CLASSICAL QUASI-PRIMARY SUBMODULES

M. BEHBOODI*, R. JAHANI-NEZHAD AND M. H. NADERI

Communicated by Omid Ali Karamzadeh

ABSTRACT. In this paper we introduce the notion of classical quasi-primary submodules that generalizes the concept of classical primary submodules. Then, we investigate decomposition and minimal decomposition into classical quasi-primary submodules. In particular, existence and uniqueness of classical quasi-primary decompositions in finitely generated modules over Noetherian rings are proved. Moreover, we show that this decomposition and the decomposition into classical primary submodules are the same when R is a domain with $\dim(R) \leq 1$.

1. Introduction

Throughout the paper all rings are commutative with an identity, and all modules are unital. Let M be an R-module. If N is a submodule (resp., proper submodule) of M, we write $N \leq M$ (resp., $N \lneq M$). For every nonempty subset X of M and every submodule N of M, the ideal $\{r \in R \mid rX \subseteq N\}$ will be denoted by $(N : X)$. When $X = \{m\}$, where $m \in M$, we use $(N : m)$ instead of $(N : X)$. Note that $(N : M)$ is the annihilator of the module M/N. Also we denote the classical Krull dimension of R by $\dim(R)$, and for an ideal I of R, $\sqrt{I} := \{r \in R \mid r^k \in I$ for some $k \in \mathbb{N}\}$.

51

We recall that a proper ideal $\mathcal{Q}$ of R is called a *primary* ideal if $ab \in \mathcal{Q}$, where $a, b \in R$, implies that either $a \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see [2,15]). The decomposition of a nonzero ideal as (shortest) intersection of a finite number of primary ideals, in a commutative Noetherian ring, was established by Noether [14]. The notion of primary ideal was generalized by Fuchs [10] through defining an ideal $\mathcal{Q}$ of a ring R to be it quasi-primary if its radical is a prime ideal, i.e., if $ab \in \mathcal{Q}$, where $a, b \in R$, then either $a^k \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see also [11]). There are some extensions of these notions to modules. We recall that a proper submodule Q of M is called a *primary submodule*, if $am \in Q$, where $a \in R$, $m \in M$, then $m \in Q$ or $a^k M \subseteq Q$ for some $k \in \mathbb{N}$ (see for example [15]). Moreover, Q is called *quasi-primary* if $\sqrt{(Q : M)}$ is a prime ideal of R (see [1]).

We define a *classical primary* submodule in M as a proper submodule Q of M such that if $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$, then either $aN \subseteq Q$ or $b^k N \subseteq Q$ for some $k \in \mathbb{N}$. Clearly, in case $M = R$, where R is any commutative ring, classical primary submodules coincide with primary ideals (see Proposition 2.1). The idea of decomposition of submodules into classical primary submodules were introduced by Baziar and Behboodi in [3]. Their definition of classical primary submodule was slightly different than ours; they defined a *classical primary* submodule in M as a proper submodule Q of M such that if $abm \in Q$, where $a, b \in R$ and $m \in M$, then either $am \in Q$ or $b^k m \in Q$ for some $k \in \mathbb{N}$. One can easily see that these two definitions coincide when M is a Noetherian module (see Proposition 2.6); but these are different in general (see Example 2.2 (e). Also, we define a *classical quasi-primary* submodule in M as a proper submodule Q of M such that if $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$, then either $a^k N \subseteq Q$ or $b^k N \subseteq Q$ for some $k \in \mathbb{N}$. Clearly, every classical quasi-primary submodule is quasi-primary, but in general, even in the case $M = R$, the converse need not be true (see Proposition 2.1). In [3], among other results, the existence and uniqueness of classical primary decompositions in finitely generated modules over domains R with $\dim(R) \leq 1$ are proved.

In this article, we continue the study of this construction via classical quasi-primary submodules. In Section 2, we study some properties of classical primary submodules and classical quasi-primary submodules. We prove that in modules over a domain R with $\dim(R) \leq 1$, classical primary submodules coincide with classical quasi-primary submodules.

We call an R-module M *primary compatible* (resp., *quasi-primary compatible*) if its primary and its classical primary (resp., quasi-primary and its classical quasi-primary) submodules are the same. A ring R is said to be primary compatible (resp., quasi-primary compatible) if every R-module is primary compatible (resp., quasi-primary compatible). Primary compatible rings are characterized in Theorem 2.14 (see also [3, Theorem 1.7). It is also shown that if $\dim(R) = 0$, then R is a quasi-primary compatible ring, and if R is quasi-primary compatible, then R is a Gelfand ring (i.e., every prime ideal of R is contained in a unique maximal ideal of R). Moreover, if also R is Noetherian, then for each minimal prime ideal $\mathcal{P}$ of R, the factor ring $R/\mathcal{P}$ has at most one nonzero prime ideal (consequently, $\dim(R) \leq 1$).

In Section 3, we investigate decompositions of submodules into intersections of classical quasi-primary submodules. In particular, the existence and uniqueness of minimal classical quasi-primary decompositions in finitely generated modules over Noetherian rings are proved (see Proposition 3.8 and Theorem 3.9).

2. Classical primary and classical quasi-primary submodules

Let R be a ring and $\mathcal{Q}$ be an ideal of R. We note that $\mathcal{Q}$ is a primary (resp., quasi-primary, classical primary, classical quasi-primary) ideal of R if and only if it is a primary (resp., quasi-primary, classical primary, classical quasi-primary) submodule of $_RR$.

It is well-known that in a Dedekind domain, the two concepts primary and quasi-primary coincide; and are equal to powers of prime ideals (see [11, p. 412]). In general, the above four concepts primary, classical primary, quasi-primary, and classical quasi-primary ideals are different in a ring R, but the following proposition more or less summarizes the overall situation.

Proposition 2.1. *Consider the following statements for a proper ideal $\mathcal{Q}$ of a ring R:*

(1) $\mathcal{Q}$ *is a primary ideal.*
(2) $\mathcal{Q}$ *is a classical primary ideal.*
(3) $(\mathcal{Q} : I)$ *is a primary ideal, for each ideal I of R such that $I \nsubseteq \mathcal{Q}$.*
(4) $\mathcal{Q}$ *is a classical quasi-primary ideal.*
(5) $\sqrt{(\mathcal{Q} : I)}$ *is a prime ideal, for each ideal I of R such that $I \nsubseteq \mathcal{Q}$.*

(6) $\mathcal{Q}$ *is a quasi-primary ideal (i.e.,* $\sqrt{\mathcal{Q}} = \sqrt{(\mathcal{Q} : R)}$ *is a prime ideal).*

(7) $\mathcal{Q}$ *is a power of a prime ideal.*

Then, $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Leftarrow (7)$*. Moreover,*

(a) *if* $dim(R) = 0$*, then* $(6) \Rightarrow (1)$*;*

(b) *if* R *is a Dedekind domain, then* $(6) \Rightarrow (7) \Rightarrow (1)$*;*

(c) *if* R *is a domain with* $dim(R) \leq 1$*, then* $(5) \Rightarrow (1)$*.*

Proof. $(1) \Rightarrow (2)$. Suppose $\mathcal{Q}$ is a primary ideal. Let $abI \subseteq \mathcal{Q}$, where a, $b \in R$ and I is an ideal of R such that $bI \nsubseteq \mathcal{Q}$. Then, there exists $x \in bI$ such that $x \notin \mathcal{Q}$. Since $\mathcal{Q}$ is primary ideal and $ax \in \mathcal{Q}$, we conclude that $a^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$. It follows that $a^k I \subseteq \mathcal{Q}$. Thus, $\mathcal{Q}$ is a classical primary ideal.

$(2) \Rightarrow (3)$ is evident.

$(3) \Rightarrow (1)$. Take $I = R$ and so by (3), $\mathcal{Q} = (\mathcal{Q} : R)$ is a primary ideal.

$(3) \Rightarrow (5)$ is evident.

$(4) \Rightarrow (5)$. Let I be an ideal of R such that $I \nsubseteq \mathcal{Q}$, and let $ab \in \sqrt{(\mathcal{Q} : I)}$, where a, $b \in R$. Then, $(ab)^k I \subseteq \mathcal{Q}$ for some $k \in \mathbb{N}$. Since $\mathcal{Q}$ is a classical quasi-primary ideal, there exists $t \in \mathbb{N}$ such that either $a^{tk} I \subseteq \mathcal{Q}$ or $b^{tk} I \subseteq \mathcal{Q}$, i.e., either $a \in \sqrt{(\mathcal{Q} : I)}$ or $b \in \sqrt{(\mathcal{Q} : I)}$. Thus, $\sqrt{(\mathcal{Q} : I)}$ is a prime ideal.

$(5) \Rightarrow (4)$. Assume that $abI \subseteq \mathcal{Q}$, where a, $b \in R$ and I is an ideal of R. Then, $ab \in (\mathcal{Q} : I) \subseteq \sqrt{(\mathcal{Q} : I)}$. Since by (5), $\sqrt{(\mathcal{Q} : I)}$ is either R or a prime ideal of R, depending on whether $I \subseteq \mathcal{Q}$ or not, we conclude that either $a \in \sqrt{(\mathcal{Q} : I)}$ or $b \in \sqrt{(\mathcal{Q} : I)}$, i.e., $a^k I \subseteq \mathcal{Q}$ or $b^k I \subseteq \mathcal{Q}$ for some $k \in \mathbb{N}$. Thus, $\mathcal{Q}$ is a classical quasi-primary ideal.

$(5) \Rightarrow (6)$ and $(7) \Rightarrow (6)$ are evident.

For Part (a), assume that $\dim(R) = 0$ and $\mathcal{Q}$ is a quasi-primary ideal. Thus, $\sqrt{\mathcal{Q}}$ is a maximal ideal and so by [15, Proposition 4.9], $\mathcal{Q}$ is a primary ideal.

For Part (b), we note that in a Dedekind domain R, the two concepts primary and quasi-primary coincide; and are equal to powers of prime ideals of R (see [11, p. 412]). Thus, $(6) \Rightarrow (7) \Rightarrow (1)$ when R is a Dedekind domain.

For Part (c), assume that R is a domain with $\dim(R) \leq 1$ and (5) holds. Take $I = R$ and then by (5), $\sqrt{\mathcal{Q}} = \sqrt{(\mathcal{Q} : R)}$ is a prime ideal. Since R is a domain and $\dim(R) \leq 1$, either $\sqrt{\mathcal{Q}} = (0)$ or $\sqrt{\mathcal{Q}}$ is a maximal ideal. If $\sqrt{\mathcal{Q}} = (0)$, then $\mathcal{Q} = (0)$; therefore, $\mathcal{Q}$ is a prime ideal

(so it is primary). If $\sqrt{\mathcal{Q}}$ is a maximal ideal, then by [15, Proposition 4.9], $\mathcal{Q}$ is a primary ideal. $\square$

Clearly, every (classical) primary ideal of a ring R is quasi-primary but the converse need not be true in general (in fact, [15, Example 4.12] shows that an ideal of a ring which has prime radical need not necessarily be primary). Also every primary submodule of an R-module M is classical (quasi) primary, but in general, the converse need not be true (see Example 2.2 (a) and (b) below). On the other hand, every classical quasi-primary submodule is quasi-primary, but in general, the converse need not be true (see Example 2.2 (c) below). Example 2.2 (d) below gives a submodule Q of a Noetherian R-module M such that Q is classical quasi primary which is not primary (Note; the main result of this paper (Theorem 3.9) is about Noetherian modules). In particular, Example 2.2 (e) below shows that the notion of classical primary submodule of this paper is different from that in [3].

Example 2.2.
(a) Assume that R is a domain and P is a nonzero prime ideal in R. Let $F = \oplus_{\lambda \in \Lambda} R$ be a free R-module, and let $N = \oplus_{\lambda \in \Lambda} A_\lambda$ be a proper submodule of F such that for every $\lambda \in \Lambda$, either $A_\lambda = P$ or $A_\lambda = (0)$. Then, N is a classical primary submodule. But, one can easily check that if there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $A_{\lambda_1} = P$ and $A_{\lambda_2} = (0)$, then N is not a primary submodule of M (see also [3, Example 1.2]).
(b) If p is a prime integer and $\mathbb{Z}(p^\infty) = \{\frac{a}{p^k} + \mathbb{Z} \mid a, k$ are integers and k is positive$\}$, then $(0) \lneqq \mathbb{Z}(p^\infty)$ is a classical primary $\mathbb{Z}$-submodule but it is not a primary submodule. In fact, we conclude that every nonzero proper submodule of $\mathbb{Z}(p^\infty)$ is classical primary but it is not primary.
(c) Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Then, each proper submodule N of M is a quasi-primary submodule since $\sqrt{(N : M)} = (0)$. Now, if $N := \mathbb{Z} + \mathbb{Z}\frac{1}{5}$, the submodule of M generated by $\{1, \frac{1}{5}\}$, then $2 \times 3 < \frac{1}{2 \times 3} > \subseteq N$, but for each $k \geq 1$, $2^k < \frac{1}{2 \times 3} > \nsubseteq N$ and $3^k < \frac{1}{2 \times 3} > \nsubseteq N$. Thus, N is not a classical quasi-primary submodule of M.
(d) Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $Q = p\mathbb{Z} \oplus (0)$, for some prime number p. Then, Q is a classical quasi-primary submodule of

the Noetherian R-module M but it is not a primary submodule of M.

(e) Let $R := \mathbb{Z}_2[x, y]$, $M = \mathbb{Z}_2[x, y, z_1, z_2, \ldots]$ and

$$Q = < \{xyz_i : i \in \mathbb{N}\}) \cup \{x^i z_i : i \in \mathbb{N}\} \cup \{y^i z_i : i \in \mathbb{N}\} >$$

as an ideal of the ring M. Clearly M is an R-module and $Q \lneqq M$. We claim that Q is not a classical primary R-submodule of M as the notion of this paper, but that is a classical primary R-submodule of M as [3]. To see this let $N = < \{z_i : i \in \mathbb{N}\} >$ as an ideal of M. Then, $N \le M$ is an R-submodule with $xyN \subseteq Q$. Clearly $x^k z_{2k} \notin Q$ and $y^k z_{2k} \notin Q$ for each $k \ge 1$. It follows that $xN \not\subseteq Q$ and $y^k N \not\subseteq Q$ for each $k \ge 1$. Thus, $Q \lneqq M$ is not a classical primary submodule as the notion of this paper. Now, we assume that $fgh \in Q$, where $f, g \in R \setminus \{0\}$ and $h \in M \setminus Q$. Without loss of generality, we can assume that $h \in \mathbb{Z}_2[x, y, z_1, \ldots, z_n]$, for some $n \ge 1$. Moreover, we can assume that $h = h_1 + h_2$, where $h_1 \in R$, $h_2 \in L$, where L is the ideal $< \{z_i : 1 \le i \le n\} >$ of the ring $\mathbb{Z}_2[x, y, z_1, z_2, \ldots, z_n]$. Clearly, $y^k h_2 \in Q$ for some $k \ge 1$. It follows that $y^k fgh_1 \in Q \cap R$. Since $Q \cap R = (0)$, $h_1 = 0$ and so $h = h_2 \in L$. If $f = 1 + xf_1 + yf_2$ and $g = 1 + xg_1 + yg_2$, where $f_1, f_2, g_1, g_2 \in R$, then $(1 + xf_1 + yf_2)(1 + xg_1 + yg_2)h = (1 + xf_3 + yf_4)h \in Q$, where $f_3, f_4 \in R$. If $xf_3 \in Q$ and $yf_4 \in Q$, then $h \in Q$, a contradiction. Thus, without loss of generality we can assume $x^t h \in Q$ but $x^{t-1}h \notin Q$ for some $t \ge 2$ (since $x^k h \in Q$ for some $k \ge 1$). Thus, $x^{t-1}(1 + xf_3 + yf_4)h = x^{t-1}h + x^t f_3 h + x^{t-1}yf_4 h \in Q$. It follows that $x^{t-1}h \in Q$, a contradiction. Thus, either $f = xf_1 + yf_2$ or $g = xg_1 + yg_2$, where $f_1, f_2, g_1, g_2 \in R$. If $g^k h \notin Q$ for each $k \ge 1$, then $g = 1 + xg_1 + yg_2$ and so $f = xf_1 + yf_2$. Thus, $(xf_1 + yf_2)(1 + xg_1 + yg_2)h \in Q$ and so $(xf_1 + yf_2 + x^2 f_1 g_1 + y^2 f_2 g_2)h \in Q$. We claim that $fh = (xf_1 + yf_2)h \in Q$, for if not, then either $xf_1 h \notin Q$ or $yf_2 h \notin Q$. If $x^2 f_1 h \in Q$ and $y^2 f_2 h \in Q$, then $(xf_1 + yf_2)h \in Q$, as we wish. Thus, without loss of generality we can assume that $x^2 f_1 h \notin Q$ and hence there exists $t \ge 3$ such that $x^t f_1 h \in Q$, but $x^{t-1}f_1 h \notin Q$. Therefore $x^{t-2}(xf_1 + yf_2 + x^2 f_1 g_1 + y^2 f_2 g_2)h \in Q$. It follows that $(x^{t-1}f_1)h \in Q$, a contradiction. Thus, $fh = (xf_1 + yf_2)h \in Q$ and so Q is a classical primary submodule as [3].

Proposition 2.3. *Let M be an R-module and Q be a proper submodule of M. Then,*

  (1) *Q is classical primary if and only if for every submodule N of M such that $N \nsubseteq Q$, $(Q : N)$ is a primary ideal of R.*

  (2) *Q is classical quasi-primary if and only if for every submodule N of M such that $N \nsubseteq Q$, $(Q : N)$ is a quasi-primary ideal of R.*

Proof. We only prove Part (2). The proof for Part (1) is similar.

(2) ($\Leftarrow$). Let $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$ such that $N \nsubseteq Q$. Then, $ab \in (Q : N)$, and since $(Q : N)$ is a quasi-primary ideal, either $b^k \in (Q : N)$ or $a^k \in (Q : N)$ for some $k \in \mathbb{N}$. Thus, either $b^k N \subseteq Q$ or $a^k N \subseteq Q$; therefore, Q is a classical quasi-primary submodule.

(2) ($\Rightarrow$) is evident. $\square$

If Q is a classical primary (resp., classical quasi-primary) submodule of an R-module M, then by Proposition 2.3, $\mathcal{P} = \sqrt{(Q : M)}$ is a prime ideal and we shall say that Q is classical $\mathcal{P}$-primary (resp., classical $\mathcal{P}$-quasi-primary).

Theorem 2.4. *Let R be a domain with $\dim(R) \leq 1$, and let M be an R-module. Then, a proper submodule Q of M is classical quasi-primary if and only if it is classical*

Proof. Assume that R is a domain with $\dim(R) \leq 1$ and M is an R-module. By Proposition 2.3, every classical primary submodule of M is classical quasi-primary. Now, let Q be a classical quasi-primary submodule of M and $N \leq M$ such that $N \nsubseteq Q$. Then, by Proposition 2.3 (2), $(Q : N)$ is a quasi-primary ideal of R. Since R is a domain with $\dim(R) \leq 1$, by Proposition 2.1 (c), every classical quasi-primary ideal of R is primary. Thus, $(Q : N)$ is a primary ideal of R. Now, by Proposition 2.3 (1), Q is a classical primary submodule of M. $\square$

Corollary 2.5. *Let M be an R-module and Q be a classical primary (or classical quasi-primary) submodule. Then,*

$$\left\{ \sqrt{(Q : N)} \,\middle|\, N \text{ is a finitely generated submodule of } M \text{ such that } N \nsubseteq Q \right\}$$

is a chain of prime ideals of R.

Proof. First, we show that $\{\sqrt{(Q:m)} \mid m \in M \setminus Q\}$ is a chain of prime ideals of R. For each $m_1, m_2 \in M \setminus Q$ we have $\sqrt{(Q:m_1)} \cap \sqrt{(Q:m_2)} \subseteq \sqrt{(Q:m_1+m_2)}$. Since by Proposition 2.3, $\sqrt{(Q:m_1+m_2)}$ is either R or a prime ideal of R, depending on whether $m_1 + m_2$ belongs to Q or not, we conclude that either $\sqrt{(Q:m_1)} \subseteq \sqrt{(Q:m_1+m_2)}$ or $\sqrt{(Q:m_2)} \subseteq \sqrt{(Q:m_1+m_2)}$. It follows that $\sqrt{(Q:m_1)} \subseteq \sqrt{(Q:m_2)}$ or $\sqrt{(Q:m_2)} \subseteq \sqrt{(Q:m_1)}$; hence, $\{\sqrt{(Q:m)} \mid m \in M \setminus Q\}$ is a chain of prime ideals of R.

Now, let $N = Rm_1 + Rm_2 + \cdots + Rm_k$ and $N' = Rm_1' + Rm_2' + \cdots + Rm_l'$, where $k, l \in \mathbb{N}$ and $m_i, m_j' \in M$ for $1 \leq i \leq k$ and $1 \leq j \leq l$, be two finitely generated submodules of M such that $N \not\subseteq Q$ and $N' \not\subseteq Q$. Since $\{\sqrt{(Q:m)} \mid m \in M \setminus Q\}$ is a chain of prime ideals of R, without loss of generality we can assume that $\sqrt{(Q:m_1)} \subseteq \sqrt{(Q:m_i)}$ for all $1 \leq i \leq k$. Thus,

$$\begin{aligned}
\sqrt{(Q:N)} &= \sqrt{(Q:m_1R + m_2R + \cdots + m_kR)} \\
&= \sqrt{(Q:m_1) \cap (Q:m_2) \cap \cdots (Q:m_k)} \\
&= \sqrt{(Q:m_1)} \cap \sqrt{(Q:m_2)} \cap \cdots \cap \sqrt{(Q:m_k)} \\
&= \sqrt{(Q:m_1)}.
\end{aligned}$$

We now apply this argument again with N' replaced by N, to obtain $\sqrt{(Q:N')} = \sqrt{(Q:m_1')}$. Now, by the first part of the proof, $\sqrt{(Q:m_1)}$ and $\sqrt{(Q:m_1')}$ are comparable prime ideals; therefore, either $\sqrt{(Q:N)} \subseteq \sqrt{(Q:N')}$ or $\sqrt{(Q:N')} \subseteq \sqrt{(Q:N)}$, which completes the proof. $\qquad\qquad\square$

Proposition 2.6. *Let M be a Noetherian R-module and Q be a proper submodule of M.*

(a) *The following statements are equivalent:*
  (1) *Q is a classical primary submodule.*
  (2) *For every $a, b \in R$ and $m \in M$, $abm \in Q$ implies that either $am \in Q$ or $b^k m \in Q$ for some $k \in \mathbb{N}$.*
  (3) *For every $m \in M \setminus Q$, $(Q:m)$ is a primary ideal of R.*

(b) *The following statements are equivalent:*
  (1) *Q is a classical quasi-primary submodule.*
  (2) *For every $a, b \in R$ and $m \in M$, $abm \in Q$ implies that either $a^k m \in Q$ or $b^k m \in Q$ for some $k \in \mathbb{N}$.*

(3) *For every $m \in M \setminus Q$, $(Q : m)$ is a quasi-primary ideal of R.*

Proof. We only prove Part (a). The proof for Part (b) is similar.

(a) $(1) \Rightarrow (2)$ and (a) $(2) \Leftrightarrow (3)$ are clear.

(a) $(2) \Rightarrow (1)$. Suppose N is a submodule of M such that $N \not\subseteq Q$. Let $ab \in (Q : N)$, where $a, b \in R$, but $b \notin (Q : N)$, i.e., $abN \subseteq Q$ and $bN \not\subseteq Q$. Thus, $bn \notin Q$, for some $n \in N$. Since $abn \in Q$, by assumption, $a^k n \in Q$ for some $k \in \mathbb{N}$. If

$$A := \{n \in N \mid bn \in Q\}, \quad B := \{n \in N \mid a^k n \in Q \text{ for some } k \in \mathbb{N}\}, \text{ then}$$

one can easily see that A and B are submodules of N and $N = A \cup B$. It follows that $N = A$ or $N = B$. If $N = A$, then $bN \subseteq Q$, a contradiction. Therefore $N = B$. Since N is finitely generated, $a^k N \subseteq Q$ for some $k \in \mathbb{N}$; hence, $b^k \in (Q : N)$. Thus, Q is a classical primary submodule of M by Proposition 2.3(1). $\square$

Now, by Proposition 2.1 and Proposition 2.6, we have the following corollary.

Corollary 2.7. *Let R be a Dedekind domain and M be a Noetherian R-module. For a proper submodule Q of M, the following statements are equivalent:*

(1) *Q is a classical primary submodule.*
(2) *Q is a classical quasi-primary submodule.*
(3) *For every $m \in M \setminus Q$, $(Q : m)$ is a power of a prime ideal of R.*

We recall that an R-module M is a *multiplication* module if each submodule of M is of the form IM, where I is an ideal of R. The following proposition shows that every multiplication module is primary compatible, but in general, it need not be quasi-primary compatible (see Proposition 2.1).

Proposition 2.8. *Let M be a multiplication R-module and Q be a proper submodule of M.*

(a) *The following statements are equivalent:*
   (1) *Q is a classical primary submodule.*
   (2) *Q is a primary submodule.*
   (3) *$\mathcal{Q} = (Q : M)$ is a primary ideal of R.*

(4) $Q = \mathcal{Q}M$, where $\mathcal{Q}$ is a primary ideal which is maximal with respect to this property (i.e., $IM = Q$ implies that $I \subseteq \mathcal{Q}$).

(b) *The following statements are equivalent:*
(1) Q *is a classical quasi-primary submodule.*
(2) $\mathcal{Q} = (Q : M)$ *is a classical quasi-primary ideal of R.*
(3) $Q = \mathcal{Q}M$, *where $\mathcal{Q}$ is a classical quasi-primary ideal which is maximal with respect to this property (i.e., $IM = Q$ implies that $I \subseteq \mathcal{Q}$).*

Proof. We only prove Part (a). The proof for Part (b) is similar.
(a) (1) $\Rightarrow$ (2). Let Q be a classical primary submodule of multiplication R-module M. Assume that $am \in Q$, where $a \in R$ and $m \in M \setminus Q$. Since M is a multiplication module, $Rm = IM$ for some ideal I of R. Hence $aIM \subseteq Q$ and $IM \nsubseteq Q$, i.e., $aI \subseteq (Q : M)$ and $I \nsubseteq (Q : M)$. By Proposition 2.3 (1), $(Q : M)$ is a primary ideal of R; hence, $a^k M \subseteq Q$ for some $k \in \mathbb{N}$. Thus, Q is a primary submodule.
(a) (2) $\Rightarrow$ (3) is clear.
(a) (3) $\Rightarrow$ (4). Since M is a multiplication module, $Q = IM$ for some ideal of R. Since $\mathcal{Q}M \subseteq Q$, $I \subseteq (Q : M) = \mathcal{Q}$ and so $Q = IM \subseteq \mathcal{Q}M$. Thus, $Q = \mathcal{Q}M$, $\mathcal{Q}$ is a primary ideal and $JM = Q$ implies that $J \subseteq \mathcal{Q}$.
(a) (4) $\Rightarrow$ (1). Let $abN \subseteq Q$, where $a, b \in R$ and $N \leq M$ such that $bN \nsubseteq Q$. Since M is a multiplication module, $N = IM$ for some ideal I of R. Thus, $abIM \subseteq Q$, i.e., $abI \subseteq (Q : M) \subseteq \mathcal{Q}$. Since $bN \nsubseteq Q$, $bI \nsubseteq \mathcal{Q}$ and so $a^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$. This implies that $a^k N \subseteq \mathcal{Q}M = Q$ and so Q is a classical primary submodule of M. $\square$

It is clear that every vector space is a (quasi) primary compatible module and every field is a (quasi) primary compatible ring. Also, if R is a (quasi) primary compatible ring, so is any factor ring of R. Next, we will show that every primary compatible ring is quasi-primary compatible (see Proposition 2.9 and Theorem 2.14). But, we have not found any examples of a quasi-primary compatible ring R that is not primary compatible. On the other hand, every cyclic R-module is primary compatible and not necessarily a quasi-primary compatible (take $M = R$ and see [15, Example 4.12]).

Proposition 2.9. *Let R be a ring with $dim(R) = 0$. Then, R is a quasi-primary compatible ring.*

Proof. Let $\dim(R) = 0$ and M be an R-module. Suppose Q is a quasi-primary submodule of M i.e., $\mathcal{P} := \sqrt{(Q : M)}$ is a maximal ideal of R. Let $N \leq M$ such that $N \nsubseteq Q$. Then, $\mathcal{P} \subseteq \sqrt{(Q : N)}$ and since $\mathcal{P}$ is a maximal ideal, $\mathcal{P} = \sqrt{(Q : N)}$. Thus, by [15, Proposition 4.9], $(Q : N)$ is a primary ideal of R. Now, by Proposition 2.3 (2), Q is a classical quasi-primary submodule of M. $\qquad\square$

A ring R is called *Gelfand* provided that, for any distinct maximal ideals $\mathcal{P}_1$ and $\mathcal{P}_2$ of R, there exist elements $a \in R \setminus \mathcal{P}_1$ and $b \in R \setminus \mathcal{P}_2$ such that $ab = 0$. Simmons in [16] proved that a ring R is Gelfand if and only if every prime ideal of R is contained in a unique maximal ideal of R. Clearly each ring R with $\dim(R) = 0$ is Gelfand. Next, we show that every quasi-primary compatible ring is Gelfand.

Theorem 2.10. *Let R be a quasi-primary compatible ring. Then, R is a Gelfand ring.*

Proof. Let R be a quasi-primary compatible ring and $\mathcal{P}$ be a prime ideal of R. Then, the ring $R' := R/\mathcal{P}$ is also quasi-primary compatible. Now, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two maximal ideals of R', and let $M = R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus R'$ as an R'-module. Since R' is a domain and $\sqrt{((0) : M)} = ((0) : M) = (0)$, we conclude that $(0) \lneq M$ is a quasi-primary R'-submodule. Now, by our hypothesis $(0) \lneq M$ is a classical quasi-primary R'-submodule. Clearly $\mathcal{P}_1 \mathcal{P}_2 (R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus (0)) = (0)$, but $R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus (0) \neq (0)$. If $\mathcal{P}_1 \neq \mathcal{P}_2$, then there exist $b \in \mathcal{P}_2 \setminus \mathcal{P}_1$ and $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$. Since $ab(R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus (0)) = (0)$, there exists $k \in \mathbb{N}$ such that either $a^k(R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus (0)) = (0)$ or $b^k(R'/\mathcal{P}_1 \oplus R'/\mathcal{P}_2 \oplus (0)) = (0)$. It follows that either $a \in \mathcal{P}_2$ or $b \in \mathcal{P}_1$, a contradiction. Thus, we must have $\mathcal{P}_1 = \mathcal{P}_2$, i.e., the prime ideal $\mathcal{P}$ of R is contained in a unique maximal ideal of R. $\qquad\square$

Proposition 2.11. *Let R be a quasi-primary compatible domain. Then, any two prime ideals of R are comparable (i.e., $Spec(R)$ is a chain).*

Proof. Let R be a quasi-primary compatible domain, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two prime ideals of R such that $\mathcal{P}_1 \nsubseteq \mathcal{P}_2$. Suppose that $M = R \oplus R \oplus R$ and $N = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus (0)$. Clearly, $\sqrt{(N : M)} = (0)$ and so N is a quasi-primary submodule of M. Now, by our hypothesis N is a classical quasi-primary submodule of M. Clearly, $\mathcal{P}_1 \mathcal{P}_2 (R \oplus R \oplus (0)) \subseteq N$, but

$R \oplus R \oplus (0) \nsubseteq N$. Now, let $a \in \mathcal{P}_1 \setminus \mathcal{P}_2$. Then, $ab(R \oplus R \oplus (0)) \subseteq N$ for each $b \in \mathcal{P}_2$, but $a^k(R \oplus R \oplus (0)) \nsubseteq N$ for each $k \in \mathbb{N}$. Thus, for each $b \in \mathcal{P}_2$ there exists $k \in \mathbb{N}$ such that $b^k(R \oplus R \oplus (0)) \subseteq N$. It follows that $b \in \mathcal{P}_1$ for each $b \in \mathcal{P}_2$, i.e., $\mathcal{P}_2 \subseteq \mathcal{P}_1$. $\square$

Theorem 2.12. *Let R be a Noetherian ring. If R is quasi-primary compatible, then for each minimal prime ideal $\mathcal{P}$ of R, the ring $R' := R/\mathcal{P}$ has at most one nonzero prime ideal. Consequently, $\dim(R) \leq 1$.*

Proof. Let R be a quasi-primary compatible Noetherian ring and $\mathcal{M}$ be a maximal ideal of R. Suppose $\mathcal{P}$ is a minimal prime ideal such that $\mathcal{P} \subseteq \mathcal{M}$. It suffices to show that there is no prime ideal of R strictly between $\mathcal{P}$ and $\mathcal{M}$. Clearly, we can assume that $\mathcal{P} \subsetneq \mathcal{M}$; therefore, by [15, Exercise 15.3], if there exists one prime ideal of R strictly between $\mathcal{P}$ and $\mathcal{M}$, then there are infinitely many. On the other hand, the domain $R' := R/\mathcal{P}$ is also quasi-primary compatible, and so by Proposition 2.11, $\mathrm{Spec}(R')$ is a chain. Since R' is a Noetherian domain, we conclude that $\mathrm{Spec}(R')$ is finite, i.e., the set of prime ideals of R between $\mathcal{P}$ and $\mathcal{M}$ is finite. Thus, there is no prime ideal of R strictly between $\mathcal{P}$ and $\mathcal{M}$. $\square$

Lemma 2.13. (See [3, Proposition 1.5]). *Let M be an R-module and Q be a submodule of M. If $\sqrt{(Q:M)} = \mathcal{P}$, where $\mathcal{P}$ is a maximal ideal of R, then Q is a primary submodule of M.*

Next, we characterize primary compatible rings (see also [3, Theorem 1.7] in which the primary compatibility property is slightly different than ours).

Theorem 2.14. *Let R be a ring. Then, the following are equivalent:*
  (1) *R is a primary compatible ring.*
  (2) *The R-module $R \oplus R$ is primary compatible.*
  (3) *Every prime ideal of R is maximal (i.e., $dim(R) = 0$).*

Proof. $(1) \Rightarrow (2)$ is evident.
$(2) \Rightarrow (3)$. Let $\mathcal{P}_1$ be a prime ideal in R and $\mathcal{P}_2$ be a maximal ideal with $\mathcal{P}_1 \subseteq \mathcal{P}_2$. We claim that $Q = \mathcal{P}_1 \oplus \mathcal{P}_2$ is a classical primary R-submodule of $M = R \oplus R$. To see this, let $a, b \in R$ and N be a submodule of M such that $N \nsubseteq \mathcal{P}_1 \oplus \mathcal{P}_2$ and $abN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$. We will show that either $aN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$ or $bN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$. Since $N \nsubseteq \mathcal{P}_1 \oplus \mathcal{P}_2$, there exists an

element $(x_0, y_0) \in N$ such that $(x_0, y_0) \notin \mathcal{P}_1 \oplus \mathcal{P}_2$; hence, we just need only consider two the cases:

Case 1. $x_0 \notin \mathcal{P}_1$. Since $abN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$, $abx_0 \in \mathcal{P}_1$ and since $\mathcal{P}_1$ is a prime ideal of R, we conclude that either $a \in \mathcal{P}_1$ or $b \in \mathcal{P}_1$. Now, $\mathcal{P}_1 \subseteq \mathcal{P}_2$ yields that $aN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$ or $bN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$.

Case 2. For each $(x, y) \in N$, $x \in \mathcal{P}_1$. Thus, $y_0 \notin \mathcal{P}_2$. Since $abN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$, $aby_0 \in \mathcal{P}_2$ and since $\mathcal{P}_2$ is a prime ideal of R, we conclude that either $a \in \mathcal{P}_2$ or $b \in \mathcal{P}_2$. It follows that $aN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$ or $bN \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$.

Thus, $\mathcal{P}_1 \oplus \mathcal{P}_2$ is a classical primary submodule. Now, by our hypothesis Q is a primary submodule of M. Clearly, $\mathcal{P}_2(0, 1) \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$, but $(0, 1) \notin \mathcal{P}_1 \oplus \mathcal{P}_2$. Thus, for each $a \in \mathcal{P}_2$ there exists $k \in \mathbb{N}$ such that $a^k(R \oplus R) \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2$, and hence, we must have $a^k \in \mathcal{P}_1$. Now, since $\mathcal{P}_1$ is prime, $a \in \mathcal{P}_1$. Therefore, $\mathcal{P}_2 \subseteq \mathcal{P}_1$ and so $\mathcal{P}_1 = \mathcal{P}_2$. Thus, every prime ideal of R is a maximal ideal i.e., $\dim(R) = 0$.

$(3) \Rightarrow (1)$ is evident by Proposition 2.3 (1) and Lemma 2.13. $\qquad\square$

3. Decomposition into classical quasi-primary submodules

The decomposition into classical primary submodules was introduced in detail in [3] and some results of the study are applied frequently in this paper. The purpose of this section is to investigate decomposition of submodules into classical quasi-primary submodules. In particular, we introduce and study minimal classical quasi-primary decomposition of submodules in Noetherian modules.

First, we need the following lemmas which are crucial in our investigation.

Lemma 3.1. *Let M be an R-module, and let $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a primary decomposition of Q, where each Q_i is a $\mathcal{P}_i$-primary submodule of M. If $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_n$, then Q is a classical $\mathcal{P}_1$-quasi-primary submodule.*

Proof. Assume that $abN \subseteq Q$, where $a, b \in R$, $N \leq M$ and $N \nsubseteq Q$. Thus, $N \nsubseteq Q_i$ for some i $(1 \leq i \leq n)$. Assume that t $(1 \leq t \leq n)$ is the smallest number such that $N \nsubseteq Q_t$. Thus, $N \subseteq Q_1 \cap \cdots \cap Q_{t-1}$. On the other hand, $abN \subseteq Q_t$ and Q_t is $\mathcal{P}_t$-primary; hence, $(ab)^{k_1} M \subseteq Q_t$ for some $k_1 \in \mathbb{N}$, i.e., $ab \in \mathcal{P}_t$. Thus, $a \in \mathcal{P}_t$ or $b \in \mathcal{P}_t$.

Now, since $\mathcal{P}_t \subseteq \mathcal{P}_{t+1} \subseteq \cdots \subseteq \mathcal{P}_n$, $a^k M \subseteq Q_t \cap Q_{t+1} \cap \cdots \cap Q_n$ or $b^k M \subseteq Q_t \cap Q_{t+1} \cap \cdots \cap Q_n$ for some $k \in \mathbb{N}$. It follows that $a^k N \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$ or $b^k N \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$. Thus, Q is a classical quasi-primary submodule of M. Now, it is clear that $\sqrt{(Q : M)} = \sqrt{(Q_1 \cap Q_2 \cap \cdots \cap Q_n : M)} = \mathcal{P}_1$; therefore, Q is a classical $\mathcal{P}_1$-quasi-primary submodule. $\square$

The following example shows that Lemma 3.1 is not necessarily true if $Q_1, \cdots, Q_n$ are only assumed to be classical (quasi) primary submodules (even if all Q_i' are classical $\mathcal{P}$-primary submodules for a prime ideal $\mathcal{P}$ of R).

Example 3.2. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$, $Q_1 = \mathbb{Z}_2 \oplus (0) \oplus (0)$, and $Q_2 = (0) \oplus \mathbb{Z}_3 \oplus (0)$. Then, one can easily see that Q_1 and Q_2 are classical (quasi) primary submodules of M. Moreover, $(0) = Q_1 \cap Q_2$ and $\sqrt{(Q_1 : M)} = \sqrt{(Q_2 : M)} = (0)$. Clearly, $2 \times 3(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (0)) = (0)$, but for each $k \geq 1$, $2^k(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (0)) \nsubseteq (0)$ and $3^k(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (0)) \nsubseteq (0)$. Thus, $(0) \subsetneqq M$ is not a classical (quasi) primary submodule.

We will show that the converse of Lemma 3.1 is also true when the decomposition $Q = Q_1 \cap \cdots \cap Q_n$ is a minimal primary decomposition.

Lemma 3.3. *Let M be an R-module and N be a proper submodule of M. Let $N = Q_1 \cap \cdots \cap Q_n$ with $\mathcal{P}_i = \sqrt{(Q_i : M)}$, $1 \leq i \leq n$, be a minimal primary decomposition of N. Then, for each $\mathcal{P} \in Spec(R)$, the following statements are equivalent:*

(1) *$\mathcal{P} = \mathcal{P}_i$ for some i ($1 \leq i \leq n$).*
(2) *There exists $m \in M \setminus N$ such that $(N : m)$ is a $\mathcal{P}$-primary ideal.*
(3) *There exists $m \in M \setminus N$ such that $\sqrt{(N : m)} = \mathcal{P}$.*

Proof. The proof is similar to [15, Theorem 4.17] and so the details are left to the reader. $\square$

Proposition 3.4. *Let M be an R-module and Q be a proper submodule of M. Let $Q = Q_1 \cap \cdots \cap Q_n$ with $\mathcal{P}_i = \sqrt{(Q_i : M)}$, $1 \leq i \leq n$, be a minimal primary decomposition of Q. Then, Q is a classical quasi-primary submodule if and only if $\{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$ is a chain of prime ideals. In that case, the radical of $(Q : M)$ is the smallest of the primes $\mathcal{P}_1, \cdots, \mathcal{P}_n$.*

Proof. ($\Rightarrow$). Since $Q = Q_1 \cap \cdots \cap Q_n$ is a minimal primary decomposition of Q, by Lemma 3.3, for each i ($1 \le i \le n$), $\mathcal{P}_i = \sqrt{(Q : m_i)}$ for some $m_i \in M \setminus Q$. Assume that $\mathcal{P}_i \nsubseteq \mathcal{P}_j$ and $\mathcal{P}_j \nsubseteq \mathcal{P}_i$ for some $i \ne j$. Let $a \in \mathcal{P}_i \setminus \mathcal{P}_j$ and $b \in \mathcal{P}_j \setminus \mathcal{P}_i$. Then, there exist positive integers k_i and k_j such that $a^{k_i} b^{k_j}(Rm_i + Rm_j) \subseteq Q$, and since Q is a classical quasi-primary submodule, $a^k(m_i + m_j) \in Q$ or $b^k(m_i + m_j) \in Q$ for some $k \ge k_i + k_j$. It follows that either $a^k m_j \in Q$ i.e., $a^k \in (Q : m_j)$ or $b^k m_i \in Q$ i.e., $b^k \in (Q : m_i)$; hence, $a \in \mathcal{P}_j$ or $b \in \mathcal{P}_i$, a contradiction. Thus, $\{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$ is a chain of prime ideals.
($\Leftarrow$) follows from Lemma 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We note that Proposition 3.4 is not necessarily true if the primary decomposition $Q = Q_1 \cap \cdots \cap Q_n$ is not minimal. See the following example:

Example 3.5. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, $Q_1 = 2\mathbb{Z} \oplus \mathbb{Z}$, $Q_2 = \mathbb{Z} \oplus 3\mathbb{Z}$, $Q_3 = \mathbb{Z} \oplus (0)$, and $Q_4 = (0) \oplus \mathbb{Z}$. Clearly, $Q_1, \cdots, Q_4$ are primary submodules of M with $\sqrt{(Q_1 : M)} = 2\mathbb{Z}$, $\sqrt{(Q_2 : M)} = 3\mathbb{Z}$, and $\sqrt{(Q_3 : M)} = \sqrt{(Q_4 : M)} = (0)$. Also $(0) = Q_1 \cap Q_2 \cap Q_3 \cap Q_4$ and (0) is a classical quasi-primary submodule of M. But, $\{(0), 2\mathbb{Z}, 3\mathbb{Z}\}$ is not a chain of prime ideals of R.

Definition 3.6. (see also [3, Definition 2.1]) Let N be a proper submodule of an R-module M. A *classical primary* (resp., *classical quasi-primary*) *decomposition* of N is an expression $N = \cap_{i=1}^n Q_i$, where each Q_i is a classical primary (resp., classical quasi-primary) submodule of M. The decomposition is called *reduced* if it satisfies the following two conditions:

(1) no $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is a classical primary (resp., classical quasi-primary) submodule, where $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, n\}$, for $t \ge 2$ with $i_1 < i_2 < \cdots < i_t$.
(2) for each j, $Q_j \nsupseteq \cap_{i \ne j} Q_i$.

Corresponding to the above definition, by Proposition 2.3, we have a *list* of prime ideals $\sqrt{(Q_1 : M)}, \cdots, \sqrt{(Q_n : M)}$. Among reduced classical primary (resp., classical quasi-primary) decompositions, any one that has the least number of distinct primes will be called *minimal*.

It is clear that every primary decomposition of a submodule N of M is classical primary. But, the converse is not true in general (see [3, Example 2.2]). On the other hand, every classical quasi-primary decomposition is a quasi-primary decomposition (an expression $N = \cap_{i=1}^{n} Q_i$, where each Q_i is a quasi-primary submodule of M). That the converse is not true in general is shown in the following example. Also, Theorem 2.4 together with [3, Example 2.2] show that not all reduced classical primary (quasi-primary) decomposition is necessarily minimal.

Example 3.7. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Clearly, $(0) \lneqq M$ is a quasi-primary submodule and so (0) is a quasi-primary decomposition of (0). But, (0) is not a classical quasi-primary submodule of M. Now, by Theorem 2.4 and [3, Example 2.2], $(0) = \mathbb{Z}_2 \oplus (0) \oplus (0) \cap (0) \oplus \mathbb{Z}_3 \oplus (0)$ is a minimal classical (quasi) primary decomposition of (0).

We recall that if N is a proper submodule of a Noetherian R-module M, then N has a primary decomposition, hence a minimal primary decomposition (see [15, Exercise 9.31], and also [3, Corollary 2.6] for the existence of classical primary decomposition of N). Now, by Proposition 3.4, and the fact that every primary submodule is classical quasi-primary, we have the following proposition:

Proposition 3.8. *Let M be a Noetherian R-module. Then, every proper submodule N of M has a classical quasi-primary decomposition; hence, it has a minimal classical quasi-primary decomposition.*

Let R be a ring. For an ideal I of R, we denote the set of all minimal prime ideals of I by $\min(I)$. Let M be a finitely generated R-module and $N \lneqq M$. In [3, Theorem 3.6], it is shown that if R is a Noetherian domain with $\dim(R) \leq 1$ and

$N = Q_1 \cap \cdots \cap Q_n$ with $\sqrt{(Q_i : M)} = \mathcal{P}_i$,   for $i = 1, 2, \cdots, n$

is a minimal classical primary decomposition of N, then

$$\{\mathcal{P}_i \mid i = 1, 2, \cdots, n\} = \min(N : M).$$

Consequently, the set $\{\mathcal{P}_i \mid i = 1, 2, \cdots, n\}$ is uniquely determined. Now, by Theorem 2.4, this uniqueness property is also true when we replace "classical primary" with "classical quasi-primary" (in fact, these two decompositions are the same when R is a domain with $\dim(R) \leq 1$). Here we extend this uniqueness property for finitely generated modules

over a Noetherian ring R without the assumption that R is a domain or $\dim(R) \leq 1$.

Theorem 3.9. [First Uniqueness Theorem]. *Let R be a Noetherian ring and M be a finitely generated R-module. Let N be a proper submodule of M and*

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{(Q_i : M)} = \mathcal{P}_i, \quad \text{for } i = 1, 2, \cdots, n$$

be a minimal classical quasi-primary decomposition of N. Then,

$$\{\mathcal{P}_i \mid i = 1, 2, \cdots, n\} = min(N : M).$$

Consequently, the set $\{\mathcal{P}_i \mid i = 1, 2, \cdots, n\}$ is uniquely determined.

Proof. First, we show that $min(N : M) \subseteq \{\mathcal{P}_i \mid i = 1, 2, \cdots, n\}$. Let $\mathcal{P}$ be a minimal prime of $(N : M)$. Then, by [15, Lemma 9.20], $\mathcal{P}$ is a minimal member of $\text{Supp}(M/N)$ and so by [15, Theorem 9.39], $\mathcal{P} \in \text{Ass}(M/N)$. Thus, $\mathcal{P} = (N : m)$ for some $0 \neq m \in M \setminus N$. Renumber the Q_i's such that $m \notin Q_i$ for $1 \leq i \leq j$ and $m \in Q_i$ for $j + 1 \leq i \leq n$. Since $\mathcal{P}_i = \sqrt{(Q_i : M)}$ and $\mathcal{P}_i$ is finitely generated, $\mathcal{P}_i^{k_i} M \subseteq Q_i$ for some $k_i \geq 1$ $(1 \leq i \leq n)$. Therefore $(\cap_{i=1}^{j} \mathcal{P}_i^{k_i})m \subseteq \cap_{i=1}^{n} Q_i = N$ and so $\cap_{i=1}^{j} \mathcal{P}_i^{k_i} \subseteq (N : m) = \mathcal{P}$. Since $\mathcal{P}$ is prime, $\mathcal{P}_t \subseteq \mathcal{P}$ for some $t \leq j$. Since $(N : M) \subseteq \sqrt{(N : M)} \subseteq \sqrt{(Q_t : M)} = \mathcal{P}_t$ and $\mathcal{P}$ is a minimal prime of $(N : M)$, we conclude that $\mathcal{P} = \mathcal{P}_t$.

Now, it is sufficient to show that each $\mathcal{P}_i$ $(1 \leq i \leq n)$ is a minimal prime of $(N : M)$. Without loss of generality, we may take $i = 1$. Clearly,

$$(N : M) \subseteq \sqrt{(N : M)} = \sqrt{(Q_1 \cap \cdots \cap Q_n : M)} = \cap_{i=1}^{n} \sqrt{(Q_i : M)} \subseteq \mathcal{P}_1.$$

On the contrary, suppose that $\mathcal{P}_1$ is not a minimal prime of $(N : M)$. Thus, there exists an $i \in \{1, 2 \cdots, n\}$ such that $\mathcal{P}_i$ is a minimal prime of $(N : M)$ with $\mathcal{P}_i \subsetneq \mathcal{P}_1$ (since $min(N : M) \subseteq \{\mathcal{P}_i \mid i = 1, 2, \cdots, n\}$). Again, without loss of generality, we may take $i = 2$. Thus, $(N : M) \subseteq \mathcal{P}_2 \subsetneq \mathcal{P}_1$. By [15, Exercise 9.31], each Q_i has a minimal primary decomposition. Suppose that $Q_1 = Q_{11} \cap \cdots \cap Q_{1s}$ with $\sqrt{(Q_{1j} : M)} = \mathcal{P}_{1j}$ $(1 \leq j \leq s)$ and $Q_2 = Q_{21} \cap \cdots \cap Q_{2t}$ with $\sqrt{(Q_{2j} : M)} = \mathcal{P}_{2j}$ $(1 \leq j \leq t)$ are minimal primary decompositions of Q_1 and Q_2, respectively. By Proposition 3.4, $\{\mathcal{P}_{1j} \mid 1 \leq j \leq s\}$ and $\{\mathcal{P}_{2j} \mid 1 \leq j \leq t\}$ are chain of prime ideals. Without loss of generality, we can assume that $\mathcal{P}_{11} \subseteq \mathcal{P}_{12} \subseteq \cdots \subseteq \mathcal{P}_{1s}$ and $\mathcal{P}_{21} \subseteq \mathcal{P}_{22} \subseteq \cdots \subseteq \mathcal{P}_{2t}$. We thus get

$\mathcal{P}_1 = \mathcal{P}_{11}$ and $\mathcal{P}_2 = \mathcal{P}_{21}$ since

$$\mathcal{P}_1 = \sqrt{(Q_1 : M)} = \sqrt{(Q_{11} \cap \cdots \cap Q_{1s} : M)} = \cap_{i=1}^{s} \sqrt{(Q_{1i} : M)} = \mathcal{P}_{11},$$

$$\mathcal{P}_2 = \sqrt{(Q_2 : M)} = \sqrt{(Q_{21} \cap \cdots \cap Q_{2t} : M)} = \cap_{i=1}^{t} \sqrt{(Q_{ti} : M)} = \mathcal{P}_{21}.$$

It follows that $\mathcal{P}_{21} \subseteq \mathcal{P}_{11} \subseteq \mathcal{P}_{12} \subseteq \cdots \subseteq \mathcal{P}_{1s}$ and so by Proposition 3.4, $Q_1' = Q_{21} \cap Q_{11} \cap \cdots \cap Q_{1s}$ is a classical quasi-primary submodule of M with $\sqrt{(Q_1' : M)} = \mathcal{P}_{21} = \mathcal{P}_2$. On the other hand,

$$N = Q_1 \cap \cdots \cap Q_n = (Q_{11} \cap \cdots \cap Q_{1s}) \cap (Q_{21} \cap \cdots \cap Q_{2t}) \cap Q_3 \cap \cdots \cap Q_n$$
$$= (Q_{21} \cap Q_{11} \cap \cdots \cap Q_{1s}) \cap (Q_{21} \cap \cdots \cap Q_{2t}) \cap Q_3 \cap \cdots \cap Q_n$$
$$= Q_1' \cap Q_2 \cap Q_3 \cap \cdots \cap Q_n.$$

Thus, $N = Q_1' \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary decomposition of N with $\sqrt{(Q_1' : M)} = \sqrt{(Q_2 : M)} = \mathcal{P}_2$ and $\sqrt{(Q_i : M)} = \mathcal{P}_i$ for $i = 3, \cdots, n$. We note that if there exists another Q_i $(3 \leq i \leq n)$ such that $\sqrt{(Q_i : M)} = \mathcal{P}_i = \mathcal{P}_1$, then by a similar argument we can replace it by Q_i' such that $\sqrt{(Q_i' : M)} = \sqrt{(Q_2 : M)} = \mathcal{P}_2$. Now by using this decomposition we can obtain a minimal classical quasi-primary decomposition $N = Q_1'' \cap Q_2'' \cap \cdots \cap Q_k''$ such that $\mathcal{P}_1 \notin \{\sqrt{(Q_i'' : M)} \mid i = 1, \cdots, k\} \subseteq \{\mathcal{P}_i | i = 2, \cdots, n\}$, contrary with the minimality of the decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ with $\{\sqrt{(Q_i : M)} | i = 1, \cdots, n\} = \{\mathcal{P}_i \mid i = 1, \cdots, n\}$. Thus, $\{\mathcal{P}_i \mid i = 1, 2, \cdots, n\} = \min(N : M)$. $\square$

Let M be an R-module. A proper submodule P of M is called a prime submodule of M if for each $a \in R$ and $m \in M$, $am \in P$ implies that either $m \in P$ or $aM \subseteq P$. Prime submodules of modules over commutative rings have been studied by various authors (see for examples, [7,9,12,13]). Also, a proper submodule P of M is called a *classical prime submodule* of M if, for every $a, b \in R$ and $m \in M$, $abm \in P$ implies that either $am \in P$ or $bm \in P$. This notion of classical prime submodule has been extensively studied in [4-6,8]. The *classical prime radical* (resp. *prime radical*) of a submodule N of M, denoted by $\sqrt[cl]{N}$ (resp. $\sqrt[p]{N}$), is defined to be the intersection of all classical prime submodules (resp. prime submodules) of M containing N. We note that for each proper ideal I of R, $\sqrt[cl]{I} = \sqrt[p]{I} = \sqrt{I}$. If $\mathcal{Q}$ is a (quasi) primary ideal of a ring, it is well-known that $\sqrt{\mathcal{Q}}$ is a prime ideal. However, in the module case, if Q is a (quasi) primary submodule, then $\sqrt[p]{Q}$ is not necessarily a prime submodule (see [17, Theorem 1.9 and Example 1.11] for more details). Also for a submodule N of M, we define

$$\sqrt[nil]{N} = \{m \mid m = \sum_{i=1}^{r} a_i m_i \text{ for some } a_i \in R, \ m_i \in M \text{ and } r \in \mathbb{N},$$
$$\text{such that } a_i{}^k m_i \in N \ (1 \leq i \leq r) \text{ for some } k \in \mathbb{N}\}.$$

This is called *(Baers) lower nilradical* of N. Clearly, $\sqrt[nil]{N}$ is a submodule of M and $N \subseteq \sqrt[nil]{N} \subseteq \sqrt[cl]{N}$ (see [6, Definition 1.4 and Lemma 2.6]).

In [3, Theorem 1.9], it is shown that for every classical primary submodule Q of a module M over a domain R with $\dim(R) \leq 1$, $\sqrt[nil]{Q}$ is a classical prime submodule and also $\sqrt[nil]{Q} = \sqrt[cl]{Q}$. Thus, by Theorem 2.4, this fact is also true when we replace "classical primary" with "classical quasi-primary".

We conclude this paper with the following fundamental conjecture:

Conjecture 3.10. [Second Uniqueness Theorem]. *Let R be a Noetherian ring and let N be a submodule of the finitely generated R-module M. Let*

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{(Q_i : M)} = \mathcal{P}_i \quad \text{for } i = 1, 2, \cdots, n$$

and

$$N = Q'_1 \cap \cdots \cap Q'_m \quad \text{with } \sqrt{(Q'_i : M)} = \mathcal{P}'_i \quad \text{for } i = 1, 2, \cdots, m$$

be two minimal classical quasi-primary decompositions of N. Then, $n = m$, and also $\sqrt[cl]{Q_1}, \cdots, \sqrt[cl]{Q_n}$ are n different classical prime submodules of M.

Remark 3.11. We note that the above conjecture is true when R is a Noetherian domain with $\dim(R) \leq 1$. In fact, since $\dim(R) \leq 1$, by Theorem 2.4, classical quasi-primary submodules of any module coincide with classical primary submodules. Now, apply [3, Theorem 3.9].

Acknowledgments

References

[1] S. E. Atani and A. Y. Darani, On quasi-primary submodules, *Chiang Mai J. Sci.* **33**(3) (2006) 249-254.

[2] M. F. Atiyah and I. G. Mc Donald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1969.

[3] M. Baziar and M. Behboodi, Classical primary submodules and decomposition theory of modules, *J. Algebra Appl.* **8**(3) (2009) 351-362.

[4] M. Behboodi, *Classical prime submodules*, Ph.D Thesis, Chamran University, Ahvaz, Iran, (2004).

[5] M. Behboodi, On weakly prime radical of modules and semi-compatible modules, *Acta Math. Hungar.* **113** (3) (2006) 243-254.

[6] M. Behboodi, A generalization of Baer's lower nilradical for modules, *J. Algebra Appl.* **6** (2) (2007), 337-353.

[7] M. Behboodi, O. A. S. Karamzadeh and H. Koohy, Modules whose certain submodules are prime, *Vietnam J. Math.* **32** (3) (2004) 303-317.

[8] M. Behboodi and H. Koohy, Weakly prime modules, *Vietnam J. Math.* **32** (2) (2004) 185-195.

[9] J. Dauns, Prime modules, *J. Reine Angew. Math.* **298** (1978) 156-181.

[10] L. Fuchs, On quasi-primary ideals, *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1947) 174-183.

[11] L. Fuchs and E. Mosteig, Ideal theory in Prüfer domains, *J. Algebra* **252** (2002) 411-430.

[12] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1992) 3593-3602.

[13] S. H. Man, On commutative Noetherian rings which have the s.p.a.r. property, Arch. Math. (Basel) **70** (1998) 31-40.

[14] E. Noether, Ideal theorie in Ringbereichen, (German) *Math. Ann.* **83** (1921) 24-66.

[15] R. Y. Sharp, *Steps in commutative algebra*, London Math. Soc. Stud. Texts, 19, Cambridge University Press, Cambridge, 1990.

[16] H. Simmons, Reticulated rings, *J. Algebra* **66** (1980) 169-192.

[17] P. F. Smith, Primary modules over commutative rings, *Glasgow. Math. J.* **43** (1) (2001) 103-111.

Mahmood Behboodi

Department of Mathematical Science, Isfahan University of Technology, P.O.Box 84156-83111, Isfahan, Iran

and

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5746, Tehran, Iran

Email: mbehbood@cc.iut.ac.ir

Reza Jahani-Nezhad

Department of Mathematics, Faculty of Science, University of Kashan, P.O.Box

87317-51167, Kashan, Iran
Email: jahanian@kashanu.ac.ir

Mohammad Hasan Naderi
Department of Mathematics, Faculty of Science, University of Kashan, P.O.Box 87317-51167, Kashan, Iran
and
Department of Mathematics, Faculty of Science, University of Qom, P.O.Box 37161-46611, Qom, Iran
Email: mnaderi@kashanu.ac.ir, mh-naderi@qom.ac.ir