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THE UNIT SUM NUMBER OF DISCRETE MODULES

N. ASHRAFI* AND N. POYAN

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Abstract. In this paper, we show that every element of a discrete module is a sum of two units if and only if its endomorphism ring has no factor ring isomorphic to $\mathbb{Z}_2$. We also characterize unit sum number equal to two for the endomorphism ring of quasi-discrete modules with finite exchange property.

1. Introduction

The study of rings generated additively by their units seems to have its beginning in 1954 with the paper by Zelinsky [14] when he showed that if V is any (finite or infinite-dimensional) vector space over a division ring D, then every linear transformation is the sum of two automorphism unless $\dim V = 1$ and $D = \mathbb{Z}_2$ is the field of two elements. Interest in this topic increased recently after Goldsmith, Pabst and Scott defined the unit sum number in [4]. Zelinsky's result motivated Skornjakov to ask in [11, Problem 31, p. 167], if every element in a (von Neumann) regular ring R can be experessed as sum of fixed (and finite) number of units. Of course one needs to add some conditions ensuring that $\mathbb{Z}_2$ is not a factor ring (for example, $1/2 \in R$) to exclude the exceptional case already noted in the result of Zelinsky. V*á*mos in [13] showed that if R is such a regular ring, then R is $2-$ *good* (for the definition see

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*Corresponding author

[13]), if it is strongly regular and every element can indeed be written as a sum of a finite number of units, if R is (right) regular. Vámos also proved that every element of a regular right self-injective ring is a sum of two units, if the ring has no nonzero corner ring which is Boolean. Recently, Ashish and Dinesh in [6] proved that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$. They extended this result to endomorphism rings of right quasi-continuous modules with finite exchange property. We investigate whether these results are true for discrete modules.
Discrete modules were first studied by Takeuchi [12] and he called them codirect modules with condition (I) [2, see Remark 27.18]. Mohamed and Singh [9] studied direct projective and lifting modules and called them dual-continuous modules. They studied the basic properties and the endomorphism ring of a discrete module. Also, a decomposition theorem for discrete modules was obtained by Mohamed and Singh [9] and later improved by Mohamed and Müller [8]. In this paper we prove that every element of a discrete module is a sum of two units if and only if no factor ring of the endomorphism ring is isomorphic to $\mathbb{Z}_2$. Then, we extend this result to the endomorphism ring of quasi-discrete modules with finite exchange property.

2. **Definitions**

All rings R in this paper are assumed to be associative and will have an identity element. We say that R has the n-sum property, for a positive integer n, if every element of R can be written as a sum of exactly n units of R. The unit sum number of a ring, denoted by $usn(R)$, is the least integer n, if any such integer exists, such that R has the n-sum property. If R has an element that is not a sum of units, then we set $usn(R)$ to be ∞, and if every element of R is a sum of units but R does not have n-sum property, for any n, then we set $usn(R) = \omega$. Clearly, $usn(R) = 1$ if and only if R is the trivial ring with $0 = 1$. The unit sum number of a module M, denoted by $usn(M)$, is the unit sum number of its endomorphism ring.
A submodule A of a module M is called small in M (denoted by $A \ll M$), if $A + B \neq M$ for any proper submodule B of M. A module H is called hollow, if every proper submodule of H is small. Let A and B be submodules of M. B is called a supplement of A, if it is minimal with the property $A + B = M$. L is called a supplement submodule,

if L is a supplement of some submodule of M. A module M is called supplemented, if for any two submodules A and B with $A + B = M$, B contains a supplement of A. A supplemented module M is called strongly discrete, if it is self-projective.

Definition 2.1. *For a module M, consider the following conditions:*

(D_1) *For every submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ is small in M.*
(D_2) *If $A \leq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.*
(D_3) *If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_1$ is a summand of M.*

M is called discrete, if it has (D_1) and (D_2); M is called quasi-discrete, if it has (D_1) and (D_3).

Definition 2.2. *A module M is said to have the (finite) exchange property, if for any (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$, for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$, for submodules $B_i \leq A_i$. A module M is said to have the lifting property, if for any index set I and any submodule X of M, if $M/X = \oplus_{i \in I} A_i$, then there exists a decomposition $M = M_0 \oplus (\oplus_{i \in I} M_i)$ such that:*

(i) $M_0 \leq X$,
(ii) $\overline{M_i} = M/M_i = A_i$,
(iii) $X \cap (\oplus_{i \in I} M_i) \ll M$.

3. The unit sum number of discrete modules

For a ring R, $J(R)$ will denote the Jacobson radical of R. Before discussing the main results we need some properties of the unit sum number of rings and modules.

Lemma 3.1. *Let D be a division ring. If $|D| \geq 3$, then $usn(D) = 2$, whereas, if $|D| = 2$, that is, $D = \mathbb{Z}_2$ the field of two elements, then $usn(\mathbb{Z}_2) = \omega$.*

Proof. See [13, Lemma 2]. □

Lemma 3.2. *Let R be a ring and let I be an ideal of R. Then, $usn(R/I) \leq usn(R)$ with equality, if I is contained in the Jacobson radical of R.*

Proof. See [13, Lemma 2]. □

Remark 3.3. *From Lemma* 1 *and Lemma* 2 *it is clear that if* R *is a local ring which has no factor ring isomorphic to* $\mathbb{Z}_2$*, then* $usn(R) = 2$*.*

Lemma 3.4. *If the ring* R_i*, for every* $i \in I$*, has the* n*-sum property, then so has the ring direct product* $\prod_{i\in I} R_i$*.*

Proof. See [4, 1.2]. □

Lemma 3.5. *Let* R *be a nonzero Boolean ring with more than two elements. Then,* $usn(R) = \infty$*.*

Proof. Since a nonzero Boolean ring with more than two elements has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows. □

Theorem 3.6. *Let* M *be a discrete* R*-module and* $S = End_R(M)$*. The following conditions are equivalent:*

(1) Every element of S *is a sum of two units.*
(2) The identity element of S *is a sum of two units.*
(3) S *has no factor ring isomorphic to* $\mathbb{Z}_2$*.*

Proof. The results (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
Now, we show (3) $\Rightarrow$ (1).
We know that a discrete module M has a decomposition, unique up to isomorphism, $M = \oplus M_i$, where the M_is, have local endomorphism rings. Since S has no factor ring isomorphic to $\mathbb{Z}_2$ and $End_R(M) = End(\oplus_{i\in I} M_i) \cong \prod_{i\in I} End(M_i)$, none of $End(M_i)$ has a factor ring isomorphic to $\mathbb{Z}_2$. Let, for $i \in I$, $T_i = End(M_i)$. Thus, T_i is a local ring which has no factor ring isomorphic to $\mathbb{Z}_2$. Therefore, by Remark 3.3, for each i, $usn(T_i) = 2$. Now, by Lemma 3.4, it is clear that $usn(M) = 2$.

□

Let M be a projective R-module with lifting property, then [1] gives that M is supplemented. Hence, by [10, Lemma 2.3], M is a semi-perfect R-module and so by [7, Corollary 4.43] it is a discrete module. Also, if R is a perfect ring and M is a quasi-projective R-module, then by [5, Proposition 2.5] we know that M is again a discrete module. Therefore, we have:

Corollary 3.7. *Let* M *be an* R*-module and* $S = End_R(M)$*. If* M *is a strongly discrete module or a projective module with lifting property or if* R *is a perfect ring and* M *is a quasi-projective* R*-module, then* $usn(M) = 2$ *if and only if* S *has a factor ring isomorphic to* $\mathbb{Z}_2$*.*

Proof. As mentioned above, in all cases M is a discrete module and therefore the result follows at once from Theorem 8. $\square$

Theorem 3.8. *Let M be a nonzero discrete R-module and $S = End_R(M)$. Then, the unit sum number of M is 2, ω or ∞. Moreover,*

(1) $usn(M) = 2$ if and only if S has no factor ring isomorphic to a nonzero Boolean ring.
(2) $usn(M) \geq \omega$, if S has a factor ring isomorphic to $\mathbb{Z}_2$. Further, if S has a factor ring isomorphic to a nonzero Boolean ring with more than two elements, then $usn(M) = \infty$.

Proof. (1) Since $\mathbb{Z}_2$ is a homomorphic image of every nonzero Boolean ring, the result follows from Theorem 3.6.
(2) Let S have a factor ring isomorphic to $\mathbb{Z}_2$, i.e., $S/I \cong \mathbb{Z}_2$. But then, since $usn(S/I) \leq usn(S)$, it follows that $usn(S) \geq \omega$. Now, if S has a factor isomorphic to a nonzero Boolean ring with more than two elements, then Lemma 3.5 implies $usn(M) = \infty$. $\square$

Remark 3.9. *Note that in Theorem 8, if S has a factor ring isomorphic to $\mathbb{Z}_2$, then $usn(M) \geq \omega$. Further, if S has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $usn(M) = \infty$.*

Theorem 3.10. *Let M be a quasi-discrete R-module with finite exchange property and $S = End_R(M)$. Then, every element of S is a sum of two units if and only if no factor ring of S is isomorphic to $\mathbb{Z}_2$.*

Proof. Suppose that no factor ring of S is isomorphic to $\mathbb{Z}_2$. Let $\triangledown = \{f \in S : Imf \ll M\}$. It is easy to check that $\triangledown$ is an ideal of S. By [7, 5.7] $\overline{S} = S/\triangledown \cong S_1 \oplus S_2$, where S_1 is a regular ring and S_2 is a reduced ring. Moreover, $\overline{S}$ has no non-zero nilpotent element and every idempotent is central. Since $S/\triangledown$ has no nontrivial idempotent, S_1 has no nontrivial idempotent. Therefore, it is a division ring which has no factor ring isomorphic to $\mathbb{Z}_2$, so each element of S_1 is a sum of two units. Now, it is enough to show that every element of S_2, which has no factor ring isomorphic to $\mathbb{Z}_2$, is a sum of two units. Let $a \in S_2$ and suppose to the contrary that a is not a sum of two units.
Let $\Omega = \{I \mid I$ is an ideal of S_2 and $a + I$ is not a sum of two units in $S_2/I\}$.
Clearly, Ω is non-empty and it can be easily checked that Ω is inductive.

So, by Zorn's Lemma, Ω has a maximal element, say, I. Clearly, S_2/I is an indecomposable ring and hence has no central idempotent. But, S_2 is an exchange ring, so S_2/I is an exchange ring too. Since it has no central idempotent, it is clean and therefore S_2/I is a local ring. Let $T_2 = S_2/I$. Since $x = a+I$ is not a sum of two units in S_2/I, $x+J(T_2)$ is not a sum of two units in $T_2/J(T_2)$, which is a division ring. Therefore, $T_2/J(T_2) \cong \mathbb{Z}_2$, a contradiction. Hence, each element of S_2 is also a sum of two units. Therefore, every element of $\overline{S}$ is a sum of two units. Since $\nabla \subseteq J(S)$, we may conclude that every element of S is a sum of two units.
The converse is obvious. □

Corollary 3.11. *Let M be a R-module with indecomposable decomposition and with finite exchange property and $S = End_R(M)$. If no factor ring of S is isomorphic to $\mathbb{Z}_2$, then $usn(M) = 2$.*

Proof. By [3, Theorem 2.8] the endomorphism ring of M is a local ring. Thus, $S/J(S)$ is a division ring which has no factor ring isomorphic to $\mathbb{Z}_2$, so $usn(M) = 2$. □

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References

[1] C. Chang, Finitely generated modules over semilocal rings and characterizations of (semi-)perfect rings, *Kyungpook Math. J.* **48**(1) (2008) 143-154.

[2] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules*, Supplements and projectivity in module theory, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[3] A. Facchini, *Module Theory, Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules*, Progress in Mathematics, 167. Birkhäuser Verlag, Basel, 1998.

[4] B. Goldsmith; S. Pabst; A. Scott, Unit sum numbers of rings and modules, *Quart. J. Math. Oxford Ser*, (2) **49**(195) (1998) 331-344.

[5] S. K. Jain, S. R. López-Permouth and S. T. Rizvi, A characterization of uniserial rings via continuous and discrete modules, *J. Austral. Math. Soc. Ser. A* **50**(2) (1991) 197-203.

[6] D. Khurana and A. K. Srivastava, Right self-injective rings in which every element is a sum of two units, *J. Algebra Appl* **6**(2) (2007) 281-286.
[7] S. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Mathematical Society Lecture Note Series, 147, Cambridge University Press, Cambridge, 1990.
[8] S. Mohamed and B. J. Müller, *Decomposition of Dual-Continuous Modules*, Module theory (Proc. Special Session, Amer. Math. Soc. Univ. Washington, Seattle, Wash., 1977), pp. 87–94, Lecture Notes in Math., 700, Springer, Berlin, 1979.
[9] S. Mohamed and S. Singh, Generalizations of decomposition theorems known over perfect rings, *J. Austral. Math. Soc. Ser. A* **24**(4) (1977) 496-510.
[10] C. Nebiyev and A. Pancar, Strongly $\oplus$-supplemented modules, *International Journal Of Computational Cognition* **2** (2004) 57-61.
[11] L. A. Skornyakov, *Complemented Modular Lattices and Regular Rings*, Oliver & Boyd, Edinburgh-London 1964.
[12] T. Takeuchi, On cofinite-dimensional modules, *Hokkaido Math. J.* **5**(1) (1976) 1-43.
[13] P. Vámos, 2-good rings, *Q. J. Math* **56**(3) (2005) 417-430.
[14] D. Zelinsky, Every linear transformation is a sum of nonsingular ones, *Proc. Amer. Math. Soc* **5** (1954) 627-630.

Nahid Ashrafi
Department of Mathematics, Semnan University, Semnan, Iran
Email: nashrafi@semnan.ac.ir and ashrafi49@yahoo.com

Neda Poyan
Department of Mathematics, Semnan University, Semnan, Iran
Email: nedapouyan@yahoo.com and neda.pouyan@gmail.com