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PROVING THE EFFICIENCY OF PRO-2-GROUPS OF FIXED CO-CLASSES

A. ARJOMANDFAR* AND H. DOOSTIE

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ABSTRACT. Among the six classes of pro-2-groups of finite and fixed co-classes and trivial Schur Multiplicator which studied by Abdolzadeh and Eick in 2009, there are two classes

$$S_5 = \langle a, b \mid [b, a^2] = 1, a^2 = [b, a]^2, (b^2)^{[b, a]} b^2 = 1 \rangle$$

and

$$S_6 = \langle a, t, b \mid a^2 = b^2, [b, a]^2 = 1, t^a = t^{-1}[b, a], b^t = aba \rangle$$

that have been conjectured to have deficiency zero presentations. In this paper we prove these conjectures. This completes the efficiency of all six classes of pro-2-groups of fixed co-classes.

1. Introduction

For detailed information on pro-*p*-groups one may see [6, 7, 8]. The pro-2-groups of fixed co-classes were first investigated in [1] and the Schur Multiplicator is used to get the appropriate presentations for such classes of groups. For a useful and prolific information on the Schur Multiplicator of a group one may consult [10]. Briefly, for a finitely presented finite group $G = \langle X \mid R \rangle$ the Schur Multiplicator of G is defined to be the group $M(G) = \frac{F' \cap \overline{R}}{[F,\overline{R}]}$, where F = F(X) is the free group

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^{*}Corresponding author

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of rank |X| and \overline{R} is the normal closure of R $(R \subseteq F)$. More detailed consideration about the relationship between the Schur Multiplicator and the deficiency of a presentation of a finitely presented group may be found in [9]. It is a classical fact that the groups with a deficiency zero presentation will have the trivial Schur Multiplicators. However, looking for a deficiency zero presentation of a group which has the trivial Schur Multiplicator, is a long-standing question and many attempts have been made during the years on finite *p*-groups and even in infinite groups. Our notations are merely standard, we use |G : H| for the index of a subgroup *H* in a group *G*, [a,b] is used for the commutator $a^{-1}b^{-1}ab$ and we will use the Modified Todd-Coxeter coset enumeration algorithm in the form as given in [2] to get a presentation for subgroups. More application of this algorithm may be found in [3, 4, 5]. Following [1] and consider the groups:

$$S_5 = \langle a, b \mid [b, a^2] = 1, a^2 = [b, a]^2, (b^2)^{[b, a]} b^2 = 1$$

and

$$S_6 = \langle a, t, b \mid a^2 = b^2, [a, b]^2 = 1, t^a = t^{-1}[b, a], b^t = aba \rangle$$

Just proved in [1], these groups have trivial Schur Multiplicators and the above presentations are the simplified presentations for them (see the Lemmas 13 and 14 of [1]). We now recall the conjectures 14 and 16 of [1] as the following propositions:

Proposition 1.1. The group S_5 has a deficiency zero presentation isomorphic to

$$\langle a, b \mid a^2 = [b, a]^2, (b^2)^{[b, a]} b^2 = 1 \rangle.$$

Proposition 1.2. The group S_6 has a deficiency zero presentation isomorphic to

$$a, t, b \mid a^2 = b^2, t^a = t^{-1}[b, a], b^t = aba \rangle.$$

2. The proofs

We give a suitable generating set for the derived subgroups of S_5 and S_6 . Then, by using the Modified Todd-Coxeter coset enumeration algorithm we get a presentation for the derived subgroups. Note that, using GAP [11] we are able to get a presentation for the derived subgroup in each case and then finding the image of the word $[b, a^2]$ in the derived subgroup of the group S_5 and the image of the word $[a, b]^2$ in the derived subgroup of S_6 are possible, however, checking that this image is the identity element of the group (in each case) is not possible every time, i.e; we checked it for S_5 and it was not possible for S_6 . For this reason we are interested in to use the combinatorial method of Modified Todd-Coxeter coset enumeration algorithm to give a clear and exact proof.

Lemma 2.1. Let $G = \langle a, b \mid a^2 = [b, a]^2, (b^2)^{[b,a]}b^2 = 1 \rangle$. Then, G', the derived subgroup of G, has a presentation isomorphic to

$$G' = \langle a_1, \dots, a_6 \mid r_1, \dots, r_{12} \rangle$$

where, the relations r_1, \ldots, r_{12} are as follows:

$$\begin{aligned} r_1 &: a_1 a_3^{-2} = 1, & r_2 : (a_1 a_6^{-1})^2 a_1^{-1} a_4 = 1, \\ r_3 &: a_1 a_5^{-2} = 1, & r_4 : (a_6 a_2)^2 a_5^{-1} a_1 a_3^{-1} = 1, \\ r_5 &: a_2 a_3 a_2 a_3^{-1} = 1, & r_6 : a_2 a_6 a_2 a_6^{-1} = 1, \\ r_7 &: a_3^2 a_6 a_3^2 a_6^{-1} = 1, & r_8 : (a_1 a_6^{-1})^2 a_2 a_3^{-1} a_4^{-1} a_1 a_2 a_3 = 1, \end{aligned}$$

$$\begin{split} r_9 &: a_4 a_3^2 a_2^{-1} a_6^{-1} a_3 a_5^{-1} a_6^{-1} a_1^{-1} = 1, \\ r_{10} &: (a_6^{-1} a_3 a_1^{-1}) a_5 (a_2^{-1} a_3)^2 a_2^{-1} (a_6^{-1} a_3 a_1^{-1}) a_4^{-1} (a_1 a_3 a_2^{-1}) = 1, \\ r_{11} &: a_1^{-1} a_4 a_1 a_6^{-1} a_2 a_3^{-1} a_2 a_5^{-1} a_1 a_6^{-1} = 1, \\ r_{12} &: (a_4^{-1} a_1) a_3 a_2^{-1} a_6^{-1} (a_3 a_1^{-1} a_5) a_1 a_6^{-1} a_2 a_3^{-1} = 1. \end{split}$$

Proof. It is easy to check that $\frac{G}{G'} \cong Z_2 \times Z_4$. We now consider the subgroup

$$K = \langle a^2, b^4, [b, a], [a^{-2}, b^{-1}], [a^{-1}, b], [b^{-1}, a] \rangle$$

of G and define eight cosets as

 $1=K,\ ib=i+1,\ (i=1,2,3),\ 1a=5,\ 4a^{-1}=6,\ 2a^{-1}=7,\ 3a^{-1}=8$ to see that K is of index 8 in G. Since $a^2,b^4\in G'$ then $K\subseteq G'$, which together with |G:H|=8 proves that K=G'. We let

$$a_1 = a^2, \ a_2 = b^4, \ a_3 = [b, a], \ a_4 = [a^{-2}, b^{-1}], \ a_5 = [a^{-1}, b], \ a_6 = [b^{-1}, a]$$

and we use the Modified Todd-Coxeter coset enumeration algorithm to get a presentation for K. In this way we have to adopt the name of a coset and its representative to get the following table of coset representatives:

$$\begin{aligned} 1a &= 5, & 1b = 2, \\ 2a &= a_4^{-1}a_1.7, & 2b = 3, \\ 3a &= a_3a_2^{-1}a_3a_2^{-1}.8, & 3b = 4, \\ 4a &= a_2a_5^{-1}a_1a_3^{-1}a_2^{-1}.6, & 4b = a_2.1, \\ 5a &= a_1.1, & 5b = a_1a_6^{-1}.7, \\ 6a &= 4, & 6b = a_2a_3a_1^{-1}.5, \\ 7a &= 2, & 7b = a_3a_2^{-1}.8, \\ 8a &= 3, & 8b = a_6^{-1}a_2^{-1}.6 \end{aligned}$$

These coset representatives have been obtained by using the eight defined cosets as 1a = 5, 1b = 2, 2b = 3, 3b = 4, $4a^{-1} = 6$, $2a^{-1} = 7$, $3a^{-1} = 8$, and by using the subgroup tables of the coset enumeration algorithm. Indeed,

$$\begin{aligned} 1a^2 &= a_1.1 &\Rightarrow 5a &= a_1.1, \\ 1b^4 &= a_2.1 &\Rightarrow 4b &= a_2.1, \\ 1[b,a] &= a_3.1 &\Rightarrow 6b &= a_2a_3a_1^{-1}.5, \\ 1[a^{-1},b] &= a_5.1 &\Rightarrow 4a &= a_2a_5^{-1}a_1a_3^{-1}a_2^{-1}.6, \\ 1[a^{-2},b^{-1}] &= a_4.1 &\Rightarrow 2a &= a_4^{-1}a_1.7, \\ 1[b^{-1},a] &= a_6.1 &\Rightarrow 5b &= a_1a_6^{-1}.7, \\ 1R_2 &= 1 &\Rightarrow 7b &= a_3a_2^{-1}.8, \\ 2R_2 &= 2 &\Rightarrow 8b &= a_6^{-1}a_2^{-1}.6, \\ 3R_1 &= 3 &\Rightarrow 3a &= (a_3a_2^{-1})^2.8. \end{aligned}$$

where, $R_1 = a^2 [a, b]^2$ and $R_2 = (b^2)^{[b,a]} b^2$.

To get a presentation for K we examine all of the equations:

$$iR_1 = i, \ (i = 1, 2, 4, 5, 6, 7, 8),$$

and

$$iR_2 = i, \ (i = 3, 4, 5, 6, 7, 8).$$

Many of them give us complicated relations, however, by using the new results in each step we will simplify the relations to get the desired

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presentation for K. A comprehensive and detailed computation may be given as follows:

The equation $1R_1 = 1$ gives us $r_1 : a_1a_3^{-2} = 1$, and the equation $2R_2 = 2$ yields the trivial relation. The equation $7R_1 = 7$ yields the relation $r_2: (a_1a_6^{-1})^2a_1^{-1}a_4 = 1.$

The equations $5R_1 = 5$, $4R_1 = 4$, $3R_2 = 3$ and $4R_2 = 4$ yield the relations:

$$r_{3} : a_{1}a_{5}^{-2} = 1,$$

$$r_{4} : (a_{6}a_{2})^{2}a_{5}^{-1}a_{1}a_{3}^{-1} = 1,$$

$$r_{5} : a_{2}a_{3}a_{2}a_{3}^{-1} = 1,$$

$$r_{6} : a_{2}a_{6}a_{2}a_{6}^{-1} = 1,$$

respectively. The preliminary relation $[(a_2^{-1}a_3)^2a_2^{-1}a_6^{-1}]^2a_3a_1^{-1}a_5 = 1$ is a result of the equation $6R_1 = 6$, and using the relations r_4 and r_5 this becomes to:

$$r_7: a_3^2 a_6 a_3^2 a_6^{-1} = 1.$$

 $r_7: a_3^2 a_6 a_3^2 a_6^{-1} = 1.$ The equation $8R_1 = 8$ gives us the relation $a_4^{-1} a_1 a_2 a_3^{-1} a_4^{-1} a_1 a_3 a_2^{-1} = 1$ that becomes to the simpler relation that becomes to the simpler relation

 $r_8: (a_1a_6^{-1})^2 a_2 a_3^{-1} a_4^{-1} a_1 a_2 a_3 = 1$

by using the relations r_2 and r_5 . The equation $5R_2 = 5$ gives the relation

$$a_4(a_3a_2^{-1})^2a_6^{-1}a_3a_1^{-1}a_5^{-1}a_1a_6^{-1}a_1^{-1} = 1$$

that becomes to $r_9: a_4 a_3^2 a_2^{-1} a_6^{-1} a_3 a_5^{-1} a_6^{-1} a_1^{-1} = 1$ by using the relations r_3 and r_5 . Finally, the equations $6R_2 = 6, 7R_2 = 7$ and $8R_2 = 8$ will give us the relations r_{10} , r_{11} and r_{12} , respectively. This completes the proof.

Proof of Proposition 1.1. By considering the presentation of K, we prove that $a_4 = 1$ holds in K. This will tends to establishing the relation $[a^2, b] = 1$ in G and so, the group S_5 has a deficiency zero presentation.

Proving $a_4 = 1$ is by using the relation r_{10} of K and we proceed as follows. First, we use the relations r_5 to get $[a_3^2, a_2] = 1$ and then the relations r_3, r_4 and r_7 to simplify the relations r_{10} , i.e;

$$\begin{aligned} &(a_6^{-1}a_3a_1^{-1})a_5(a_2^{-1}a_3a_2^{-1}a_3)a_2^{-1}(a_6^{-1}a_3a_1^{-1})a_4^{-1}(a_1a_3a_2^{-1}) = 1 \\ \Rightarrow & (a_6^{-1}a_3a_5a_1^{-1}).a_3^2a_2^{-1}.(a_6^{-1}a_3a_1^{-1}).a_4^{-1}a_1.a_3a_2^{-1} = 1, \ (by \ r_3), \\ \Rightarrow & (a_6^{-1}a_6a_2a_6a_2).a_3^2a_2^{-1}.a_6^{-1}a_3a_1^{-1}.(a_4^{-1}a_1).a_3a_2^{-1} = 1, \ (by \ r_4d), \end{aligned}$$

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$$\Rightarrow a_6(a_2a_3^2a_2^{-1}).a_6^{-1}a_3a_1^{-1}.(a_4^{-1}a_1)a_3 = 1, (a_6a_3^2a_6^{-1}a_3)a_1^{-1}.(a_4^{-1}a_1)a_3 = 1, (by \ [a_3^2, a_2] = 1), \Rightarrow a_3^{-1}a_1^{-1}a_4^{-1}a_1a_3 = 1, (by \ r_7), \Rightarrow a_4 = 1.$$

Lemma 2.2. Let $G = \langle a, t, b \mid a^2 = b^2, t^a = t^{-1}[b, a], b^t = aba \rangle$. Then, $G' = \langle A, B, C \rangle$, where $A = a^2, B = t^2$ and C = [a, b]. Moreover, $C^2 = 1$ holds in G'.

Proof. Obviously, $\frac{G}{G'} \cong Z_2 \times Z_2 \times Z_2$ and letting $K = \langle A, B, C \rangle$ and defining eight cosets as

$$K=1,\ 1a=2,\ 1t=3,\ 1b=4,\ 5b=2,\ 3b=6,\ 3a=7,\ 6a=8$$

shows that |G:H| = 8. Since $a^2, t^2 \in G'$ then $K \subseteq G'$, so, $G' \cong K$. To prove the equation $C^2 = 1$ we use the Modified Todd-Coxeter coset enumeration algorithm as well as in Lemma 2.1 and get the coset representatives as,

$$\begin{array}{ll} 1a=2, & 1b=4, & 1t=3, \\ 2a=A.1, & 2b=A.5, & 2t=CB^{-1}.7, \\ 3a=7, & 3b=6, & 3t=B.1, \\ 4a=AC^{-1}.5, & 4b=A.1, & 4t=BC^{-1}AB^{-1}.6, \\ 5a=C.4, & 5b=2, & 5t=C^{-1}A^{-6}B^{-1}.8, \\ 6a=8, & 6b=BA^3B^{-1}.3, & 6t=BAC.4, \\ 7a=BA^3B^{-1}.3, & 7b=BC^{-1}A^{-2}B^{-1}.8, & 7t=BA^3B^{-1}A^{-1}C^{-1}.2, \\ 8a=BA^3B^{-1}.6, & 8b=BA^5CB^{-1}.7, & 8t=BA^5B^{-1}AC^{-2}.5. \end{array}$$

To get a presentation for G' we have to consider all of the equations

$$iR_j = i, \ (i = 1, 2, ..., 8, \ j = 1, 2, 3)$$

where, $R_1 = a^2 b^{-2}$, $R_2 = t^a [a, b]t$ and $R_3 = bta^{-1}b^{-1}a^{-1}t^{-1}$. To prove the relation $C^2 = 1$ we do not need to calculate all of the relations of G'we just examine the suitable equation $5R_3 = 5$ to get the result. Indeed, by using the above coset representatives we get:

$$5bta^{-1}b^{-1}a^{-1}t^{-1} = 5 \quad \Rightarrow CB^{-1}.BA^{-3}B^{-1}.BA^{-3}B^{-1}.BA^{6}C = 1 \\ \Rightarrow C^{2} = 1.$$

This completes the proof.

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Proof of Proposition 1.2. Using the result of Lemma 2.2 and substituting for *C* will conclude the validity of the relation $[a, b]^2 = 1$ in the group *G*.

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A. Arjomandfar

Department of Mathematics, Science and Research Branch, Islamic Azad University P.O.Box 14515/1775, Tehran, Iran

Email: ab.arj44@gmail.com

H. Doostie

Mathematics Department, Tarbiat Moallem University, 49 Mofateh Ave., Tehran 15614, Iran

Email: doostih@saba.tmu.ac.ir



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