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ON MODULE EXTENSION BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra and X be a Banach A-bimodule. Then $\mathcal{S} = A \oplus X$, the l^1-direct sum of A and X becomes a module extension Banach algebra when equipped with the algebra product $(a, x).(a', x') = (aa', ax'+xa')$. In this paper, we investigate biflatness and biprojectivity for these Banach algebras. We also discuss on automatic continuity of derivations on $\mathcal{S} = A \oplus A$.

1. Introduction

In this paper we shall focus on an especial kind of Banach algebras which are constructed from a Banach algebra A and a Banach A-bimodule X, called *module extension Banach algebras*. The module extension Banach algebra corresponding to A and X is $\mathcal{S} = A \oplus X$, the l^1-direct sum of A and X, with the algebra product defined as follows:

$$(a, x).(a', x') = (aa', ax' + xa') \qquad (a, a' \in A, x, x' \in X).$$

The article is organized as follows: In section 2, we bring some preliminaries. In section 3, we verify the biflatness and biprojectivity of $\mathcal{S}$. Using these Banach algebras, we construct a class of non-biflat and non-biprojective Banach algebras. In section 4, we discuss on automatic continuity of derivations on $\mathcal{S} = A \oplus A$ and we prove that $\mathcal{S} = A \oplus A$ has

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automatically continuous derivations if and only if A has automatically continuous derivations.

2. Preliminaries

Let A be a Banach algebra. The product map on A extends to a map $\Delta_A : A\hat{\otimes}A \to A$, determined by $\Delta_A(a\otimes b) = ab$ for all $a, b \in A$, where $A\hat{\otimes}A$ denotes the projective tensor product. The projective tensor product $A\hat{\otimes}A$ becomes a Banach A-bimodule with the following module actions:

$$a.(b\otimes c) = ab\otimes c, \quad (b\otimes c).a = b\otimes ca \qquad (a, b, c \in A).$$

By above actions, Δ_A becomes an A-bimodule homomorphism.

A Banach algebra A is called biprojective if Δ_A has a bounded right inverse which is an A-bimodule homomorphism and is called biflat if Δ_A^* has a bounded left inverse which is an A-bimodule homomorphism.

Let $L_x : A \to X$ and $R_x : A \to X$ be bounded linear maps defined by $L_x(a) = xa$ and $R_x(a) = ax$ for all $a \in A$. We consider the set $\mathcal{M}(X)$ of all double multipliers of A-bimodule X, i.e., the set of all pairs (L, R) of bounded linear maps from A to X such that L is a right A-module homomorphism, R is a left A-module homomorphism and $aL(b) = R(a)b$ for all $a, b \in A$. Linear operations in $\mathcal{M}(X)$ are defined as usual. The norm of $(L, R) \in \mathcal{M}(X)$ is $\|(L, R)\| = \max\{\|L\|, \|R\|\}$ where $\|L\|$ and $\|R\|$ are the operator norms of L and R. With operations

$$a.(L, R) = (L_{R(a)}, R_{R(a)}), \qquad (L, R).a = (L_{L(a)}, R_{L(a)}),$$

$\mathcal{M}(X)$ becomes a Banach A-bimodule. We say that a derivation $D : A \to X$ is determined by a multiplier of X if there exists $(L, R) \in \mathcal{M}(X)$ such that $D = L - R$.

Let A and X be as above. Then X^*, the dual space of X, is also a Banach A-bimodule as well via

$$\langle x, a.\phi\rangle := \langle x.a, \phi\rangle, \quad \langle x, \phi.a\rangle := \langle a.x, \phi\rangle \qquad (a \in A,\ \phi \in X^*,\ x \in X).$$

A derivation from A into X is a linear map satisfying

$$D(ab) = a.(Db) + (Da).b \qquad (a, b \in A).$$

For each $x \in X$ we denote by ad_x the derivation $\mathrm{ad}_x(a) = a.x - x.a$ for all $a \in A$, which is called an inner derivation. We denote by $Z^1(A, X)$ the space of all bounded derivations from A into X, and by $B^1(A, X)$

the space of all inner derivations from A into X. The first cohomology group of A and X, denoted by $H^1(A,X)$, is the quotient space $Z^1(A,X)/B^1(A,X)$. A Banach algebra A is called weakly amenable if $H^1(A,A^*)=0$.

Throughout this paper, A will denote a Banach algebra, X a Banach A-bimodule and $\mathcal{S}$ the corresponding module extension Banach algebra of A and X. AX, XA and AXA will denote the subsets $\{ax \,|\, a\in A, x\in X\}$, $\{xa \,|\, a\in A, x\in X\}$ and $\{axb \,|\, a,b\in A, x\in X\}$ of X, respectively.

3. Biflatness and Biprojectivity of Module Extension Banach Algebras

Our aim in this section is to state the biflatness and biprojectivity of $\mathcal{S}$ in terms of biflatness and biprojectivity of A and some conditions on X. The next two theorems give some conditions on A and X which are necessary for biflatness and biprojectivity of $\mathcal{S}$. We use the notation $\langle E\rangle$ for the linear span of a subset E of X.

Theorem 3.1. *Let the module extension Banach algebra $\mathcal{S}$ be biflat. Then*

(*i*) *A is biflat;*
(*ii*) *if ρ_A is a left inverse of Δ_A^*, then $\rho_A^*(a)$ vanishes on the subset $X(\mathcal{S}\hat{\otimes}\mathcal{S})^*X$ of $(A\hat{\otimes}A)^*$, for all $a\in A$;*
(*iii*) *$\langle AX+XA\rangle$ is dense in X;*
(*iv*) *$AXA=0$.*

Proof. (i) Let $\rho_{\mathcal{S}}$ be a bounded left inverse for $\Delta_{\mathcal{S}}^*$ which is a $\mathcal{S}$-bimodule homomorphism and let $q_1 : A^*\to\mathcal{S}^*$ and $q_2 : (A\hat{\otimes}A)^*\to(\mathcal{S}\hat{\otimes}\mathcal{S})^*$ be the canonical embeddings and let $r:\mathcal{S}^*\to A^*$ be the restriction map. Define $\rho_A = r\circ\rho_{\mathcal{S}}\circ q_2$. Obviously ρ_A is a bounded A-bimodule homomorphism. For $f\in A^*$ and $(a_1,x_1),(a_2,x_2)\in\mathcal{S}$ we have

$$\begin{aligned}\langle(a_1,x_1)\otimes(a_2,x_2), q_2\circ\Delta_A^*(f)\rangle &= \langle a_1\otimes a_2, \Delta_A^*(f)\rangle\\ &= \langle\Delta_A(a_1\otimes a_2), f\rangle\\ &= \langle a_1a_2, f\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle(a_1,x_1)\otimes(a_2,x_2), \Delta_{\mathcal{S}}^*\circ q_1(f)\rangle &= \langle\Delta_{\mathcal{S}}((a_1,x_1)\otimes(a_2,x_2)), q_1(f)\rangle\\ &= \langle(a_1a_2, a_1x_2+x_1a_2), q_1(f)\rangle\\ &= \langle a_1a_2, f\rangle.\end{aligned}$$

Therefore, $q_2 \circ \Delta_A^* = \Delta_{\mathcal{S}}^* \circ q_1$ and so

$$\begin{aligned} \rho_A \circ \Delta_A^* &= r \circ \rho_{\mathcal{S}} \circ q_2 \circ \Delta_A^* = r \circ \rho_{\mathcal{S}} \circ \Delta_{\mathcal{S}}^* \circ q_1 \\ &= r \circ id_{\mathcal{S}^*} \circ q_1 = r \circ q_1 = id_{A^*}. \end{aligned}$$

Hence A is biflat.

(ii) First note that every element of $X(\mathcal{S}\hat{\otimes}\mathcal{S})^*X$ vanishes on the subspace $(A\hat{\otimes}X) \oplus (X\hat{\otimes}A) \oplus (X\hat{\otimes}X)$ of $\mathcal{S}\hat{\otimes}\mathcal{S}$, and thus it can be viewed as an element of $(A\hat{\otimes}A)^*$. Now let $a \in A$, $x, y \in X$ and $f \in (\mathcal{S}\hat{\otimes}\mathcal{S})^*$. Then

$$\begin{aligned} \langle xfy, \rho_A^*(a)\rangle &= \langle \rho_A(xfy), a\rangle = \langle r \circ \rho_{\mathcal{S}} \circ q_2(xfy), a\rangle = \langle r \circ \rho_{\mathcal{S}}(xfy), a\rangle \\ &= \langle r(x\rho_{\mathcal{S}}(f)y), a\rangle = \langle x\rho_{\mathcal{S}}(f)y, a\rangle = \langle \rho_{\mathcal{S}}(f), yax\rangle = 0. \end{aligned}$$

(iii) Since $\mathcal{S}$ is biflat, $\langle \mathcal{S}^2\rangle$ is dense in $\mathcal{S}$. Hence $\langle AX + XA\rangle$ is dense in X.

(iv) Since $\mathcal{S}$ is biflat, by [9, Theorem 5.9(ii)], for every $\mathcal{S}$-bimodule Y each derivation $D : \mathcal{S} \to Y$, is determined by a multiplier of Y^{**}. So there exists $(L, R) \in \mathcal{M}(Y^{**})$ such that $D = L - R$. Now consider X as an $\mathcal{S}$-bimodule by the following module actions:

$$(a, x).y = ay, \quad y.(a, x) = ya \qquad (a \in A, x, y \in X), \tag{3.1}$$

and define $D : \mathcal{S} \to X$ by $D(a, x) = x$. It is easy to check that D is a bounded derivation. Thus there exists $(L, R) \in \mathcal{M}(X^{**})$ such that $L(a, x) - R(a, x) = x$, for all $a \in A$ and $x \in X$. By the derivation property of D and the multiplier property of (L, R) we have

$$ay + xb = L(a, x)b - aR(b, y) \qquad (a, b \in A,\ x, y \in X),$$

note that X^{**} is an $\mathcal{S}$-bimodule with the actions same as (3.1). By letting $a = 0$ we obtain

$$L(0, x)b = xb \qquad (b \in A,\ x \in X). \tag{3.2}$$

On the other hand, by the property of multipliers we have $aL(b, x) = R(a, y)b$. So by setting $b = 0$ we get

$$aL(0, x) = 0 \qquad (a \in A,\ x \in X). \tag{3.3}$$

Now by (3.2) and (3.3) we have $axb = a(L(0, x)b) = (aL(0, x))b = 0$ for all $a, b \in A$ and $x \in X$. $\square$

Corollary 3.2. *If X is a non-trivial symmetric A-bimodule, then $\mathcal{S} = A \oplus X$ is not biflat.*

Theorem 3.3. *Let the module extension Banach algebra $\mathcal{S}$ be biprojective. Then*

(i) *A is biprojective;*
(ii) *if ρ_A is a right inverse of Δ_A, then $X\rho_A(A)X = 0$;*
(iii) *$\langle AX + XA\rangle$ is dense in X;*
(iv) *$AXA = 0$.*

Proof. Since every biprojective Banach algebra is biflat, by Theorem 3.1 we have to show (i) and (ii). Let $\rho_{\mathcal{S}}$ be a bounded right inverse for $\Delta_{\mathcal{S}}$ which is an $\mathcal{S}$-bimodule homomorphism and let $p : \mathcal{S} \to A$ and $\iota : A \to \mathcal{S}$ be defined by $p(a, x) = a$ and $\iota(a) = (a, 0)$ respectively. Define $\rho_A = (p \otimes p) \circ \rho_{\mathcal{S}} \circ \iota$. Obviously ρ_A is a bounded A-bimodule homomorphism and $\Delta_A \circ \rho_A = id_A$. For (ii), first note that A is an $\mathcal{S}$-bimodule via p and so $p \otimes p$ is an $\mathcal{S}$-bimodule homomorphism. Now let $a \in A$ and $x, y \in X$, then

$$\begin{aligned} x\rho_A(a)y &= x(p \otimes p) \circ \rho_{\mathcal{S}} \circ \iota(a)y = x(p \otimes p) \circ \rho_{\mathcal{S}}(a, 0)y \\ &= (p \otimes p) \circ \rho_{\mathcal{S}}(x(a, 0)y) = 0. \end{aligned}$$

□

Corollary 3.4. *The module extension Banach algebra $\mathcal{S} = A \oplus X$ fails to be biflat or biprojective whenever $AXA \neq 0$.*

Corollary 3.5. *For each non-trivial Banach algebra A and each non-negative integer n, the module extension Banach algebra $\mathcal{S} = A \oplus A^{(n)}$ fails to be biflat or biprojective, where $A^{(n)}$ is the n-th dual space of A for $n \in \mathbb{N}$ and $A^{(0)} = A$.*

Proof. Assume that $\mathcal{S}$ is biflat. By Theorem 3.1, $\langle A^2\rangle$ is dense in A. Let $a, b \in A$ and let (F_n) be a sequence in $\langle A^2\rangle$ with $F_n = \sum_{i=1}^{m(n)} a_{i,n}b_{i,n}$ such that $\lim_n F_n = a$. Then $ab = \lim_n F_n b$, so $\langle A^2\rangle \subseteq \langle A^3\rangle$. Thus $\langle A^3\rangle$ is dense in A. Now let n be even. Then from $AA^{(n)}A = 0$ we have $A^3 = 0$ which implies $A = 0$. For odd n we have $AA^*A = 0$. Thus every element of A^* vanishes on A^3 and so on A. Therefore $A^* = 0$, which implies $A = 0$. □

Example 3.6. *Let G be a locally compact group, and let $L^1(G)$ and $M(G)$ be its group algebra and measure algebra respectively. Then the module extension Banach algebras $L^1(G)\oplus M(G)$, $M(G)\oplus L^1(G)$, $M(G)\oplus M(G)$, $L^1(G) \oplus L^1(G)$, $L^1(G) \oplus L^\infty(G)$, are neither biprojective nor biflat.*

Now we proceed the converse of Theorems 3.1 and 3.3. We show that if A has a bounded approximate identity, then the converses of Theorems 3.1 and 3.3 are true. Also, with some examples we show that this is the best one can get.

Proposition 3.7. *Suppose that A and X satisfy the conditions (i), (ii), (iii) and (iv) of Theorem 3.1 and A has a bounded approximate identity (in fact, A is amenable). Then the module extension Banach algebra $\mathcal{S}$ is biflat.*

Proof. Let (e_α) be a bounded approximate identity for A. First we show that $\langle AX\rangle \bigcap \langle XA\rangle = 0$. Let $F \in \langle AX\rangle \bigcap \langle XA\rangle$. Then

$$F = \sum_{i=1}^{n} a_i x_i = \sum_{j=1}^{m} y_j b_j \qquad (a_i, b_j \in A, x_i, y_j \in X),$$

and so

$$F = \lim_{\alpha} e_\alpha F = \lim_{\alpha} e_\alpha \sum_{j=1}^{m} y_j b_j = \lim_{\alpha} \sum_{j=1}^{m} e_\alpha y_j b_j = 0,$$

by condition (iv). Secondly we show that $X = AX + XA$. Let E be the closure of $\langle AX\rangle$ in X. Then E is a left essential A-module and so $E = AX$ by Cohen's factorization theorem [1, Theorem 11.10]. So AX is a closed submodule of X. Similarly XA is a closed submodule of X and $AX + XA$ is dense in X. Now projections $p : AX + XA \to AX$ and $q : AX + XA \to XA$ defined by $p(u) = \lim_\alpha e_\alpha u$ and $q(u) = \lim_\alpha u e_\alpha$ are continuous and so we can extend continuously to $\tilde{p} : X \to AX$ and $\tilde{q} : X \to XA$. The restriction of $\tilde{p} + \tilde{q} : X \to AX + XA$ on $AX + XA$ is the identity map. For $x \in X$ let (u_n) be a sequence in $AX + XA$ which converges to x. We have

$$x = \lim_{n} u_n = \lim_{n} (\tilde{p} + \tilde{q})(u_n) = \lim_{n} (\tilde{p} + \tilde{q})(x),$$

and so $x \in AX + XA$. Thus $AX + XA = X$. Let $\iota : A \to \mathcal{S}$ be the canonical injection as in the proof of Theorem 3.3 and set $\tilde{\rho} = (\iota \otimes \iota)^{**} \circ \rho_A^*$. It is obvious that $\tilde{\rho}$ is a bounded A-bimodule homomorphism from A^{**} into $(\mathcal{S}\hat{\otimes}\mathcal{S})^{**}$. Now define $\rho_{\mathcal{S}} : \mathcal{S} \to (\mathcal{S}\hat{\otimes}\mathcal{S})^{**}$ by

$$\rho_{\mathcal{S}}(a, bx + yc) = \tilde{\rho}(a) + \tilde{\rho}(b)x + y\tilde{\rho}(c) \qquad (a, b, c \in A, x, y \in X).$$

Since the norm of X is equivalent with the l^1-norm of $AX + XA$ and $\tilde{\rho}(b)x = \lim_\alpha \tilde{\rho}(e_\alpha)bx$ and $y\tilde{\rho}(c) = \lim_\alpha yc\tilde{\rho}(e_\alpha)$, $\rho_{\mathcal{S}}$ is a well defined bounded linear map. We show that $\rho_{\mathcal{S}}$ is an $\mathcal{S}$-bimodule homomorphism.

For $a,b \in A$ and $x \in X$ we have $ax\tilde{\rho}(b) = \lim_\alpha axb\tilde{\rho}(e_\alpha) = 0$ and $\tilde{\rho}(b)xa = \lim_\alpha \tilde{\rho}(e_\alpha)bxa = 0$ by condition (iv). So for $a,b,c,d \in A$ and $x,y,z \in X$ we have

$$\begin{aligned} \rho_{\mathcal{S}}(d(a,bx+yc)) &= \rho_{\mathcal{S}}(da, dbx+dyc) = \tilde{\rho}(da) + \tilde{\rho}(db)x \\ &= d(\tilde{\rho}(a) + \tilde{\rho}(b)x) = d(\tilde{\rho}(a) + \tilde{\rho}(b)x + y\tilde{\rho}(c)) \\ &= d\rho_{\mathcal{S}}(a, bx+yc), \end{aligned}$$

and by using condition (ii) we have

$$\begin{aligned} \rho_{\mathcal{S}}(z(a,bx+yc)) &= \rho_{\mathcal{S}}(za) = z\tilde{\rho}(a) = z(\tilde{\rho}(a) + \tilde{\rho}(b)x + y\tilde{\rho}(c)) \\ &= z\rho_{\mathcal{S}}(a, bx+yc). \end{aligned}$$

Therefore, $\rho_{\mathcal{S}}$ is a left $\mathcal{S}$-module homomorphism. Similarly, $\rho_{\mathcal{S}}$ is a right $\mathcal{S}$-module homomorphism. Since $\Delta_A^{**} \circ (\rho_A^*|_A)$ is the canonical embedding of A into A^{**}, for $a,b,c \in A$ and $x,y \in X$ we have

$$\Delta_{\mathcal{S}}^{**}(\rho_{\mathcal{S}}(a)) = \Delta_{\mathcal{S}}^{**}(\tilde{\rho}(a)) = \Delta_{\mathcal{S}}^{**}((\iota \otimes \iota)^{**} \circ \rho_A^*(a)) = \Delta_A^{**}(\rho_A^*(a)) = a,$$

and

$$\begin{aligned} \Delta_{\mathcal{S}}^{**}(\rho_{\mathcal{S}}(bx+yc)) &= \Delta_{\mathcal{S}}^{**}(\tilde{\rho}(b)x + y\tilde{\rho}(c)) \\ &= \Delta_{\mathcal{S}}^{**}\left((\iota \otimes \iota)^{**} \circ \rho_A^*(b)x + y(\iota \otimes \iota)^{**} \circ \rho_A^*(c)\right) \\ &= \Delta_A^{**}(\rho_A^*(b))x + y\Delta_A^{**}(\rho_A^*(c)) \\ &= bx + yc. \end{aligned}$$

Therefore, $\Delta_{\mathcal{S}}^{**} \circ \rho_{\mathcal{S}}$ is the canonical embedding of $\mathcal{S}$ into $\mathcal{S}^{**}$ and so $\mathcal{S}$ is biflat. □

Proposition 3.8. *Suppose that A and X satisfy the conditions (i), (ii), (iii) and (iv) of Theorem 3.3 and A has a bounded approximate identity. Then the module extension Banach algebra $\mathcal{S}$ is biprojective.*

Proof. As in the proof of Proposition 3.7, AX and XA are closed A-submodules of X with trivial intersection and $AX+XA = X$. Therefore we can define $\rho_{\mathcal{S}} : \mathcal{S} \to (A\hat{\otimes}A) \oplus (A\hat{\otimes}X) \oplus (X\hat{\otimes}A) \subseteq \mathcal{S}\hat{\otimes}\mathcal{S}$ by

$$\rho_{\mathcal{S}}(a, bx+yc) = \rho_A(a) + \rho_A(b)x + y\rho_A(c) \qquad (a,b,c \in A, x,y \in X).$$

As in Proposition 3.7, we see that $\rho_{\mathcal{S}}$ is a well defined bounded linear map. We show that $\rho_{\mathcal{S}}$ is an $\mathcal{S}$-bimodule homomorphism. Since A has a bounded approximate identity, we have $AX\rho_A(A) = 0$, and so by using conditions (ii) and (iv), for $a,b,c,d \in A$ and $x,y,z \in X$ we have

$$\begin{aligned}\rho_{\mathcal{S}}(d(a,bx+yc)) &= \rho_{\mathcal{S}}(da,dbx+dyc) = \rho_A(da)+\rho_A(db)x\\ &= d(\rho_A(a)+\rho_A(b)x) = d(\rho_A(a)+\rho_A(b)x+y\rho_A(c))\\ &= d\rho_{\mathcal{S}}(a,bx+yc),\end{aligned}$$

and

$$\begin{aligned}\rho_{\mathcal{S}}(z(a,bx+yc)) &= \rho_{\mathcal{S}}(za) = z\rho_A(a) = z(\rho_A(a)+\rho_A(b)x+y\rho_A(c))\\ &= z\rho_{\mathcal{S}}(a,bx+yc).\end{aligned}$$

Thus $\rho_{\mathcal{S}}$ is a left $\mathcal{S}$-module homomorphism. Similarly, one can show that $\rho_{\mathcal{S}}$ is a right $\mathcal{S}$-module homomorphism. Obviously, $\rho_{\mathcal{S}}$ is a right inverse for $\Delta_{\mathcal{S}}$ and so $\mathcal{S}$ is biprojective. $\square$

We point out that there is a large class of modules that satisfies in conditions of Propositions 3.7 and 3.8. Let X and Y be Banach A-bimodules. We denote by X_0 (respectively, ${}_0Y$) the A-bimodules with right (respectively, left) trivial module action. Let X (respectively, Y) be a left (respectively, right) essential A-module. Consider the direct sum $Z = X_0 \oplus {}_0Y$ with the following module actions:

$$a.(x,y) = a.x, \quad (x,y).a = y.a \qquad (a \in A, x \in X, y \in Y).$$

Then Z satisfies in conditions (iii) and (iv) of Propositions 3.7 and 3.8. Also, every essential left (respectively, right) A-module with trivial right (respectively, left) module action satisfies in conditions (ii), (iii) and (iv) of Propositions 3.7 and 3.8.

By the following examples we show that if the Banach algebra A has an approximate identity (not necessarily bounded), then the conclusion of Propositions 3.7 and 3.8 may fail. We also show that the biflatness or biprojectivity of $\mathcal{S}$ does not imply that A has a bounded approximate identity.

Example 3.9. *Let $A = l^1$, $X = l_0^p$ for $1 \le p \le \infty$ and $\mathcal{S} = l^1 \oplus l_0^p$. We know that l^1 is a commutative biprojective Banach algebra (with pointwise multiplication) with approximate identity.*

- (i) *Let $p = 1$. Define $\rho : \mathcal{S} \to (l^1 \hat{\otimes} l^1) \oplus (l^1 \hat{\otimes} l_0^1) \subseteq \mathcal{S} \hat{\otimes} \mathcal{S}$ by $\rho(\delta_n, \delta_m) = \delta_n \otimes \delta_n + \delta_m \otimes \delta_m$ and extend by linearity. It is easy to see that ρ is a left inverse of $\Delta_{\mathcal{S}}$ which is a bounded $\mathcal{S}$-bimodule homomorphism. So $\mathcal{S}$ is a biprojective (and so biflat) Banach algebra and $A = l^1$ has no bounded approximate identity.*

(*ii*) *Let* $1 < p < \infty$. *If* $\mathcal{S}$ *is a biflat (biprojective) Banach algebra, then by* [2, Proposition 2.8.62] *it is weakly amenable and so* $H^1(l^1, {}_0l^q) = 0$ *by* [11, Theorem 2.1], *where* $1/p + 1/q = 1$. *Now consider the bounded derivation* $D : l^1 \to {}_0l^q$ *by* $Df = f$. *If* D *is inner, then there is an element* $g \in l^q$ *such that* $gf = f$ *for all* $f \in l^1$, *which implies that* $g(n) = 1$ *for all* $n \in \mathbb{N}$ *and so* $g \notin l^q$. *Hence* $\mathcal{S}$ *can not be biflat (biprojective).*

(*iii*) *Let* $p = \infty$. *Define* $D : l^1 \to ({}_0l^1)^{**} = {}_0(l^{1**})$ *with* $D(f) = f$ *as in* (*ii*). *If* D *is inner, then there is a* $F \in (l^1)^{**}$ *such that* $F\Box f = f$, $\Box$ *denotes the first Arens product on* $(l^1)^{**}$, *for all* $f \in l^1$. *Now it is easy that one can find a bounded approximate identity for* l^1 *which is a contradiction.*

4. Automatic Continuity of Derivations on $\mathcal{S} = A \oplus A$

Our aim in this section is to discuss on automatic continuity of derivations on $\mathcal{S} = A \oplus A$. We show that if A has a one-sided approximate identity and $H^1(A, A)$ is trivial, then the module extension Banach algebra $\mathcal{S} = A \oplus A$ has automatically continuous derivations if and only if A has automatically continuous derivations.

Notation 4.1. *If* A *is a Banach algebra and* X *is a Banach* A*-bimodule, we denote by*

(*i*) $\mathrm{Z}(A)$ *the algebraic center of* A,
(*ii*) $C_A(X, X)$ *the set* $\{\mathrm{ad}_a : X \to X \mid a \in \mathrm{Z}(A)\}$,
(*iii*) $\mathrm{Hom}_A(X, X)$ *the set of all* A*-bimodule homomorphisms (not necessarily bounded) from* X *to* X.

The following Proposition characterizes derivations on $\mathcal{S}$ [7, Proposition 2.2]. Note that the continuity property of derivations is not necessary for this characterizing.

Proposition 4.2. *Let* $\mathcal{S} = A \oplus X$. *Then* $D : \mathcal{S} \to \mathcal{S}$ *is a (bounded) derivation if and only if*

$$D(a, x) = (D_A(a) + T_1(x), D_X(a) + T_2(x)) \quad (a \in A, x \in X), \tag{4.1}$$

such that

(*i*) $D_A : A \to A$ *is a (bounded) derivation,*
(*ii*) $D_X : A \to X$ *is a (bounded) derivation,*

(*iii*) *$T_1 : X \to A$ is a (bounded) A-bimodule homomorphism such that $T_1(x)y + xT_1(y) = 0$ for all $x, y \in X$,*

(*iv*) *$T_2 : X \to X$ is a (bounded) linear map such that*

$$T_2(ax) = aT_2(x)+D_A(a)x, \quad T_2(xa) = T_2(x)a+xD_A(a) \quad (a \in A, x \in X).$$

Moreover, D is inner if and only if D_A and D_X are inner, $T_1 = 0$ and if $D_A = \mathrm{ad}_a$, then $T_2 = \mathrm{ad}_a$.

By [11, Lemma 3.2] again note that the continuity property of homomorphisms is not necessary. For each (bounded) $\varphi \in \mathrm{Hom}_A(X, X)$, $D_\varphi : \mathcal{S} \to \mathcal{S}$ defined by $D_\varphi(a, x) = (0, \varphi(x))$ is a (bounded) derivation. Moreover, D_φ is inner if and only if $\varphi \in C_A(X, X)$. Also, by [7, Proposition 2.4] for $\mathcal{S} = A \oplus A$, every (bounded) derivation $D : A \to A$ gives rise a (bounded) derivation $\tilde{D} : \mathcal{S} \to \mathcal{S}$ with $\tilde{D}(a, b) = (D(a), D(b))$. Moreover, $\tilde{D}$ is inner if and only if D is inner. So, we have the following corollary:

Corollary 4.3. *Let A be a Banach algebra and $\mathcal{S} = A \oplus A$. Then there exists a linear isomorphism from $H^1(A, A)$ onto a subspace of $H^1(\mathcal{S}, \mathcal{S})$. In particular, if $H^1(\mathcal{S}, \mathcal{S}) = 0$, then $H^1(A, A) = 0$.*

For presenting the main theorem of this section we need a Lemma.

Lemma 4.4. *Let A be a Banach algebra with a one-sided approximate identity, and let $T \in \mathrm{Hom}_A(A, A)$. Then T is automatically continuous.*

Proof. Let (e_i) be a left approximate identity for A (the other case is similar). Let $a, b \in A$, and let $(a_n) \subseteq A$ be a sequence such that $\lim_n a_n = a$ and $\lim_n T(a_n) = b$. We have

$$\begin{aligned} b &= \lim_i e_i b = \lim_i \lim_n e_i T(a_n) = \lim_i \lim_n T(e_i a_n) \\ &= \lim_i \lim_n T(e_i) a_n = \lim_i T(e_i) a = \lim_i T(e_i a) \\ &= \lim_i e_i T(a) = T(a). \end{aligned}$$

Therefore, the closed graph theorem [4, Theorem 5.12] establishes the continuity of T. $\square$

Theorem 4.5. *Let A be a Banach algebra with a one-sided approximate identity, $\mathcal{S} = A \oplus A$ and $H^1(A, A) = 0$. Then $\mathcal{S}$ has automatically continuous derivations if and only if A has automatically continuous derivations.*

Proof. One direction is an immediate consequence of Corollary 4.3. For the other, let $D : \mathcal{S} \to \mathcal{S}$ be a derivation. Then D is of the form (4.1). Let (e_i) be a left approximate identity for A (the other case is similar). By Lemma 4.4, T_1 is continuous. For $a \in A$ we have $2T_1(ae_i) = T_1(a)e_i + aT_1(e_i) = 0$, thus by letting $i \to \infty$ we obtain $T_1 = 0$. Since A has automatically continuous derivations, D_A and D_X are continuous and since $H^1(A, A) = 0$, we have $D_A = \mathrm{ad}_a$ and $D_X = \mathrm{ad}_y$ for some $a, y \in A$. Thus

$$D(b, x) = (\mathrm{ad}_a(b), \mathrm{ad}_y(b) + T_2(x)).$$

If we let

$$D_0(b, x) = (\mathrm{ad}_a(b), \mathrm{ad}_y(b) + \mathrm{ad}_a(x)) = \mathrm{ad}_{(a,y)}(b, x),$$

then D_0 is inner and therefore is continuous. Moreover, the derivation $\tilde{D} = D - D_0$ is of the following form:

$$\tilde{D}(b, x) = (0, T_2(x) - \mathrm{ad}_a(x)).$$

Obviously D is continuous if and only if $\tilde{D}$ is so and therefore $T : X \to X$ defined by $T(x) = T_2(x) - \mathrm{ad}_a(x)$ is continuous. For if $b, x \in A$ then

$$\begin{aligned} T(bx) &= T_2(bx) - \mathrm{ad}_a(bx) = bT_2(x) + \mathrm{ad}_a(b)x - \mathrm{ad}_a(bx) \\ &= bT_2(x) + (bax - abx) - (bxa - abx) \\ &= bT_2(x) + bax - bxa = bT_2(x) - b\mathrm{ad}_a(x) = bT(x), \end{aligned}$$

and

$$\begin{aligned} T(xb) &= T_2(xb) - \mathrm{ad}_a(xb) = T_2(x)b + x\mathrm{ad}_a(b) - \mathrm{ad}_a(xb) \\ &= T_2(x)b + (xba - xab) - (xba - axb) \\ &= T_2(x)b + axb - xab = T_2(x)b - \mathrm{ad}_a(x)b = T(x)b. \end{aligned}$$

Thus $T \in \mathrm{Hom}_A(A, A)$ and so by Lemma 4.4 is continuous. □

Corollary 4.6. *Let A be a unital, commutative, semisimple Banach algebra and $\mathcal{S} = A \oplus A$. Then every derivation $D : \mathcal{S} \to \mathcal{S}$ is continuous.*

Proof. This follows immediately from Theorem 4.5, since every derivation on A is continuous by [5] and [10]. □

Note that, as we have discussed in [7, Remark 2.1], there are large classes of Banach algebras A satisfy $H^1(A, A) = 0$. For example:

(i) If A is a von-Neumann algebra, a commutative C^*-algebra, a W^*-algebra or a simple unital C^*-algebra (i.e. A has no proper closed two-sided ideal.), then $H^1(A, A) = 0$ [8].

(ii) If A is a semi-simple commutative Banach algebra, then $H^1(A, A) = 0$ [10].

(iii) For a locally compact group G, $H^1(M(G), M(G)) = 0$ [6].

Proposition 4.7. *Suppose that $\mathcal{S} = A \oplus X$ has automatically continuous derivations. Then every derivation from A into X is automatically continuous.*

Proof. As in the proof of [7, Theorem 2.9] every derivation $D : A \to X$ can be lift to a derivation $\delta_D : \mathcal{S} \to \mathcal{S}$ defined by $\delta_D(a, x) = (0, D(a))$. Therefore, every derivation $D : A \to X$ is automatically continuous. $\square$

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References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, New York, 1973.

[2] H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, 2000.

[3] H. G. Dales, F. Ghahramani and N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), 19-54. MR 99g:46064

[4] G. B. Folland, *Real Analysis*, John Wiley, New York, 1999.

[5] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067-1073.

[6] V. Losert, *The derivation problem for group algebras*, Ann. of Math. **168** (2008), 221-246.

[7] A. R. Medghalchi and H. Pourmahmood-Aghababa, *The First Cohomology Group of Module Extension Banach Algebras*, to appear in the Rocky Mountain Journal of Mathematics.

[8] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin Heidelberg, New York, 1971.

[9] Yu. V. Selivanov, *Cohomological characterizations of biprojective and biflat Banach algebras*, Monatsh. Math. **128** (1999), 35-60.

[10] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260-264.

[11] Y. Zhang, *Weak Amenability of Module Extensions of Banach Algebras*, Trans. Amer. Math. Soc. **354** (2002), 4131-4151.

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