $\tau$ -essential in M.

**Proof.** Assume that A is a  $\tau$ -dense submodule of M. Let  $A \cap B \leq \tau(M)$  for some submodule B of M, and  $b \in B$ . As  $\tau(M/A) = M/A$ , there exists a  $\tau$ -dense right ideal I of R such that  $bI \leq A$ . Then,  $bI \leq A \cap B \leq \tau(M)$ , hence  $b + \tau(M) \in \tau(M/\tau(M)) = 0$  and so  $b \in \tau(M)$ .

**Proposition 2.2.** Let A be a  $\tau$ -torsionfree submodule of M and B a submodule of M which is maximal with respect to the property  $A \cap B = 0$ . Then,  $A \oplus B \leq_{p.\tau.e} M$  and  $(A \oplus B)/B \leq_{p.\tau.e} M/B$ .

**Proof.** Assume that C is a submodule of M such that  $(A \oplus B) \cap C \leq \tau(M)$ . Let  $a \in A$ ; if a = b + c, for some  $b \in B$  and  $c \in C$ , then  $a-b \in (A \oplus B) \cap C$  and so (a-b)I = 0 for some dense right ideal I of R. Thus,  $aI = bI \leq A \cap B = 0$ , hence  $a \in \tau(M)$  and so a = 0. This implies that  $A \cap (B + C) = 0$ , hence by hypothesis  $C \leq B$  and so  $C \leq \tau(M)$ . This shows that  $A \oplus B \leq_{p.\tau,e} M$ . The pseudo  $\tau$ -essentiality of  $(A \oplus B)/B$ 

is clear as it is a  $\tau$ -torsionfree essential submodule of M/B.

A hereditary torsion theory is called *stable* if the torsion class is closed under injective envelopes; equivalently,  $\tau(M)$  is (essentially) closed, for every module M.

**Proposition 2.3.** The following statements are equivalent for a submodule A of M.

(1)  $A \leq_{p.\tau.e} M$ .

(2)  $(A + \tau(M))/\tau(M) \leq_e M/\tau(M).$ 

(3) For all  $m \in M \setminus \tau(M)$ , there exists  $r \in R$  such that  $mr \in A \setminus \tau(A)$ .

If the hereditary torsion theory  $\tau$  is stable, then the above statements are equivalent to

- (4)  $A + \tau(M) \leq_e M$ .
- (5)  $A \oplus B \leq_e M$ , for some  $\tau$ -torsion submodule B of M.
- (6)  $A \cap B \neq 0$ , for every non-zero  $\tau$ -torsionfree submodule B of M.

**Proof.** Clearly  $(1) \Rightarrow (3)$  and  $(3) \Rightarrow (2)$ .

 $(2) \Rightarrow (1)$ . Let *B* be a submodule of *M*, for which  $A \cap B \leq \tau(M)$ . If  $a \in A$ ,  $b \in B$  with  $a + \tau(M) = b + \tau(M)$  then there exists a  $\tau$ -dense right ideal *I* such that (a - b)I = 0. Then,  $aI = bI \leq A \cap B \leq \tau(M)$ , hence  $a + \tau(M) \in \tau(M/\tau(M)) = 0$ . This implies that  $(A + \tau(M))/\tau(M) \cap (B + \tau(M))/\tau(M) = 0$ , thus by hypothesis  $B \leq \tau(M)$ .

Now, assume that the hereditary torsion theory  $\tau$  is stable.

 $(1) \Rightarrow (6)$ . Let B be a non-zero  $\tau$ -torsionfree submodule of M. Then, by  $(1), A \cap B$  is non- $\tau$ -torsion, hence  $A \cap B \neq 0$ .

 $(6) \Rightarrow (5)$ . There exists a submodule B of M such that  $A \oplus B \leq_e M$ . Also, there exists a submodule B' of B such that  $B' \oplus \tau(B) \leq_e B$ . Then, (6) implies that B' = 0, hence  $\tau(B) \leq_e B$ . Since  $\tau(B)$  is closed in B we conclude that  $B = \tau(B)$ .

 $(5) \Rightarrow (4)$ . This is obvious.

(4)  $\Rightarrow$  (2). This is clear as  $A + \tau(M) \leq_e M$  and  $\tau(M)$  is a closed submodule of M.

**Corollary 2.4.** Every essential submodule of M is pseudo  $\tau$ -essential in M if and only if  $\tau(M)$  is (essentially) closed. Consequently, for every

module M the set of essential submodules in M is a subset of pseudo  $\tau$ -essential submodules in M if and only if the hereditary torsion theory  $\tau$  is stable.

**Proof.** This follows by Proposition 2.3-(2) and [4, Proposition 1.27- $((1) \Leftrightarrow (3))$ ].

**Corollary 2.5.** The following statements are equivalent for a module M.

- (1) Every submodule of M is pseudo  $\tau$ -essential.
- (2) Every submodule of M is  $\tau$ -dense.
- (3) Every submodule of M is  $\tau$ -torsion.
- (4) M has a  $\tau$ -torsion pseudo  $\tau$ -essential submodule.
- (5)  $\tau(M)$  is pseudo  $\tau$ -essential in M.
- (6) M is  $\tau$ -torsion.

**Proposition 2.6.** (1) Suppose  $A \leq B \leq C$  are modules. Then,  $A \leq_{p.\tau.e} C$  if and only if  $A \leq_{p.\tau.e} B$  and  $B \leq_{p.\tau.e} C$ .

- (2) Let  $A_1$ ,  $A_2$ ,  $B_1$  and  $B_2$  be modules such that  $A_1 \leq_{p.\tau.e} B_1$  and  $A_2 \leq_{p.\tau.e} B_2$ . Then,  $A_1 \cap A_2 \leq_{p.\tau.e} B_1 \cap B_2$ .
- (3) Assume that  $f : B \to C$  is a homomorphism of modules, and  $A \leq_{p.\tau.e} C$ . Then,  $f^{-1}(A) \leq_{p.\tau.e} B$ .
- (4) Let  $A_{\lambda}$  be a submodule of  $B_{\lambda}$ , for all  $\lambda$  in a set  $\Lambda$ . Then,  $\bigoplus_{\Lambda} A_{\lambda} \leq_{p.\tau.e} \bigoplus_{\Lambda} B_{\lambda}$  if and only if  $A_{\lambda} \leq_{p.\tau.e} B_{\lambda}$ , for all  $\lambda \in \Lambda$ .

**Proof.** (1) and (2) follow easily from the definition.

(3). Let  $b \in B \setminus \tau(B)$ . By Proposition 2.3-(3) it suffices to show that there exists  $r \in R$  such that  $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$ . If  $f(b) \notin \tau(C)$ , as  $A \leq_{p,\tau,e} C$ , there exists  $r \in R$  such that  $f(b)r \in A \setminus \tau(A)$ . Hence,  $br \in f^{-1}(A) \setminus \tau(f^{-1}(A))$ . Now, assume that  $f(b) \in \tau(C)$ . There exists a  $\tau$ -dense right ideal I of R such that f(b)I = 0. If  $bI \leq \tau(B)$ then  $b + \tau(B) \in \tau(B/\tau(B)) = 0$  which is a contradiction. Thus,  $bI \not\leq \tau(B)$  and so  $bI \not\leq \tau(f^{-1}(A))$ . Hence, there exists  $x \in I$  such that  $bx \in f^{-1}(A) \setminus \tau(f^{-1}(A))$ .

(4). The implication  $(\Rightarrow)$  follows from Proposition 2.3-(3). For the converse implication  $(\Leftarrow)$ , by Proposition 2.3-(3) it is enough to check the case of a finite direct sum, and by induction it suffices to check the

case  $\Lambda = \{1, 2\}$ . The latter follows from (2) and (3) for projections  $B_1 \oplus B_2 \to B_\lambda \ (\lambda = 1, 2)$ .

Let us call a module M quasi- $\tau$ -torsion if  $\tau(M) \leq_e M$ . Clearly every submodule of a quasi- $\tau$ -torsion module is quasi- $\tau$ -torsion, and every direct sum of quasi- $\tau$ -torsion modules is quasi- $\tau$ -torsion. Every  $\tau$ -torsion module is quasi- $\tau$ -torsion, however the next result shows that the class of  $\tau$ -torsion modules coincides with the class of quasi- $\tau$ -torsion modules precisely when the hereditary torsion theory  $\tau$  is stable.

**Proposition 2.7.** The class of  $\tau$ -torsion modules is equal to the class of quasi- $\tau$ -torsion modules if and only if the hereditary torsion theory  $\tau$  is stable.

**Proof.** For  $(\Rightarrow)$ , it suffices to show that  $\tau(M)$  is (essentially) closed in M, for any module M. Then, assume that  $\tau(M) \leq_e K \leq M$ . Clearly  $\tau(K) = \tau(M)$ , hence K is quasi- $\tau$ -torsion and so it is  $\tau$ -torsion by hypothesis. Thus,  $K = \tau(M)$ . The converse implication ( $\Leftarrow$ ) is clear.  $\Box$ 

**Proposition 2.8.** A module M is quasi- $\tau$ -torsion if and only if M has a pseudo  $\tau$ -essential submodule which is quasi- $\tau$ -torsion.

**Proof.** The implication  $(\Rightarrow)$  is clear. For  $(\Leftarrow)$ , let M have a pseudo  $\tau$ -essential submodule A which is quasi- $\tau$ -torsion. If  $\tau(M)$  is not essential in M then there exists a non-zero submodule K of M such that  $\tau(M) \cap K = 0$ . Since  $A \leq_{p.\tau.e} M$  we conclude that  $A \cap K \neq 0$ . Thus,  $\tau(A) \leq_e A$  implies that  $\tau(A) \cap K \neq 0$  which is a contradiction.  $\Box$ 

## 3. $\tau$ -Ranks

Let M be a non-quasi- $\tau$ -torsion module. We say that M is pseudo  $\tau$ -uniform if every non-quasi- $\tau$ -torsion submodule of M is pseudo  $\tau$ essential. Equivalently, a pseudo  $\tau$ -uniform module is a non-quasi- $\tau$ torsion module M such that for every  $A, B \leq M$  if  $A \cap B$  is  $\tau$ -torsion then A is quasi- $\tau$ -torsion or B is  $\tau$ -torsion. For a  $\tau$ -torsionfree module the properties of uniform and pseudo  $\tau$ -uniform are equivalent. By Proposition 2.3, if M is a non-quasi- $\tau$ -torsion module for which  $M/\tau(M)$  is uniform then M is pseudo  $\tau$ -uniform. Moreover, by Proposition 2.1, if M is a non-quasi- $\tau$ -torsion module which is  $\tau$ -uniform (see [1, Definition 3.18]) then M is pseudo  $\tau$ -uniform.

**Theorem 3.1.** Let  $A_1 \oplus \cdots \oplus A_m \oplus K$  and  $B_1 \oplus \cdots \oplus B_n \oplus L$  be pseudo  $\tau$ -essential submodules of a module M such that each  $A_i$  and each  $B_j$  is pseudo  $\tau$ -uniform and K and L are quasi- $\tau$ -torsion. Then, m = n.

**Proof.** Clearly a complement to  $\tau(A_i)$  in  $A_i$  is a non-zero  $\tau$ -torsionfree submodule and so it is pseudo  $\tau$ -essential in  $A_i$ . Thus, by Proposition 2.6-(4), we can assume that each  $A_i$  is  $\tau$ -torsionfree and uniform. Similarly we can assume that each  $B_j$  is  $\tau$ -torsionfree and uniform. Now, let  $m \leq n$  and set  $A = A_2 \oplus \cdots \oplus A_m$ . If  $A \cap B_j \neq 0$ , for all j, then  $A \cap B_j \leq_{p,\tau,e} B_j$  and so

$$(A \cap B_1) \oplus \cdots \oplus (A \cap B_n) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L_2$$

hence  $(A \cap (B_1 \oplus \cdots \oplus B_n)) \oplus L \leq_{p.\tau.e} B_1 \oplus \cdots \oplus B_n \oplus L \leq_{p.\tau.e} M$ . Thus, by Proposition 2.6-(1),  $A \oplus L \leq_{p.\tau.e} M$ ; note that  $A \cap L$  is a  $\tau$ -torsionfree submodule of the quasi- $\tau$ -torsion module L and so it is zero. However  $(A_1 \oplus \cdots \oplus A_m) \cap L = 0$  and so  $(A \oplus L) \cap A_1 = 0$  which is impossible since  $A_1$  is non- $\tau$ -torsion. Hence,  $A \cap B_j = 0$ , for some j, say j = 1 and set  $B = A \oplus B_1$ . If  $A_1 \cap B = 0$  then  $A_1 + A + B_1$  would be a direct sum and so it is  $\tau$ -torsionfree, hence  $(A_1 \oplus A \oplus B_1) \cap K = 0$ . Thus,  $(A_1 \oplus A \oplus K) \cap B_1 = 0$  which is impossible as  $A_1 \oplus A \oplus K \leq_{p.\tau.e} M$ . Therefore,  $A_1 \cap B \neq 0$  and so

$$(A_1 \cap B) \oplus A_2 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} A_1 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

hence by Proposition 2.6-(1),  $B \oplus K \leq_{p.\tau.e} M$ . This shows that we can replace the summand  $A_1$  of  $A_1 \oplus \cdots \oplus A_m \oplus K$  by  $B_1$ . By repeating this process we obtain

$$B_1 \oplus B_2 \oplus A_3 \oplus \cdots \oplus A_m \oplus K \leq_{p.\tau.e} M,$$

and after m steps we will arrive at  $B_1 \oplus \cdots \oplus B_m \oplus K \leq_{p.\tau.e} M$  which is impossible if m < n, since  $(B_1 \oplus \cdots \oplus B_m \oplus K) \cap B_{m+1} = 0$  and  $B_{m+1}$ is non- $\tau$ -torsion. Thus, m = n as desired. **Proposition 3.2.** Let  $M_1 \oplus \cdots \oplus M_n \oplus N$  be a pseudo  $\tau$ -essential submodule of a module M such that each  $M_i$  is pseudo  $\tau$ -uniform and Nis quasi- $\tau$ -torsion. Then, M does not contain any direct sum of n + 1non-quasi- $\tau$ -torsion submodules.

**Proof.** If n = 0 then M is quasi- $\tau$ -torsion by Proposition 2.8 and so the conclusion is clear. Now, let n > 0 and assume that the statement holds for n-1. Let M contain a direct sum  $A_1 \oplus \cdots \oplus A_{n+1}$  of n+1 nonquasi- $\tau$ -torsion submodules. As every non-quasi- $\tau$ -torsion module has a nonzero  $\tau$ -torsionfree submodule, we can assume that  $A_1, \ldots, A_{n+1}$  are non-zero  $\tau$ -torsionfree. Moreover,  $B_i = (M_1 \oplus \cdots \oplus M_n \oplus N) \cap A_i$  is non-quasi- $\tau$ -torsion since  $M_1 \oplus \cdots \oplus M_n \oplus N$  is pseudo  $\tau$ -essential, and clearly  $B_1 \oplus \cdots \oplus B_{n+1} \leq M_1 \oplus \cdots \oplus M_n \oplus N$ . Hence, we may assume that  $M = M_1 \oplus \cdots \oplus M_n \oplus N$ . Now, set  $A = A_1 \oplus \cdots \oplus A_n$ . If  $A \cap M_1$ is quasi- $\tau$ -torsion then  $A \cap M_1 = 0$  since A is  $\tau$ -torsionfree. Then, we can embed A in  $M_2 \oplus \cdots \oplus M_n \oplus N$  by using the natural projection  $M \to M_2 \oplus \cdots \oplus M_n \oplus N$ . Thus,  $M_2 \oplus \cdots \oplus M_n \oplus N$  contains a direct sum of n non-quasi- $\tau$ -torsion submodules, contradicting the induction hypothesis. Therefore,  $A \cap M_1$  is non-quasi- $\tau$ -torsion and similarly so is  $A \cap M_i$ , for all *i*. Thus,  $A \cap M_i \leq_{p.\tau.e} M_i$  and so 

$$(A \cap M_1) \oplus \cdots \oplus (A \cap M_n) \oplus N \leq_{p.\tau,e} M_1 \oplus \cdots \oplus M_n \oplus N \leq_{p.\tau,e} M.$$

Consequently  $A \oplus N \leq_{p,\tau,e} M$ . However  $(A_1 \oplus \cdots \oplus A_{n+1}) \cap N$  is a  $\tau$ -torsionfree submodule of the quasi- $\tau$ -torsion module N, hence it is zero and so  $(A \oplus N) \cap A_{n+1} = 0$  which is impossible as  $A_{n+1}$  is non- $\tau$ -torsion. Hence, M does not contain a direct sum of n + 1 non-quasi- $\tau$ -torsion submodules.

**Corollary 3.3.** For any module M, the uniform dimensions of all complements to  $\tau(M)$  (in M) are equal.

**Proof.** Assume that there exists a complement C to  $\tau(M)$  of finite uniform dimension n. Then, C contains an essential submodule  $C_1 \oplus \cdots \oplus C_n$  such that each  $C_i$  is uniform. By Proposition 2.2, there exists a submodule D such that  $C \oplus D \leq_{p.\tau.e} M$ , hence by Proposition 2.6-(4), (1),  $C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p.\tau.e} M$ . If D is non-quasi- $\tau$ -torsion then it contains a non-zero  $\tau$ -torsionfree submodule B. Thus,  $(B \oplus C) \cap \tau(M) = 0$  which is impossible, hence D is quasi- $\tau$ -torsion. Therefore, by Proposition 3.2,

if a complement to  $\tau(M)$  is of finite uniform dimension then every complement to  $\tau(M)$  is of finite uniform dimension and by Theorem 3.1, the uniform dimensions of all complements to  $\tau(M)$  are equal.

As Corollary 3.3 shows, for any module M either all complements to  $\tau(M)$  are not of finite uniform dimension or all complements to  $\tau(M)$  are of finite uniform dimension n. Let us call this integer n, the  $\tau$ -rank of M and denote this by  $\mathbf{r}_{\tau}(M)$ . Note that  $\mathbf{r}_{\tau}(M) = 0$  if and only if M is quasi- $\tau$ -torsion. If a complement (hence, every complement) to  $\tau(M)$  is not of finite uniform dimension, we say that M is not of finite  $\tau$ -rank and write  $\mathbf{r}_{\tau}(M) = \infty$ . Let u.dim(M) denote the uniform dimension of M. Clearly u.dim $(M) = \mathbf{r}_{\tau}(M) + u.dim(\tau(M))$ , hence  $\mathbf{r}_{\tau}(M) = u.dim(M)$  if M is  $\tau$ -torsionfree and the converse holds if M is of finite uniform dimension.

**Proposition 3.4.** The following statements are equivalent for a module M.

- (1) M has finite  $\tau$ -rank n.
- (2) *M* has a pseudo  $\tau$ -essential submodule which is a finite direct sum of  $n \tau$ -torsionfree uniform submodules and a quasi- $\tau$ -torsion submodule.
- (3) M has a pseudo τ-essential submodule which is a finite direct sum of n pseudo τ-uniform submodules and a quasi-τ-torsion submodule.
- (4) *M* contains a direct sum of *n* non-quasi- $\tau$ -torsion submodules, but no direct sum of n + 1 non-quasi- $\tau$ -torsion submodules.
- (5) *M* contains a direct sum of *n* non-zero  $\tau$ -torsionfree submodules, but no direct sum of n + 1 non-zero  $\tau$ -torsionfree submodules.

**Proof.** (1)  $\Rightarrow$  (2). Assume that *C* is a complement to  $\tau(M)$ . As the proof of Corollary 3.3 shows there exist some  $\tau$ -torsionfree uniform submodules  $C_1, \ldots, C_n$  of *C* and a quasi- $\tau$ -torsion submodule *D* of *M* such that  $C_1 \oplus \cdots \oplus C_n \oplus D \leq_{p,\tau,e} M$ .

 $(2) \Rightarrow (3)$ . This implication is clear.

 $(3) \Rightarrow (4)$ . This follows by Proposition 3.2.

(4)  $\Rightarrow$  (5). This implication is clear as every non-quasi- $\tau$ -torsion submodule has a non-zero  $\tau$ -torsionfree submodule and every non-zero

 $\tau$ -torsionfree submodule is non-quasi- $\tau$ -torsion.

 $(5) \Rightarrow (1)$ . By hypothesis there exists a direct sum of n non-zero  $\tau$ torsionfree submodules  $K_1 \oplus \cdots \oplus K_n$ . This direct sum can be enlarged
into a complement C of  $\tau(M)$ . Then, C contains a direct sum of nnon-zero  $\tau$ -torsionfree submodules, but no direct sum of n + 1 non-zero  $\tau$ -torsionfree submodules and so u.dim(C) = n.

**Corollary 3.5.** The following statements are equivalent for a module M.

- (1) M is of finite  $\tau$ -rank.
- (2) M contains no infinite direct sum of non-quasi-τ-torsion submodules.
- (3) M contains no infinite direct sum of non-zero τ-torsionfree submodules.

**Proof.** The implication  $(1) \Rightarrow (2)$  follows by Proposition 3.4, and  $(2) \Rightarrow (3)$  is clear.

 $(3) \Rightarrow (1)$ . Let C be a complement to  $\tau(M)$ . By hypothesis C contains no infinite direct sum of non-zero submodules, hence C is of finite uniform dimension.

#### Corollary 3.6. For any module M,

 $\mathbf{r}_{\tau}(M) = \sup\{k : M \text{ contains a direct sum of } k \text{ non-quasi-}\tau\text{-torsion } submodules}\} = \sup\{k : M \text{ contains a direct sum of } k \text{ non-zero } \tau\text{-torsionfree submodules}\}.$ 

**Corollary 3.7.**  $\mathbf{r}_{\tau}(M) \leq \mathbf{r}_{\tau}(M/N) \leq \mathrm{u.dim}(M/N)$ , for every  $\tau$ -torsion submodule N of M. In particular,  $\mathbf{r}_{\tau}(M) \leq \mathrm{u.dim}(M/\tau(M))$ . Moreover,  $\mathbf{r}_{\tau}(M) = \mathrm{u.dim}(M/\tau(M))$  for every module M, if and only if the hereditary torsion theory  $\tau$  is stable.

**Proof.** Assume that M contains a direct sum  $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ of non-zero  $\tau$ -torsionfree submodules. If N is a  $\tau$ -torsion submodule of M then M/N has a direct sum of non-zero  $\tau$ -torsionfree submodules  $(A_1 + N)/N \oplus (A_2 + N)/N \oplus \cdots \oplus (A_k + N)/N$ . Thus,  $\mathbf{r}_{\tau}(M) \leq$ 

 $\mathbf{r}_{\tau}(M/N)$  by Corollary 3.6. If  $\tau$  is stable and  $A_1/\tau(M) \oplus \cdots \oplus A_k/\tau(M)$ is a direct sum of non-zero submodules of  $M/\tau(M)$ , then  $B_1 \oplus \cdots \oplus B_k$ is a direct sum of non-zero  $\tau$ -torsionfree submodules of M, where  $B_i$ is a complement to  $\tau(M)$  in  $A_i$ . Thus,  $\operatorname{u.dim}(M/\tau(M)) \leq \mathbf{r}_{\tau}(M)$  and so  $\mathbf{r}_{\tau}(M) = \operatorname{u.dim}(M/\tau(M))$ . Now, let  $\mathbf{r}_{\tau}(M) = \operatorname{u.dim}(M/\tau(M))$ , for every module M. Then,  $\operatorname{u.dim}(M/\tau(M)) = 0$ , for every quasi- $\tau$ -torsion module M. Hence, every quasi- $\tau$ -torsion module is  $\tau$ -torsion and so the hereditary torsion theory is stable by Proposition 2.7.

Corollary 3.8. Let A be a submodule of M.

- (1)  $\mathbf{r}_{\tau}(A) \leq \mathbf{r}_{\tau}(M)$ .
- (2)  $\mathbf{r}_{\tau}(A) = \mathbf{r}_{\tau}(M)$  if and only if  $\mathbf{r}_{\tau}(A) = \infty$  or if  $\mathbf{r}_{\tau}(A) = k < \infty$ then every complement in M of each direct sum of k non-zero  $\tau$ -torsionfree submodules of A is quasi- $\tau$ -torsion.
- (3)  $\mathbf{r}_{\tau}(A) = \mathbf{r}_{\tau}(M)$  if  $\mathbf{r}_{\tau}(A) = \infty$  or  $A \leq_{p.\tau.e} M$ . The converse holds if the hereditary torsion theory  $\tau$  is stable.

**Proof.** Clearly (1) follows by Corollary 3.6.

(2). ( $\Leftarrow$ ). By Corollary 3.5, if  $\mathbf{r}_{\tau}(A) = \infty$  then  $\mathbf{r}_{\tau}(M) = \infty$ . Now, assume that  $\mathbf{r}_{\tau}(A) = k < \infty$ . By Proposition 3.4, A contains a direct sum of k non-zero  $\tau$ -torsionfree submodules  $A_1 \oplus \cdots \oplus A_k$  and so by Proposition 2.2,  $A_1 \oplus \cdots \oplus A_k \oplus B \leq_{p.\tau.e} M$  for a submodule B of Mwhich is maximal with respect to the property  $(A_1 \oplus \cdots \oplus A_k) \cap B = 0$ . By hypothesis B is quasi- $\tau$ -torsion, hence  $\mathbf{r}_{\tau}(M) = k$ .

 $(\Rightarrow)$ . Let  $\mathbf{r}_{\tau}(A) = k < \infty$  and  $A_1 \oplus \cdots \oplus A_k$  be a direct sum of k non-zero  $\tau$ -torsionfree submodules of A. If a complement B in M of  $A_1 \oplus \cdots \oplus A_k$  is non-quasi- $\tau$ -torsion, then there exists a non-zero  $\tau$ -torsionfree submodule C of B and so M contains the direct sum  $A_1 \oplus \cdots \oplus A_k \oplus C$  of k + 1 non-zero  $\tau$ -torsionfree submodules which is impossible as  $\mathbf{r}_{\tau}(M) = k$ .

(3). The first statement is clear by Proposition 3.4 and Corollary 3.5. Now, assume that  $\tau$  is stable and  $\mathbf{r}_{\tau}(A) = k < \infty$ , moreover A is not pseudo  $\tau$ -essential in M. Then, A contains a direct sum  $A_1 \oplus \cdots \oplus A_k$  of knon-zero  $\tau$ -torsionfree submodules. Since A is not pseudo  $\tau$ -essential in M, there exists a non- $\tau$ -torsion submodule B such that  $A \oplus B \leq_e M$  by Proposition 2.3-(5). Thus, M contains the direct sum  $A_1 \oplus \cdots \oplus A_k \oplus B$ of non- $\tau$ -torsion submodules. Hence,  $\mathbf{r}_{\tau}(M) \geq k + 1$  as the notions of non- $\tau$ -torsion and non-quasi- $\tau$ -torsion are the same whenever a hereditary torsion theory is stable.

Note that by Corollaries 3.7 and 3.8,  $\mathbf{r}_{\tau}(M) \leq \mathbf{r}_{\tau}(A) + \mathbf{r}_{\tau}(M/A)$  if A is a  $\tau$ -torsion submodule or a pseudo  $\tau$ -essential submodule of M. The next corollary shows that the inequality holds for some other submodules of M. Recall that a submodule A of M is called  $\tau$ -pure (or  $\tau$ -closed) if M/A is  $\tau$ -torsionfree.

**Corollary 3.9.** Let A be a  $\tau$ -torsionfree submodule or a  $\tau$ -pure submodule of M. Then,

$$\mathbf{r}_{\tau}(M) \leq \mathbf{r}_{\tau}(A) + \mathbf{r}_{\tau}(M/A).$$

**Proof.** Let A be a  $\tau$ -torsionfree submodule. There exists a submodule B such that  $A \oplus B \leq_{p.\tau.e} M$ . Then,  $\mathbf{r}_{\tau}(M) = \mathbf{r}_{\tau}(A \oplus B) = \mathbf{r}_{\tau}(A) + \mathbf{r}_{\tau}(B)$ . But,  $B \cong (A \oplus B)/A \leq M/A$ , hence  $\mathbf{r}_{\tau}(B) \leq \mathbf{r}_{\tau}(M/A)$ . Now, assume that A is  $\tau$ -pure, moreover C is a complement to  $\tau(M)$  in M. Clearly  $C \cap A$  can be enlarged to a complement D to  $\tau(A)$  in A. Then,

$$\begin{aligned} \text{u.dim}(C) &\leq \quad \text{u.dim}(C \cap A) + \text{u.dim}(C/(C \cap A)) \\ &\leq \quad \text{u.dim}(D) + \text{u.dim}(M/A). \end{aligned}$$

Since A is  $\tau$ -pure,  $\mathbf{r}_{\tau}(M/A) = \text{u.dim}(M/A)$  and so  $\mathbf{r}_{\tau}(M) \leq \mathbf{r}_{\tau}(A) + \mathbf{r}_{\tau}(M/A)$ .

A module M is called  $\tau$ -injective if for any  $\tau$ -dense submodule A of B, any homomorphism  $A \to M$  extends to a homomorphism  $B \to M$ . If  $E_{\tau}(M)$  is a  $\tau$ -injective  $\tau$ -essential extension of M, then  $E_{\tau}(M)$  is the smallest  $\tau$ -injective module containing M. Moreover, it is unique up to isomorphism.  $E_{\tau}(M)$  is called the  $\tau$ -injective hull of M. More properties of the  $\tau$ -injective hull of a module can be found in [2, § 3]. Note that by Proposition 2.1,  $M \leq_{p.\tau.e} E_{\tau}(M)$ . Proposition 3.11 below interprets the finiteness of the  $\tau$ -rank of M via a certain decomposition length of  $E_{\tau}(M)$ . The following lemma is helpful.

**Lemma 3.10.** A module M is pseudo  $\tau$ -uniform if and only if  $E_{\tau}(M)$  is pseudo  $\tau$ -uniform.

**Proof.** Clearly if M is non-quasi- $\tau$ -torsion then  $E_{\tau}(M)$  is non-quasi- $\tau$ -torsion and the converse holds by Proposition 2.8. For  $(\Rightarrow)$ , assume that  $A \cap B \leq \tau(E_{\tau}(M))$ . Then,  $(A \cap M) \cap (B \cap M) \leq \tau(M)$ , hence by hypothesis  $A \cap M$  is quasi- $\tau$ -torsion or  $B \cap M$  is  $\tau$ -torsion. However  $M \leq_{p.\tau.e} E_{\tau}(M)$  and so  $A \cap M \leq_{p.\tau.e} A$ , therefore A is quasi- $\tau$ -torsion by Proposition 2.8 or B is  $\tau$ -torsion. The converse implication ( $\Leftarrow$ ) is clear.  $\Box$ 

**Proposition 3.11.**  $\mathbf{r}_{\tau}(M) = n < \infty$  if and only if  $E_{\tau}(M)$  is a direct sum of n pseudo  $\tau$ -uniform modules and a quasi- $\tau$ -torsion module.

**Proof.** ( $\Rightarrow$ ). By hypothesis M contains pseudo  $\tau$ -uniform submodules  $A_1, \ldots, A_n$  and a quasi- $\tau$ -torsion submodule B such that

$$A_1 \oplus \cdots \oplus A_n \oplus B \leq_{p.\tau.e} M.$$

There exists a submodule C of M for which  $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_e M$ . Then, C is  $\tau$ -torsion and by Proposition 2.6-(1),

$$A_1 \oplus \cdots \oplus A_n \oplus B \oplus C \leq_{p.\tau.e} M.$$

Thus,  $A_1 \oplus \cdots \oplus A_n \oplus B \oplus C$  is essential and pseudo  $\tau$ -essential in  $E_{\tau}(M)$ . Thus,

$$E_{\tau}(M) = E_{\tau}(A_1 \oplus \cdots \oplus A_n \oplus B \oplus C)$$
  
=  $E_{\tau}(A_1) \oplus \cdots \oplus E_{\tau}(A_n) \oplus E_{\tau}(B) \oplus E_{\tau}(C),$ 

where, each  $E_{\tau}(A_i)$  is pseudo  $\tau$ -uniform by Lemma 3.10 and  $E_{\tau}(B)$  and  $E_{\tau}(C)$  are quasi- $\tau$ -torsion by Proposition 2.8.

(⇐). By Proposition 3.4,  $\mathbf{r}_{\tau}(E_{\tau}(M)) = n$  and so by Corollary 3.8-(3),  $\mathbf{r}_{\tau}(M) = \mathbf{r}_{\tau}(E_{\tau}(M)) = n$ .

Corollary 3.12.  $\mathbf{r}_{\tau}(\bigoplus_{i=1}^{k} M_i) = \sum_{i=1}^{k} \mathbf{r}_{\tau}(M_i).$ 

#### 4. Complements and $\tau$ -ranks

Recall that for a module M, if  $u.\dim(M) = n < \infty$ , then any chain of complements has length  $\leq n$ . In addition,  $u.\dim(M) = \infty$  if and only if there exists an infinite strictly ascending chain of complements in M if and only if there exists an infinite strictly descending chain of complements (See [3, Propositions (6.29) and (6.30)]). In this section we obtain similar relations for  $\tau$ -rank of a module M in terms of certain complement submodules.

**Proposition 4.1.** Let M be a module and  $\mathbf{r}_{\tau}(M) = n < \infty$ . Then, in M any chain of complements to  $\tau$ -torsionfree submodules has length  $\leq n$ .

**Proof.** Let  $C_0 < C_1 < \cdots < C_k$ , where each  $C_{i-1}$  is a complement to some  $\tau$ -torsionfree submodule  $T_i$  of M. Then, each  $C_{i-1}$  is a complement to the  $\tau$ -torsionfree submodule  $T_i \cap C_i$  of  $C_i$ . Set  $S_i = T_i \cap C_i$ , for all  $i = 1, \ldots, k$ . Since  $C_{i-1} \neq C_i$ , we have  $S_i \neq 0$ . Then,  $S_1 \oplus \cdots \oplus S_k$  is a direct sum of k non-zero  $\tau$ -torsionfree submodules of M, hence  $k \leq n$  by Corollary 3.6.

**Theorem 4.2.** The following statements are equivalent for a module M.

- (1)  $\mathbf{r}_{\tau}(M) = \infty$ .
- (2) There exists an infinite strictly ascending chain of complements to  $\tau$ -torsionfree submodules in M.
- (3) There exists an infinite strictly descending chain of complements to  $\tau$ -torsionfree submodules in M.

**Proof.** (1)  $\Rightarrow$  (2). By Corollary 3.5, M contains an infinite direct sum  $T_1 \oplus T_2 \oplus \cdots$ , where  $T_i$  is a non-zero  $\tau$ -torsionfree submodule. Enlarge  $T_1$  into a complement to  $T_2 \oplus T_3 \oplus \cdots$ , say  $C_1$ . Then, enlarge  $C_1 \oplus T_2$  into a complement to  $T_3 \oplus T_4 \oplus \cdots$ , say  $C_2$ . In this way, we get an ascending chain  $C_1 \leq C_2 \leq \cdots$ , where each  $C_i$  is a complement to a  $\tau$ -torsionfree submodule in M. Since  $T_i \leq C_i$  and  $T_i \cap C_{i-1} = 0$ , we have  $C_{i-1} \neq C_i$ , for all i.

 $(2) \Rightarrow (3)$ . Assume that  $C_0 < C_1 < \cdots$ , where each  $C_i$  is a complement to a  $\tau$ -torsionfree submodule in M. If  $C_k$  is  $\tau$ -torsion then  $C_k = \tau(M)$  and so only  $C_0$  can be  $\tau$ -torsion. Moreover, similar to the proof of Proposition 4.1,  $C_{i-1}$  is a complement to some non-zero  $\tau$ -torsionfree submodule  $S_i$  in  $C_i$ . Enlarge  $S_2 \oplus S_3 \oplus \cdots$  into a complement to  $S_1$ , let  $L_1$  be this complement. Then, enlarge  $S_3 \oplus S_4 \oplus \cdots$ 

into a complement to  $S_2$  in  $L_1$ , say  $L_2$ . Clearly  $L_2$  is a complement to the non-zero  $\tau$ -torsionfree submodule  $S_1 \oplus S_2$  in M. Moreover,  $L_2 < L_1$ since  $S_2 \leq L_1$  and  $L_2 \cap S_2 = 0$ . By this process we get a strictly descending chain of complements to  $\tau$ -torsionfree submodules in M, i.e.,  $L_1 > L_2 > \cdots$ .

 $(3) \Rightarrow (1)$  is clear by Proposition 4.1.

Recall that u.dim  $(M) = \sup\{k : M \text{ contains a chain of complements} of length k\}$ . A similar result holds for the  $\tau$ -rank of M.

### Corollary 4.3. For any module M,

 $\mathbf{r}_{\tau}(M) = \sup\{k : M \text{ contains a chain of length } k \text{ of complements to } \tau \text{-torsionfree submodules}\}.$ 

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