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THE UNIT SUM NUMBER OF DISCRETE MODULES

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ABSTRACT. In this paper, we show that every element of a discrete module is a sum of two units if and only if its endomorphism ring has no factor ring isomorphic to \mathbb{Z}_2 . We also characterize unit sum number equal to two for the endomorphism ring of quasi-discrete modules with finite exchange property.

1. Introduction

The study of rings generated additively by their units seems to have its beginning in 1954 with the paper by Zelinsky [14] when he showed that if V is any (finite or infinite-dimensional) vector space over a division ring D, then every linear transformation is the sum of two automorphism unless dim V = 1 and $D = \mathbb{Z}_2$ is the field of two elements. Interest in this topic increased recently after Goldsmith, Pabst and Scott defined the unit sum number in [4]. Zelinsky's result motivated Skornjakov to ask in [11, Problem 31, p. 167], if every element in a (von Neumann) regular ring R can be experessed as sum of fixed (and finite) number of units. Of course one needs to add some conditions ensuring that \mathbb{Z}_2 is not a factor ring (for example, $1/2 \in R$) to exclude the exceptional case already noted in the result of Zelinsky. Vámos in [13] showed that if R is such a regular ring, then R is 2 - good (for the definition see

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[13]), if it is strongly regular and every element can indeed be written as a sum of a finite number of units, if R is (right) regular. Vámos also proved that every element of a regular right self-injective ring is a sum of two units, if the ring has no nonzero corner ring which is Boolean. Recently, Ashish and Dinesh in [6] proved that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to \mathbb{Z}_2 . They extended this result to endomorphism rings of right quasi-continuous modules with finite exchange property. We investigate whether these results are true for discrete modules.

Discrete modules were first studied by Takeuchi [12] and he called them codirect modules with condition (I) [2, see Remark 27.18]. Mohamed and Singh [9] studied direct projective and lifting modules and called them dual-continuous modules. They studied the basic properties and the endomorphism ring of a discrete module. Also, a decomposition theorem for discrete modules was obtained by Mohamed and Singh [9] and later improved by Mohamed and Müller [8]. In this paper we prove that every element of a discrete module is a sum of two units if and only if no factor ring of the endomorphism ring is isomorphic to \mathbb{Z}_2 . Then, we extend this result to the endomorphism ring of quasi-discrete modules with finite exchange property.

2. Definitions

All rings R in this paper are assumed to be associative and will have an identity element. We say that R has the *n*-sum property, for a positive integer n, if every element of R can be written as a sum of exactly nunits of R. The unit sum number of a ring, denoted by usn(R), is the least integer n, if any such integer exists, such that R has the *n*-sum property. If R has an element that is not a sum of units, then we set usn(R) to be ∞ , and if every element of R is a sum of units but R does not have *n*-sum property, for any n, then we set $usn(R) = \omega$. Clearly, usn(R) = 1 if and only if R is the trivial ring with 0 = 1. The unit sum number of a module M, denoted by usn(M), is the unit sum number of its endomorphism ring.

A submodule A of a module M is called small in M (denoted by $A \ll M$), if $A + B \neq M$ for any proper submodule B of M. A module H is called hollow, if every proper submodule of H is small. Let A and B be submodules of M. B is called a supplement of A, if it is minimal with the property A + B = M. L is called a supplement submodule,

if L is a supplement of some submodule of M. A module M is called supplemented, if for any two submodules A and B with A + B = M, B contains a supplement of A. A supplemented module M is called strongly discrete, if it is self-projective.

Definition 2.1. For a module M, consider the following conditions:

- (D_1) For every submodule A of M, there exists a decomposition M = $M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ is small in M.
- (D_2) If $A \leq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.
- (D_3) If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_1$ is a summand of M.

M is called discrete, if it has (D_1) and (D_2) ; M is called quasi-discrete, if it has (D_1) and (D_3) .

Definition 2.2. A module M is said to have the (finite) exchange property, if for any (finite) index set I, whenever $M \oplus N = \bigoplus_{i \in I} A_i$, for modules N and A_i , then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$, for submodules $B_i \leq A_i$. A module M is said to have the lifting property, if for any index set I and any submodule X of M, if $M/X = \bigoplus_{i \in I} A_i$, then there exists a decomposition $M = M_0 \oplus (\bigoplus_{i \in I} M_i)$ such that:

- (i) $M_0 \leq X$, (ii) $\overline{M_i} = M/M_i = A_i$, (iii) $X \cap (\bigoplus_{i \in I} M_i) \ll M$.

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For a ring R, J(R) will denote the Jacobson radical of R. Before discussing the main results we need some properties of the unit sum number of rings and modules.

Lemma 3.1. Let D be a division ring. If $|D| \ge 3$, then usn(D) = 2, whereas, if |D| = 2, that is, $D = \mathbb{Z}_2$ the field of two elements, then $usn(\mathbb{Z}_2) = \omega.$

Proof. See [13, Lemma 2].

Lemma 3.2. Let R be a ring and let I be an ideal of R. Then, $usn(R/I) \leq$ usn(R) with equality, if I is contained in the Jacobson radical of R.

Proof. See [13, Lemma 2].

Remark 3.3. From Lemma 1 and Lemma 2 it is clear that if R is a local ring which has no factor ring isomorphic to \mathbb{Z}_2 , then usn(R) = 2.

Lemma 3.4. If the ring R_i , for every $i \in I$, has the n-sum property, then so has the ring direct product $\prod_{i \in I} R_i$.

Proof. See [4, 1.2].

Lemma 3.5. Let R be a nonzero Boolean ring with more than two elements. Then, $usn(R) = \infty$.

Proof. Since a nonzero Boolean ring with more than two elements has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows.

Theorem 3.6. Let M be a discrete R-module and $S = End_R(M)$. The following conditions are equivalent:

- (1) Every element of S is a sum of two units.
- (2) The identity element of S is a sum of two units.
- (3) S has no factor ring isomorphic to \mathbb{Z}_2 .

Proof. The results $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious. Now, we show $(3) \Rightarrow (1)$.

We know that a discrete module M has a decomposition, unique up to isomorphism, $M = \oplus M_i$, where the M_i s, have local endomorphism rings. Since S has no factor ring isomorphic to \mathbb{Z}_2 and $End_R(M) =$ $End(\bigoplus_{i \in I} M_i) \cong \prod_{i \in I} End(M_i)$, none of $End(M_i)$ has a factor ring isomorphic to \mathbb{Z}_2 . Let, for $i \in I$, $T_i = End(M_i)$. Thus, T_i is a local ring which has no factor ring isomorphic to \mathbb{Z}_2 . Therefore, by Remark 3.3, for each i, $usn(T_i) = 2$. Now, by Lemma 3.4, it is clear that usn(M) = 2.

Let M be a projective R-module with lifting property, then [1] gives that M is supplemented. Hence, by [10, Lemma 2.3], M is a semi-perfect R-module and so by [7, Corollary 4.43] it is a discrete module. Also, if R is a perfect ring and M is a quasi-projective R-module, then by [5, Proposition 2.5] we know that M is again a discrete module. Therefore, we have:

Corollary 3.7. Let M be an R-module and $S = End_R(M)$. If M is a strongly discrete module or a projective module with lifting property or if R is a perfect ring and M is a quasi-projective R-module, then usn(M) = 2 if and only if S has a factor ring isomorphic to \mathbb{Z}_2 .

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Proof. As mentioned above, in all cases M is a discrete module and therefore the result follows at once from Theorem 8.

Theorem 3.8. Let M be a nonzero discrete R-module and $S = End_R(M)$. Then, the unit sum number of M is 2, ω or ∞ . Moreover,

- (1) usn(M) = 2 if and only if S has no factor ring isomorphic to a nonzero Boolean ring.
- (2) $usn(M) \ge \omega$, if S has a factor ring isomorphic to \mathbb{Z}_2 . Further, if S has a factor ring isomorphic to a nonzero Boolean ring with more than two elements, then $usn(M) = \infty$.

Proof. (1) Since \mathbb{Z}_2 is a homomorphic image of every nonzero Boolean ring, the result follows from Theorem 3.6.

(2) Let S have a factor ring isomorphic to \mathbb{Z}_2 , i.e., $S/I \cong \mathbb{Z}_2$. But then, since $usn(S/I) \leq usn(S)$, it follows that $usn(S) \geq \omega$. Now, if S has a factor isomorphic to a nonzero Boolean ring with more than two elements, then Lemma 3.5 implies $usn(M) = \infty$.

Remark 3.9. Note that in Theorem 8, if S has a factor ring isomorphic to \mathbb{Z}_2 , then $usn(M) \ge \omega$. Further, if S has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $usn(M) = \infty$.

Theorem 3.10. Let M be a quasi-discrete R-module with finite exchange property and $S = End_R(M)$. Then, every element of S is a sum of two units if and only if no factor ring of S is isomorphic to \mathbb{Z}_2 .

Proof. Suppose that no factor ring of S is isomorphic to \mathbb{Z}_2 . Let $\nabla = \{f \in S : Imf \ll M\}$. It is easy to check that ∇ is an ideal of S. By [7, 5.7] $\overline{S} = S/\nabla \cong S_1 \oplus S_2$, where S_1 is a regular ring and S_2 is a reduced ring. Moreover, \overline{S} has no non-zero nilpotent element and every idempotent is central. Since S/∇ has no nontrivial idempotent, S_1 has no nontrivial idempotent. Therefore, it is a division ring which has no factor ring isomorphic to \mathbb{Z}_2 , so each element of S_1 is a sum of two units. Now, it is enough to show that every element of S_2 , which has no factor ring isomorphic to \mathbb{Z}_2 , is a sum of two units. Let $a \in S_2$ and suppose to the contrary that a is not a sum of two units.

Let $\Omega = \{I \mid I \text{ is an ideal of } S_2 \text{ and } a + I \text{ is not a sum of two units in } S_2/I\}.$

Clearly, Ω is non-empty and it can be easily checked that Ω is inductive.

So, by Zorn's Lemma, Ω has a maximal element, say, I. Clearly, S_2/I is an indecomposable ring and hence has no central idempotent. But, S_2 is an exchange ring, so S_2/I is an exchange ring too. Since it has no central idempotent, it is clean and therefore S_2/I is a local ring. Let $T_2 = S_2/I$. Since x = a+I is not a sum of two units in S_2/I , $x+J(T_2)$ is not a sum of two units in $T_2/J(T_2)$, which is a division ring. Therefore, $T_2/J(T_2) \cong \mathbb{Z}_2$, a contradiction. Hence, each element of S_2 is also a sum of two units. Therefore, every element of \overline{S} is a sum of two units. Since $\nabla \subseteq J(S)$, we may conclude that every element of S is a sum of two units.

The converse is obvious.

Corollary 3.11. Let M be a R-module with indecomposable decomposition and with finite exchange property and $S = End_R(M)$. If no factor ring of S is isomorphic to \mathbb{Z}_2 , then usn(M) = 2.

Proof. By [3, Theorem 2.8] the endomorphism ring of M is a local ring. Thus, S/J(S) is a division ring which has no factor ring isomorphic to \mathbb{Z}_2 , so usn(M) = 2.

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