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ONE-POINT EXTENSIONS OF LOCALLY COMPACT PARACOMPACT SPACES

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Abstract. A space Y is called an *extension* of a space X, if Y contains X as a dense subspace. Two extensions of X are said to be *equivalent*, if there is a homeomorphism between them which fixes X point-wise. For two (equivalence classes of) extensions Y and Y' of X let $Y \leq Y'$, if there is a continuous function of Y' into Y which fixes X point-wise. An extension Y of X is called a *one-point extension*, if $Y \backslash X$ is a singleton. An extension Y of X is called *first-countable*, if Y is first-countable at points of $Y \backslash X$. Let $\mathcal{P}$ be a topological property. An extension Y of X is called a $\mathcal{P}$-*extension*, if it has $\mathcal{P}$.

In this article, for a given locally compact paracompact space X, we consider the two classes of one-point Čech-complete; $\mathcal{P}$-extensions of X and one-point first-countable locally-$\mathcal{P}$ extensions of X, and we study their order-structures, by relating them to the topology of a certain subspace of the outgrowth $\beta X \backslash X$. Here $\mathcal{P}$ is subject to some requirements and include σ-compactness and the Lindelöf property as special cases.

1. Introduction

A space Y is called an *extension* of a space X, if Y contains X as a dense subspace. If Y is an extension of X, then the subspace $Y \backslash X$ of

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Y is called the *remainder* of Y. Extensions with a one-point remainder are called *one-point extensions*. Two extensions of X are said to be *equivalent*, if there exists a homeomorphism between them which fixes X point-wise. This defines an equivalence relation on the class of all extensions of X. The equivalence classes will be identified with individuals when this causes no confusion. For two extensions Y and Y' of X we let $Y \leq Y'$, if there exists a continuous function of Y' into Y which fixes X point-wise. The relation $\leq$ defines a partial order on the set of extensions of X (see Section 4.1 of [16] for more details). An extension Y of X is called *first-countable*, if Y is first-countable at points of $Y \backslash X$, that is, Y has a countable local base at every point of $Y \backslash X$. Let $\mathcal{P}$ be a topological property. An extension Y of X is called a *$\mathcal{P}$-extension*, if it has $\mathcal{P}$. If $\mathcal{P}$ is compactness, then $\mathcal{P}$-extensions are called *compactifications*.

This work was mainly motivated by our previous work [9] (see [1], [7], [8], [11], [12] and [13] for related results) in which we have studied the partially ordered set of one-point $\mathcal{P}$-extensions of a given locally compact space X by relating it to the topologies of certain subspaces of its outgrowth $\beta X \backslash X$. In this article, we continue our studies by considering the classes of one-point Čech-complete $\mathcal{P}$-extensions and one-point first-countable locally-$\mathcal{P}$ extensions of a given locally compact paracompact space X. The topological property $\mathcal{P}$ is subject to some requirements and include σ-compactness, the Lindelöf property and the linearly Lindelöf property as special cases.

We review some of the terminology, notation and well-known results that will be used in the sequel. Our definitions mainly come from the standard text [3] (thus, in particular, compact spaces are Hausdorff, etc.). Other useful sources are [5] and [16].

The letters $\mathbf{I}$ and $\mathbf{N}$ denote the closed unit interval and the set of all positive integers, respectively. For a subset A of a space X we let $\mathrm{cl}_X A$ and $\mathrm{int}_X A$ denote the closure and the interior of A in X, respectively. A subset of a space is called *clopen*, if it is simultaneously closed and open. A *zero-set* of a space X is a set of the form $Z(f) = f^{-1}(0)$ for some continuous $f : X \to \mathbf{I}$. Any set of the form $X \backslash Z$, where Z is a zero-set of X, is called a *cozero-set* of X. We denote the set of all zero-sets of X by $\mathscr{Z}(X)$ and the set of all cozero-sets of X by $Coz(X)$.

For a Tychonoff space X the *Stone-Čech compactification* of X is the largest (with respect to the partial order $\leq$) compactification of X and is denoted by βX. The Stone-Čech compactification of X can be characterized among all compactifications of X by either of the following properties:

(1) Every continuous function of X to a compact space is continuously extendible over βX.
(2) Every continuous function of X to $\mathbf{I}$ is continuously extendible over βX.
(3) For every $Z, S \in \mathscr{Z}(X)$ we have

$$\mathrm{cl}_{\beta X}(Z \cap S) = \mathrm{cl}_{\beta X} Z \cap \mathrm{cl}_{\beta X} S.$$

A Tychonoff space is called *zero-dimensional*, if it has an open base consisting of its clopen subsets. A Tychonoff space is called *strongly zero-dimensional*, if its Stone-Čech compactification is zero-dimensional. A Tychonoff space X is called *Čech-complete*, if its outgrowth $\beta X \backslash X$ is an F_σ in βX. Locally compact spaces are Čech-complete, and in the realm of metrizable spaces X, Čech-completeness is equivalent to the existence of a compatible complete metric on X.

Let $\mathcal{P}$ be a topological property. A topological space X is called *locally-$\mathcal{P}$*, if for every $x \in X$ there exists an open neighborhood U_x of x in X such that $\mathrm{cl}_X U_x$ has $\mathcal{P}$.

A topological property $\mathcal{P}$ is said to be *hereditary with respect to closed subsets*, if each closed subset of a space with $\mathcal{P}$ also has $\mathcal{P}$. A topological property $\mathcal{P}$ is said to be *preserved under finite (closed) sums of subspaces*, if a Hausdorff space has $\mathcal{P}$, provided that it is the union of a finite collection of its (closed) $\mathcal{P}$-subspaces.

Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets. A mapping $f : (P, \leq) \to (Q, \leq)$ is said to be an *order-homomorphism* (*anti-order-homomorphism*, respectively), if $f(a) \leq f(b)$ ($f(b) \leq f(a)$, respectively) whenever $a \leq b$. An order-homomorphism (anti-order-homomorphism, respectively) $f : (P, \leq) \to (Q, \leq)$ is said to be an *order-isomorphism* (*anti-order-isomorphism*, respectively), if $f^{-1} : (Q, \leq) \to (P, \leq)$ (exists and) is an order-homomorphism (anti-order-homomorphism, respectively). Two partially ordered sets $(P, \leq)$ and $(Q, \leq)$ are called *order-isomorphic* (*anti-order-isomorphic*, respectively), if there exists an order-isomorphism (anti-order-isomorphism, respectively) between them.

2. Motivations, notations and definitions

In this article we will be dealing with various sets of one-point extensions of a given topological space X. For the reader's convenience we list all these sets at the beginning.

Notation 2.1. Let X be a topological space. Denote

- $\mathscr{E}(X)=\{Y: Y \text{ is a one-point Tychonoff extension of } X\}$
- $\mathscr{E}^*(X)=\{Y\in\mathscr{E}(X): Y \text{ is first-countable at } Y\backslash X\}$
- $\mathscr{E}^C(X)=\{Y\in\mathscr{E}(X): Y \text{ is Čech-complete}\}$
- $\mathscr{E}^K(X)=\{Y\in\mathscr{E}(X): Y \text{ is locally compact}\}$

and when $\mathcal{P}$ is a topological property

- $\mathscr{E}_{\mathcal{P}}(X)=\{Y\in\mathscr{E}(X): Y \text{ has } \mathcal{P}\}$
- $\mathscr{E}_{local-\mathcal{P}}(X)=\{Y\in\mathscr{E}(X): Y \text{ is locally-}\mathcal{P}\}$.

Also, we may use notations which are obtained by combinations of the above notations, e.g.

$$\mathscr{E}^*_{local-\mathcal{P}}(X)=\mathscr{E}^*(X)\cap\mathscr{E}_{local-\mathcal{P}}(X).$$

Definition 2.2 ([10])**.** For a Tychonoff space X and a topological property $\mathcal{P}$, let

$$\lambda_{\mathcal{P}}X=\bigcup\big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}C: C\in Coz(X) \text{ and } \mathrm{cl}_X C \text{ has } \mathcal{P}\big\}.$$

Definition 2.3 ([14])**.** We say that a topological property $\mathcal{P}$ *satisfies Mrówka's condition* (W), if it satisfies the following: If X is a Tychonoff space in which there exists a point p with an open base $\mathscr{B}$ for X at p such that $X\backslash B$ has $\mathcal{P}$, for each $B\in\mathscr{B}$, then X has $\mathcal{P}$.

Mrówka's condition (W) is satisfied by a large number of topological properties; among them are (regularity +) the Lindelöf property, paracompactness, metacompactness, subparacompactness, the para-Lindelöf property, the σ-para-Lindelöf property, weak θ-refinability, θ-refinability (or submetacompactness), weak $\delta\theta$-refinability, $\delta\theta$-refinability (or the submeta-Lindelöf property), countable paracompactness, $[\theta,\kappa]$-compactness, κ-boundedness, screenability, σ-metacompactness, Dieudonné completeness, N-compactness [15], realcompactness, almost realcompactness [4] and zero-dimensionality (see [10], [12] and [13] for proofs and [2], [17] and [18] for definitions).

In [11] we have obtained the following result.

Theorem 2.4 ([11]). *Let X and Y be locally compact locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces where $\mathcal{P}$ is either pseudocompactness or a closed hereditary topological property which is preserved under finite closed sums of subspaces and satisfies Mrówka's condition* (W). *Then, the following are equivalent:*

(1) *$\lambda_{\mathcal{P}} X \backslash X$ and $\lambda_{\mathcal{P}} Y \backslash Y$ are homeomorphic.*
(2) *$(\mathscr{E}_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}_{\mathcal{P}}(Y), \leq)$ are order-isomorphic.*
(3) *$(\mathscr{E}^C_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^C_{\mathcal{P}}(Y), \leq)$ are order-isomorphic.*
(4) *$(\mathscr{E}^K_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^K_{\mathcal{P}}(Y), \leq)$ are order-isomorphic, provided that X and Y are moreover strongly zero-dimensional.*

There are topological properties, however, which do not satisfy the assumption of Theorem 2.4 (σ-compactness, for example, does not satisfy Mrówka's condition (W); see [10]). The purpose of this article is to prove the following version of Theorem 2.4. Specific topological properties $\mathcal{P}$ which satisfy the requirements of Theorem 2.5 below are σ-compactness, the Lindelöf property and the linearly Lindelöf property. Note that in Theorem 3.19 of [9] we have shown that conditions (1) and (3) of Theorem 2.5 are equivalent, if $\mathcal{P}$ is σ-compactness, and in Theorem 3.21 of [9] we have shown that conditions (1) and (2) of Theorem 2.5 are equivalent, if $\mathcal{P}$ is the Lindelöf property. Thus, in some sense, Theorem 2.5 generalizes Theorems 3.19 and 3.21 of [9], and at the same time, brings them under a same umbrella.

Theorem 2.5. *Let X and Y be locally compact paracompact spaces and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. Then, the following are equivalent:*

(1) *$\lambda_{\mathcal{P}} X \backslash X$ and $\lambda_{\mathcal{P}} Y \backslash Y$ are homeomorphic.*
(2) *$(\mathscr{E}^C_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^C_{\mathcal{P}}(Y), \leq)$ are order-isomorphic.*
(3) *$(\mathscr{E}^*_{local-\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^*_{local-\mathcal{P}}(Y), \leq)$ are order-isomorphic.*

We now introduce some notation which will be widely used in this article.

Notation 2.6. Let X be a Tychonoff space X. For a subset A of X denote

$$A^* = \mathrm{cl}_{\beta X} A \backslash X.$$

In particular, $X^* = \beta X \backslash X$.

Remark 2.7. *Note that the notation given in Notation 2.6 can be ambiguous, as A^* can mean either $\beta A \backslash A$ or $\mathrm{cl}_{\beta X} A \backslash X$. However, since for C^*-embedded subsets these two notions coincide, this will not cause any confusion.*

Definition 2.8 ([7])**.** For a Tychonoff space X, let

$$\sigma X = \bigcup \{\mathrm{cl}_{\beta X} H : H \subseteq X \text{ is } \sigma\text{-compact}\}.$$

Notation 2.9. Let X be a locally compact paracompact non-compact space. Then, X can be represented as

$$X = \bigoplus_{i \in I} X_i$$

for some index set I, with each X_i, for $i \in I$, being σ-compact and non-compact (see Theorem 5.1.27 and Exercise 3.8.C of [3]). For $J \subseteq I$ denote

$$X_J = \bigcup_{i \in J} X_i.$$

Thus, using the notation of 2.6, we have

$$X_J^* = \mathrm{cl}_{\beta X}\Big(\bigcup_{i \in J} X_i\Big) \backslash X.$$

Remark 2.10. *Note that in Notation 2.9 the set X_J^* is homeomorphic to $\beta X_J \backslash X_J$, as $\mathrm{cl}_{\beta X} X_J$ is homeomorphic to βX_J (see Corollary 3.6.8 of* [3]*). Thus, when J is countable (since X_J is σ-compact and locally compact) X_J^* is a zero-sets in $\mathrm{cl}_{\beta X} X_J$ (see 1B of* [19]*). But, $\mathrm{cl}_{\beta X} X_J$ is clopen in βX, as X_J is clopen in X (see Corollary 3.6.5 of* [3]*) therefore, X_J^* is a zero-set in βX. Also, note that with the notation given in 2.9, we have*

$$\sigma X = \bigcup \{\mathrm{cl}_{\beta X} X_J : J \subseteq I \textit{ is countable}\}.$$

Note that σX is open in βX and it contains X.

3. Partially ordered set of one-point extensions as related to topologies of subspaces of outgrowth

In Lemma 3.5 we establish a connection between one-point Tychonoff extensions of a given space X and compact non-empty subsets of its outgrowth X^*. Lemma 3.5 (and its preceding lemmas) is known (see e.g. [12]). It is included here for the sake of completeness.

Lemma 3.1. *Let X be a Tychonoff space and let C be a non-empty compact subset of X^*. Let T be the space which is obtained from βX by contracting C to a point p. Then, the subspace $Y = X \cup \{p\}$ of T is Tychonoff and $\beta Y = T$.*

Proof. Let $q : \beta X \to T$ be the quotient mapping. Note that T is Hausdorff, and thus, being a continuous image of βX, it is compact. Also, note that Y is dense in T. Therefore, T is a compactification of Y. To show that $\beta Y = T$, it suffices to verify that every continuous $h : Y \to \mathbf{I}$ is continuously extendable over T. Let $h : Y \to \mathbf{I}$ be continuous. Let $G : \beta X \to \mathbf{I}$ continuously extend $hq|(X \cup C) : X \cup C \to \mathbf{I}$ (note that $\beta(X \cup C) = \beta X$, as $X \subseteq X \cup C \subseteq \beta X$, see Corollary 3.6.9 of [3]). Define $H : T \to \mathbf{I}$ such that $H|(\beta X \backslash C) = G|(\beta X \backslash C)$ and $H(p) = h(p)$. Then, $H|Y = h$, and since $Hq = G$ is continuous, the function H is continuous.

Notation 3.2. Let X be a Tychonoff space and let $Y \in \mathscr{E}(X)$. Denote by

$$\tau_Y : \beta X \to \beta Y$$

the (unique) continuous extension of id_X.

Lemma 3.3. *Let X be a Tychonoff space and let $Y = X \cup \{p\} \in \mathscr{E}(X)$. Let T be the space which is obtained from βX by contracting $\tau_Y^{-1}(p)$ to the point p, and let $q : \beta X \to T$ be the quotient mapping. Then, $T = \beta Y$ and $\tau_Y = q$.*

Proof. We need to show that Y is a subspace of T. Since βY is also a compactification of X and $\tau_Y|X = \mathrm{id}_X$, by Theorem 3.5.7 of [3], we have $\tau_Y(X^*) = \beta Y \backslash X$. For an open subset W of βY, the set $q(\tau_Y^{-1}(W))$

is open in T, as $q^{-1}(q(\tau_Y^{-1}(W))) = \tau_Y^{-1}(W)$ is open in βX. Therefore,

$$Y \cap W = Y \cap q\big(\tau_Y^{-1}(W)\big)$$

is open in Y, when Y is considered as a subspace of T. For the converse, note that if V is open in T, since

$$Y \cap V = Y \cap \big(\beta Y \backslash \tau_Y\big(\beta X \backslash q^{-1}(V)\big)\big)$$

and $\tau_Y(\beta X \backslash q^{-1}(V))$ is compact and thus closed in βY, the set $Y \cap V$ is open in Y in its original topology. By Lemma 3.1 we have $T = \beta Y$. This also implies that $\tau_Y = q$, as $\tau_Y, q : \beta X \to \beta Y$ are continuous and coincide with id_X on the dense subset X of βX.

Lemma 3.4. *Let X be a Tychonoff space. Let $Y_i \in \mathscr{E}(X)$, for $i = 1, 2$, and denote by $\tau_i = \tau_{Y_i} : \beta X \to \beta Y_i$ the continuous extension of* id_X. *Then, the following are equivalent:*

(1) $Y_1 \leq Y_2$.
(2) $\tau_2^{-1}(Y_2 \backslash X) \subseteq \tau_1^{-1}(Y_1 \backslash X)$.

Proof. Let $Y_i = X \cup \{p_i\}$, for $i = 1, 2$. (1) *implies* (2). Suppose that (1) holds. By the definition, there exists a continuous $f : Y_2 \to Y_1$ such that $f|X = \mathrm{id}_X$. Let $f_\beta : \beta Y_2 \to \beta Y_1$ continuously extend f. Note that the continuous functions $f_\beta \tau_2, \tau_1 : \beta X \to \beta Y_1$ coincide with id_X on the dense subset X of βX, and thus $f_\beta \tau_2 = \tau_1$. Note that X is dense in βY_i (for $i = 1, 2$), as it is dense in Y_i, and therefore, βY_i is a compactification of X. Since $f_\beta | X = \mathrm{id}_X$, by Theorem 3.5.7 of [3], we have $f_\beta(\beta Y_2 \backslash X) = \beta Y_1 \backslash X$, and thus $f_\beta(p_2) \in \beta Y_1 \backslash X$. But, $f_\beta(p_2) = f(p_2)$, which implies that $f_\beta(p_2) \in Y_1 \backslash X = \{p_1\}$. Therefore,

$$\begin{aligned} \tau_2^{-1}(p_2) &\subseteq \tau_2^{-1}\big(f_\beta^{-1}\big(f_\beta(p_2)\big)\big) \\ &= (f_\beta \tau_2)^{-1}\big(f_\beta(p_2)\big) = \tau_1^{-1}\big(f_\beta(p_2)\big) = \tau_1^{-1}(p_1). \end{aligned}$$

(2) *implies* (1). Suppose that (2) holds. Let $f : Y_2 \to Y_1$ be defined such that $f(p_2) = p_1$ and $f|X = \mathrm{id}_X$. We show that f is continuous, this will show that $Y_1 \leq Y_2$. Note that by Lemma 3.3, the space βY_2 is the quotient space of βX which is obtained by contracting $\tau_2^{-1}(p_2)$ to a point, and τ_2 is its corresponding quotient mapping. Thus, in particular, Y_2 is the quotient space of $X \cup \tau_2^{-1}(p_2)$, and therefore, to show that f is continuous, it suffices to show that $f\tau_2 | (X \cup \tau_2^{-1}(p_2))$ is continuous. We show this by verifying that $f\tau_2(t) = \tau_1(t)$, for each $t \in X \cup \tau_2^{-1}(p_2)$. This obviously holds if $t \in X$. If $t \in \tau_2^{-1}(p_2)$, then $\tau_2(t) = p_2$, and

thus $f\tau_2(t) = p_1$. But, since $t \in \tau_2^{-1}(\tau_2(t))$, we have $t \in \tau_1^{-1}(p_1)$, and therefore $\tau_1(t) = p_1$. Thus, $f\tau_2(t) = \tau_1(t)$ in this case as well.

Lemma 3.5. *Let X be a Tychonoff space. Define a function*

$$\Theta : \big(\mathscr{E}(X), \leq\big) \to \big(\{C \subseteq X^* : C \text{ is compact}\}\backslash\{\emptyset\}, \subseteq\big)$$

by

$$\Theta(Y) = \tau_Y^{-1}(Y\backslash X),$$

for $Y \in \mathscr{E}(X)$. Then, Θ is an anti-order-isomorphism.

Proof. To show that Θ is well-defined, let $Y \in \mathscr{E}(X)$. Note that since X is dense in Y, the space X is dense in βY. Thus, $\tau_Y : \beta X \to \beta Y$ is onto, as $\tau_Y(\beta X)$ is a compact (and therefore closed) subset of βY and it contains $X = \tau_Y(X)$. Thus, $\tau_Y^{-1}(Y\backslash X) \neq \emptyset$. Also, since $\tau_Y|X = \mathrm{id}_X$ we have $\tau_Y^{-1}(Y\backslash X) \subseteq X^*$, and since the singleton $Y\backslash X$ is closed in βY, its inverse image $\tau_Y^{-1}(Y\backslash X)$ is closed in βX, and therefore it is compact. Now, we show that Θ is onto, Lemma 3.4 will then complete the proof. Let C be a non-empty compact subset of X^*. Let T be the quotient space of βX which is obtained by contracting C to a point p. Consider the subspace $Y = X \cup \{p\}$ of T. Then, $Y \in \mathscr{E}(X)$, and thus, by Lemma 3.1 we have $\beta Y = T$. The quotient mapping $q : \beta X \to T$ is identical to τ_Y, as it coincides with id_X on the dense subset X of βX. Therefore,

$$\Theta(Y) = \tau_Y^{-1}(p) = q^{-1}(p) = C.$$

Notation 3.6. For a Tychonoff space X denote by

$$\Theta_X : \big(\mathscr{E}(X), \leq\big) \to \big(\{C \subseteq X^* : C \text{ is compact}\}\backslash\{\emptyset\}, \subseteq\big)$$

the anti-order-isomorphism defined by

$$\Theta_X(Y) = \tau_Y^{-1}(Y\backslash X),$$

for $Y \in \mathscr{E}(X)$.

Lemmas 3.7 and 3.8 below are known results (see [9]).

Lemma 3.7. *Let X be a Tychonoff space. For $Y \in \mathscr{E}(X)$ the following are equivalent:*

(1) $Y \in \mathscr{E}^*(X)$.
(2) $\Theta_X(Y) \in \mathscr{Z}(\beta X)$.

Proof. Let $Y = X \cup \{p\}$. (1) *implies* (2). Suppose that (1) holds. Let $\{V_n : n \in \mathbf{N}\}$ be an open base at p in Y. For each $n \in \mathbf{N}$, let V'_n be an open subset of βY such that $Y \cap V'_n = V_n$, and let $f_n : \beta Y \to \mathbf{I}$ be continuous and such that $f_n(p) = 0$ and $f_n(\beta Y \backslash V'_n) \subseteq \{1\}$. Let

$$Z = \bigcap_{n=1}^{\infty} Z(f_n) \in \mathscr{Z}(\beta Y).$$

We show that $Z = \{p\}$. Obviously, $p \in Z$. Let $t \in Z$ and suppose to the contrary that $t \neq p$. Let W be an open neighborhood of p in βY such that $t \notin \mathrm{cl}_{\beta Y} W$. Then, $Y \cap W$ is an open neighborhood of p in Y. Let $k \in \mathbf{N}$ be such that $V_k \subseteq Y \cap W$. We have

$$\begin{aligned} t \in Z(f_k) \subseteq V'_k &\subseteq \mathrm{cl}_{\beta Y} V'_k \\ &= \mathrm{cl}_{\beta Y}(Y \cap V'_k) \\ &= \mathrm{cl}_{\beta Y} V_k \subseteq \mathrm{cl}_{\beta Y}(Y \cap W) \subseteq \mathrm{cl}_{\beta Y} W \end{aligned}$$

which is a contradiction. This shows that $t = p$ and therefore $Z \subseteq \{p\}$. Thus, $\{p\} = Z \in \mathscr{Z}(\beta Y)$, which implies that $\tau_Y^{-1}(p) \in \mathscr{Z}(\beta X)$.

(2) *implies* (1). Suppose that (2) holds. Let $\tau_Y^{-1}(p) = Z(f)$ where $f : \beta X \to \mathbf{I}$ is continuous. Note that by Lemma 3.3 the space βY is obtained from βX by contracting $\tau_Y^{-1}(p)$ to p with $\tau_Y : \beta X \to \beta Y$ as the quotient mapping. Then, for each $n \in \mathbf{N}$, the set $\tau_Y(f^{-1}([0, 1/n)))$ is an open neighborhood of p in βY. We show that the collection

$$\Big\{ Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{n}\Big)\Big)\Big) : n \in \mathbf{N} \Big\}$$

of open neighborhoods of p in Y constitutes an open base at p in Y. This will show (1). Let V be an open neighborhood of p in Y. Let V' be an open subset of βY such that $Y \cap V' = V$. Then, $p \in V'$ and thus

$$\bigcap_{n=1}^{\infty} f^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) = Z(f) = \tau_Y^{-1}(p) \subseteq \tau_Y^{-1}(V').$$

By compactness we have $f^{-1}([0, 1/k]) \subseteq \tau_Y^{-1}(V')$, for some $k \in \mathbf{N}$. Therefore,

$$\begin{aligned} Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{k}\Big)\Big)\Big) &\subseteq Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) \\ &\subseteq Y \cap \tau_Y\big(\tau_Y^{-1}(V')\big) \subseteq Y \cap V' = V. \end{aligned}$$

Lemma 3.8. *Let X be a locally compact space. For $Y \in \mathscr{E}(X)$ the following are equivalent:*

(1) $Y \in \mathscr{E}^C(X)$.
(2) $\Theta_X(Y) \in \mathscr{Z}(X^*)$.

Proof. Let $Y = X \cup \{p\}$. (1) *implies* (2). Suppose that (1) holds. Then, Y^* is an F_σ in βY. Let $Y^* = \bigcup_{n=1}^\infty K_n$ where each K_n is closed in βY, for $n \in \mathbf{N}$. Then,

$$X^* = \tau_Y^{-1}(p) \cup \bigcup_{n=1}^\infty K_n$$

(recall that βY is the quotient space of βX which is obtained by contracting $\tau_Y^{-1}(p)$ to p and τ_Y is its quotient mapping; see Lemma 3.3). For each $n \in \mathbf{N}$, let $f_n : \beta X \to \mathbf{I}$ be continuous and such that

$$f_n\big(\tau_Y^{-1}(p)\big) = \{0\} \text{ and } f_n(K_n) \subseteq \{1\}.$$

Let $f = \sum_{n=1}^\infty f_n/2^n$. Then, $f : \beta X \to \mathbf{I}$ is continuous and

$$\tau_Y^{-1}(p) = Z(f) \cap X^* \in \mathscr{Z}(X^*).$$

(2) *implies* (1). Suppose that (2) holds. Let $\tau_Y^{-1}(p) = Z(g)$ where $g : X^* \to \mathbf{I}$ is continuous. Then, using Lemma 3.3, we have

$$\begin{aligned} Y^* = X^* \backslash \tau_Y^{-1}(p) &= X^* \backslash Z(g) \\ &= g^{-1}\big((0,1]\big) = \bigcup_{n=1}^\infty g^{-1}\Big(\Big[\frac{1}{n}, 1\Big]\Big) \end{aligned}$$

and each set $g^{-1}([1/n, 1])$, for $n \in \mathbf{N}$, being closed in X^*, is compact (note that since X is locally compact, X^* is compact) and thus closed in βY. Therefore, Y^* is an F_σ in βY, that is, Y is Čech-complete.

Then, the following lemma justifies our requirement on $\mathcal{P}$ in Theorem 3.16. We simply need $\lambda_{\mathcal{P}} X$ to have a more familiar structure.

Lemma 3.9. *Let $\mathcal{P}$ be a topological property which is preserved under finite closed sums of subspaces. The following are equivalent:*

(1) *The topological property $\mathcal{P}$ coincides with σ-compactness in the realm of locally compact paracompact spaces.*
(2) *For every locally compact paracompact space X we have*

$$\lambda_{\mathcal{P}} X = \sigma X.$$

Proof. (1) *implies* (2). Suppose that (1) holds. Let X be a locally compact paracompact space. Assume the notation of 2.9. Let $J \subseteq I$ be countable. Then, X_J is σ-compact and thus (since it is also locally compact and paracompact) it has $\mathcal{P}$. Note that X_J is clopen in X thus it has a clopen closure in βX, therefore

$$\mathrm{cl}_{\beta X} X_J = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} X_J \subseteq \lambda_{\mathcal{P}} X$$

that is, $\sigma X \subseteq \lambda_{\mathcal{P}} X$. To see the reverse inclusion, let $C \in Coz(X)$ be such that $\mathrm{cl}_X C$ has $\mathcal{P}$. Then, (since $\mathrm{cl}_X C$ being closed in X is also locally compact and paracompact) $\mathrm{cl}_X C$ is σ-compact. Therefore,

$$\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C \subseteq \mathrm{cl}_{\beta X} C \subseteq \sigma X$$

which shows that $\lambda_{\mathcal{P}} X \subseteq \sigma X$. Thus, $\lambda_{\mathcal{P}} X = \sigma X$.

(2) *implies* (1). Suppose that (2) holds. Let X be a locally compact paracompact space. By the assumption we have $\lambda_{\mathcal{P}} X = \sigma X$. We verify that X has $\mathcal{P}$ if and only if X is σ-compact. Assume the notation of Notation 2.9. Suppose that X has $\mathcal{P}$. Then, $\lambda_{\mathcal{P}} X = \beta X$ and thus $\sigma X = \beta X$. Now, by compactness, we have

$$\beta X = \mathrm{cl}_{\beta X} X_{J_1} \cup \cdots \cup \mathrm{cl}_{\beta X} X_{J_n},$$

for some $n \in \mathbf{N}$ and some countable $J_1, \ldots, J_n \subseteq I$. Therefore,

$$X = X_{J_1} \cup \cdots \cup X_{J_n}$$

is σ-compact. For the converse, suppose that X is σ-compact. Then, $\sigma X = \beta X$ and (since $\lambda_{\mathcal{P}} X = \sigma X$) we have $\beta X = \lambda_{\mathcal{P}} X$. Thus, by compactness, we have

$$\beta X = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C_1 \cup \cdots \cup \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} C_n,$$

for some $n \in \mathbf{N}$ and some $C_1, \ldots, C_n \in Coz(X)$ such that $\mathrm{cl}_X C_i$ has $\mathcal{P}$, for $i = 1, \ldots, n$. Now, using our assumption, the space

$$X = \mathrm{cl}_X C_1 \cup \cdots \cup \mathrm{cl}_X C_n$$

being a finite union of its closed $\mathcal{P}$-subspaces, has $\mathcal{P}$.

Lemma 3.10. *Let X be a locally compact paracompact space and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. For $Y \in \mathscr{E}(X)$ the following are equivalent:*

(1) $Y \in \mathscr{E}^C_{\mathcal{P}}(X)$.
(2) $\Theta_X(Y) \in \mathscr{Z}(X^*)$ *and* $\beta X \backslash \lambda_{\mathcal{P}} X \subseteq \Theta_X(Y)$.

Thus, in particular

$$\Theta_X\big(\mathscr{E}^C_{\mathcal{P}}(X)\big)=\big\{Z\in\mathscr{Z}(X^*):\beta X\backslash\lambda_{\mathcal{P}}X\subseteq Z\big\}\backslash\{\emptyset\}.$$

Proof. Let $Y=X\cup\{p\}$. (1) *implies* (2). Suppose that (1) holds. By Lemma 3.8 we have $\tau_Y^{-1}(p)\in\mathscr{Z}(X^*)$. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}}X=\sigma X$. Let $t\in\beta X\backslash\sigma X$ and suppose to the contrary that $t\notin\tau_Y^{-1}(p)$. Let $f:\beta X\to\mathbf{I}$ be continuous and such that $f(t)=0$ and $f(\tau_Y^{-1}(p))=\{1\}$. Since $\tau_Y(f^{-1}([0,1/2]))$ is compact, the set

$$T=X\cap f^{-1}\Big(\Big[0,\frac{1}{2}\Big]\Big)=Y\cap\tau_Y\Big(f^{-1}\Big(\Big[0,\frac{1}{2}\Big]\Big)\Big)$$

being closed in Y, has $\mathcal{P}$. But, T, being closed in X, is locally compact and paracompact, and thus, having $\mathcal{P}$, it is σ-compact. Therefore, by definition of σX we have $\mathrm{cl}_{\beta X}T\subseteq\sigma X$. But, since

$$\begin{aligned}t\in f^{-1}\Big(\Big[0,\frac{1}{2}\Big)\Big) &\subseteq \mathrm{cl}_{\beta X}f^{-1}\Big(\Big[0,\frac{1}{2}\Big)\Big)\\ &= \mathrm{cl}_{\beta X}\Big(X\cap f^{-1}\Big(\Big[0,\frac{1}{2}\Big)\Big)\Big)\\ &\subseteq \mathrm{cl}_{\beta X}\Big(X\cap f^{-1}\Big(\Big[0,\frac{1}{2}\Big]\Big)\Big)=\mathrm{cl}_{\beta X}T\end{aligned}$$

we have $t\in\sigma X$, which contradicts the choice of t. Thus, $t\in\tau_Y^{-1}(p)$ and therefore $\beta X\backslash\sigma X\subseteq\tau_Y^{-1}(p)$.

(2) *implies* (1). Suppose that (2) holds. Note that since X is locally compact, the set X^* is closed in (the normal space) βX and thus, since $\tau_Y^{-1}(p)\in\mathscr{Z}(X^*)$ (using the Tietze-Urysohn Theorem) we have $\tau_Y^{-1}(p)=Z\cap X^*$, for some $Z\in\mathscr{Z}(\beta X)$. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}}X=\sigma X$. Now, since $\beta X\backslash\sigma X\subseteq\tau_Y^{-1}(p)\subseteq Z$ we have $\beta X\backslash Z\subseteq\sigma X$. Therefore, assuming the notation of 2.9 (since $\beta X\backslash Z$, being a cozero-set in βX, is σ-compact) we have

$$\beta X\backslash Z\subseteq\bigcup_{n=1}^{\infty}\mathrm{cl}_{\beta X}X_{J_n}\subseteq\mathrm{cl}_{\beta X}X_J$$

where $J_1,J_2,\ldots\subseteq I$ are countable and $J=J_1\cup J_2\cup\cdots$. But,

$$Y=\tau_Y(Z)\cup(X\backslash Z)\subseteq\tau_Y(Z)\cup X_J$$

and thus we have

$$Y=\tau_Y(Z)\cup X_J. \tag{3.1}$$

Now, since X_J has $\mathcal{P}$, as it is σ-compact (and being closed in X, it is locally compact and paracompact) and $\tau_Y(Z)$ has $\mathcal{P}$, as it is compact, from (3.1) it follows that the space Y, being a finite union of its $\mathcal{P}$-subspaces, has $\mathcal{P}$. The fact that Y is Čech-complete follows from Lemma 3.8.

The following generalizes Lemma 3.18 of [9].

Lemma 3.11. *Let X be a locally compact paracompact space and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. For $Y \in \mathscr{E}(X)$ the following are equivalent:*

(1) $Y \in \mathscr{E}^*_{local-\mathcal{P}}(X)$.
(2) $\Theta_X(Y) \in \mathscr{Z}(\beta X)$ *and* $\Theta_X(Y) \subseteq \lambda_{\mathcal{P}} X$.

Thus, in particular

$$\Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big) = \big\{Z \in \mathscr{Z}(\beta X) : Z \subseteq \lambda_{\mathcal{P}} X \backslash X\big\} \backslash \{\emptyset\}.$$

Proof. Let $Y = X \cup \{p\}$. (1) *implies* (2). Suppose that (1) holds. Since $Y \in \mathscr{E}^*(X)$, by Lemma 3.7 we have $\tau_Y^{-1}(p) \in \mathscr{Z}(\beta X)$. Let $\tau_Y^{-1}(p) = Z(f)$, for some continuous $f : \beta X \to \mathbf{I}$. Since Y is locally-$\mathcal{P}$, there exists an open neighborhood V of p in Y such that $\mathrm{cl}_Y V$ has $\mathcal{P}$. Let V' be an open subset of βY such that $Y \cap V' = V$. Then, $p \in V'$, and thus since

$$\bigcap_{n=1}^{\infty} f^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) = Z(f) = \tau_Y^{-1}(p) \subseteq \tau_Y^{-1}(V')$$

by compactness, we have $f^{-1}([0, 1/k]) \subseteq \tau_Y^{-1}(V')$, for some $k \in \mathbf{N}$. Now, for each $n \geq k$, since

$$\begin{aligned} Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) \backslash f^{-1}\Big(\Big[0, \frac{1}{n+1}\Big)\Big)\Big) &\subseteq Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) \\ &\subseteq Y \cap \tau_Y\big(\tau_Y^{-1}(V')\big) \\ &\subseteq Y \cap V' = V \subseteq \mathrm{cl}_Y V \end{aligned}$$

the set

$$\begin{aligned} K_n &= X \cap \Big(f^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) \backslash f^{-1}\Big(\Big[0, \frac{1}{n+1}\Big)\Big)\Big) \\ &= Y \cap \tau_Y\Big(f^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) \backslash f^{-1}\Big(\Big[0, \frac{1}{n+1}\Big)\Big)\Big) \end{aligned}$$

being closed in $\mathrm{cl}_Y V$, has $\mathcal{P}$, and therefore (since being closed in X it is locally compact and paracompact) it is σ-compact. (It might be helpful to recall that by Lemma 3.3 the space βY is obtained from βX by contracting $\tau_Y^{-1}(p)$ to p with τ_Y as its quotient mapping.) Thus, the set

$$X \cap f^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big) = \bigcup_{n=k}^{\infty} K_n$$

is σ-compact, and therefore, by the definition of σX, we have

$$\mathrm{cl}_{\beta X}\Big(X \cap f^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) \subseteq \sigma X.$$

But,

$$\begin{aligned} Z(f) \subseteq f^{-1}\Big(\Big[0, \frac{1}{k}\Big)\Big) &\subseteq \mathrm{cl}_{\beta X} f^{-1}\Big(\Big[0, \frac{1}{k}\Big)\Big) \\ &= \mathrm{cl}_{\beta X}\Big(X \cap f^{-1}\Big(\Big[0, \frac{1}{k}\Big)\Big)\Big) \\ &\subseteq \mathrm{cl}_{\beta X}\Big(X \cap f^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) \end{aligned}$$

from which it follows that $\tau_Y^{-1}(p) \subseteq \sigma X$. Finally, note that by Lemma 3.9 we have $\lambda_{\mathcal{P}} X = \sigma X$.

(2) *implies* (1). Suppose that (2) holds. By Lemma 3.7 we have $Y \in \mathscr{E}^*(X)$. Therefore, it suffices to verify that Y is locally-$\mathcal{P}$. Also, since by the assumption X is locally compact, it is locally-$\mathcal{P}$, as $\mathcal{P}$ is assumed to be a topological property of compact spaces. Thus, we only need to verify that p has an open neighborhood in Y whose closure in Y has $\mathcal{P}$. Let $g : \beta X \to \mathbf{I}$ be continuous and such that $Z(g) = \tau_Y^{-1}(p)$. Then, since

$$\bigcap_{n=1}^{\infty} g^{-1}\Big(\Big[0, \frac{1}{n}\Big]\Big) = Z(g) \subseteq \lambda_{\mathcal{P}} X$$

by compactness (and since $\lambda_{\mathcal{P}} X$ is open in βX) we have $g^{-1}([0, 1/k]) \subseteq \lambda_{\mathcal{P}} X$, for some $k \in \mathbf{N}$. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}} X = \sigma X$. Assume the notation of Notation 2.9. By compactness, we have

$$g^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big) \subseteq \mathrm{cl}_{\beta X} X_{J_1} \cup \cdots \cup \mathrm{cl}_{\beta X} X_{J_n} = \mathrm{cl}_{\beta X} X_J$$

where $n \in \mathbf{N}$, the sets $J_1, \ldots, J_n \subseteq I$ are countable and $J = J_1 \cup \cdots \cup J_n$. The set $X \cap g^{-1}([0, 1/k]) \subseteq X_J$, being closed in the latter (σ-compact space) is σ-compact, and therefore (since being closed in X, it is locally compact and paracompact) it has $\mathcal{P}$. Let

$$V = Y \cap \tau_Y\Big(g^{-1}\Big(\Big[0, \frac{1}{k}\Big)\Big)\Big).$$

Then, V is an open neighborhood of p in Y. We show that $\mathrm{cl}_Y V$ has $\mathcal{P}$. But, this follows, since

$$\begin{aligned}\mathrm{cl}_Y V \subseteq Y \cap \tau_Y\Big(g^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) &= \Big(X \cap \tau_Y\Big(g^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big)\Big) \cup \{p\} \\ &= \Big(X \cap g^{-1}\Big(\Big[0, \frac{1}{k}\Big]\Big)\Big) \cup \{p\}\end{aligned}$$

and the latter, being a finite union of its $\mathcal{P}$-subspaces (note that the singleton $\{p\}$, being compact, has $\mathcal{P}$) has $\mathcal{P}$, and thus, its closed subset $\mathrm{cl}_Y V$, also has $\mathcal{P}$.

Lemmas 3.12–3.14 are from [8].

Lemma 3.12. *Let X be a locally compact paracompact space. If $Z \in \mathscr{Z}(\beta X)$ in non-empty, then $Z \cap \sigma X \neq \emptyset$*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in σX. Assume the notation of 2.9. Then, $\{x_n : n \in \mathbf{N}\} \subseteq \mathrm{cl}_{\beta X} X_J$, for some countable $J \subseteq I$. Therefore, $\{x_n : n \in \mathbf{N}\}$ has a limit point in $\mathrm{cl}_{\beta X} X_J \subseteq \sigma X$. Thus, σX is countably compact, and therefore is pseudocompact, and $\upsilon(\sigma X) = \beta(\sigma X) = \beta X$ (note that the latter equality holds, as $X \subseteq \sigma X \subseteq \beta X$). The result now follows, as for any Tychonoff space T, any non-empty zero-set of υT meets T (see Lemma 5.11 (f) of [16]).

Lemma 3.13. *Let X be a locally compact paracompact space. If $Z \in \mathscr{Z}(X^*)$ is non-empty, then $Z \cap \sigma X \neq \emptyset$.*

Proof. Let $S \in \mathscr{Z}(\beta X)$ be such that $S \cap X^* = Z$ (which exists, as X^* is closed in (the normal space) βX, as X is locally compact, and thus, by the Tietze-Urysohn Theorem, every continuous function from X^* to $\mathbf{I}$ is continuously extendible over βX). By Lemma 3.12 we have $S \cap \sigma X \neq \emptyset$. Suppose that $S \cap (\sigma X \backslash X) = \emptyset$. Then, $S \cap \sigma X = X \cap S$. Assume the notation of 2.9. Let $J = \{i \in I : X_i \cap S \neq \emptyset\}$. Then, J is finite. Note that since X_J is clopen in X, it has a clopen closure in βX. Now,

$$T = S \cap (\beta X \backslash \mathrm{cl}_{\beta X} X_J) \in \mathscr{Z}(\beta X)$$

misses σX, and therefore, by Lemma 3.12 we have $T = \emptyset$. But, this is a contradiction, as $Z = S \cap (\beta X \backslash \sigma X) \subseteq T$. This shows that

$$Z \cap (\sigma X \backslash X) = S \cap (\sigma X \backslash X) \neq \emptyset.$$

Lemma 3.14. *Let X be a locally compact paracompact space. For $S, T \in \mathscr{Z}(X^*)$, if $S \cap \sigma X \subseteq T \cap \sigma X$, then $S \subseteq T$.*

Proof. Suppose to the contrary that $S \backslash T \neq \emptyset$, let $s \in S \backslash T$. Let $f : \beta X \to \mathbf{I}$ be continuous and such that $f(s) = 0$ and $f(T) \subseteq \{1\}$. Then, $Z(f) \cap S$ is non-empty, and thus by Lemma 3.13 it follows that $Z(f) \cap S \cap \sigma X \neq \emptyset$. But, this is not possible, as

$$Z(f) \cap S \cap \sigma X \subseteq Z(f) \cap T = \emptyset.$$

The following lemma is from [9].

Lemma 3.15. *Let X and Y be locally compact spaces. The following are equivalent:*

(1) *X^* and Y^* are homeomorphic.*
(2) *$(\mathscr{E}^C(X), \leq)$ and $(\mathscr{E}^C(Y), \leq)$ are order-isomorphic.*

Proof. This follows from the fact that in a compact space the order-structure of the set of its all zero-sets (partially ordered with $\subseteq$) determines its topology.

The proof of the following theorem is essentially a combination of the proofs we have given for Theorems 3.19 and 3.21 in [9] with the appropriate usage of the preceding lemmas. The reasonably detailed proof is included here for the reader's convenience.

Theorem 3.16. *Let X and Y be locally compact paracompact (non-compact) spaces and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. Then, the following are equivalent:*

(1) *$\lambda_{\mathcal{P}} X \backslash X$ and $\lambda_{\mathcal{P}} Y \backslash Y$ are homeomorphic.*
(2) *$(\mathscr{E}^C_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^C_{\mathcal{P}}(Y), \leq)$ are order-isomorphic.*
(3) *$(\mathscr{E}^*_{local-\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^*_{local-\mathcal{P}}(Y), \leq)$ are order-isomorphic.*

Proof. Let

$$X = \bigoplus_{i\in I} X_i \text{ and } Y = \bigoplus_{j\in J} Y_j,$$

for some index sets I and J with each X_i and Y_j, for $i \in I$ and $j \in J$ being σ-compact and non-compact. We will use notation of 2.9 and Remark 2.10 without mentioning. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}} X = \sigma X$ and $\lambda_{\mathcal{P}} Y = \sigma Y$. Let

$$\omega\sigma X = \sigma X \cup \{\Omega\} \text{ and } \omega\sigma Y = \sigma Y \cup \{\Omega'\}$$

denote the one-point compactifications of σX and σY, respectively.

(1) *implies* (2). Suppose that (1) holds. Suppose that either X or Y, say X, is σ-compact. Then, $\sigma Y\backslash Y$ is compact, as it is homeomorphic to $\sigma X \backslash X = X^*$, and the latter is compact, as X is locally compact. Thus,

$$\sigma Y\backslash Y = Y^*_{H_1} \cup \cdots \cup Y^*_{H_n} = Y^*_H$$

where $n \in \mathbf{N}$, the sets $H_1, \ldots, H_n \subseteq J$ are countable and

$$H = H_1 \cup \cdots \cup H_n.$$

Now, if there exists some $u \in J\backslash H$, then since $Y_u \cap Y_H = \emptyset$ we have

$$\mathrm{cl}_{\beta Y} Y_u \cap \mathrm{cl}_{\beta Y} Y_H = \emptyset.$$

Therefore, $\mathrm{cl}_{\beta Y} Y_u \subseteq Y$, contradicting the fact that Y_u is non-compact. Thus, $J = H$ and Y is σ-compact. Therefore, $\sigma Y\backslash Y = Y^*$. Note that by Lemmas 3.8 and 3.10 we have $\mathscr{E}^C_{\mathcal{P}}(X) = \mathscr{E}^C(X)$ and $\mathscr{E}^C_{\mathcal{P}}(Y) = \mathscr{E}^C(Y)$. The result now follows from Lemma 3.15.

Suppose that X and Y are non-σ-compact. Let $f : \sigma X\backslash X \to \sigma Y \backslash Y$ denote a homeomorphism. We define an order-isomorphism

$$\phi : \big(\Theta_X\big(\mathscr{E}^C_{\mathcal{P}}(X)\big), \subseteq\big) \to \big(\Theta_Y\big(\mathscr{E}^C_{\mathcal{P}}(Y)\big), \subseteq\big).$$

Since Θ_X and Θ_Y are anti-order-isomorphisms, this will prove (2). Let $D \in \Theta_X(\mathscr{E}^C_{\mathcal{P}}(X))$. By Lemma 3.10 we have $D \in \mathscr{Z}(X^*)$ and $\beta X\backslash\sigma X \subseteq D$. Since $X^*\backslash D \subseteq \sigma X$, being a cozero-set in X^* is σ-compact, there exists a countable $G \subseteq I$ such that $X^*\backslash D \subseteq X^*_G$. Now, since $D \cap X^*_G \in \mathscr{Z}(X^*_G)$, we have

$$f(D \cap X^*_G) \in \mathscr{Z}\big(f(X^*_G)\big).$$

Since X^*_G is open in $\sigma X\backslash X$, its homeomorphic image $f(X^*_G)$ is open in $\sigma Y\backslash Y$, and thus, is open in Y^*. But, $f(X^*_G)$ is compact, as it is a continuous image of a compact space, and therefore, $f(X^*_G)$ is clopen in Y^*. Thus,

$$f(D \cap X^*_G) \cup \big(Y^*\backslash f(X^*_G)\big) \in \mathscr{Z}(Y^*).$$

Let

$$\phi(D) = f\big(D \cap (\sigma X \backslash X)\big) \cup (\beta Y \backslash \sigma Y).$$

Note that since

$$\begin{aligned} f\big(D \cap (\sigma X \backslash X)\big) &= f\big((D \cap X^*_G) \cup \big((\sigma X \backslash X) \backslash X^*_G\big)\big) \\ &= f(D \cap X^*_G) \cup \big((\sigma Y \backslash Y) \backslash f(X^*_G)\big) \end{aligned}$$

we have

$$\begin{aligned} \phi(D) &= f\big(D \cap (\sigma X \backslash X)\big) \cup (\beta Y \backslash \sigma Y) \\ &= f(D \cap X^*_G) \cup \big((\sigma Y \backslash Y) \backslash f(X^*_G)\big) \cup (\beta Y \backslash \sigma Y) \\ &= f(D \cap X^*_G) \cup \big(Y^* \backslash f(X^*_G)\big) \end{aligned}$$

which shows that ϕ is well-defined. The function ϕ is clearly an order-homomorphism. Since $f^{-1} : \sigma Y \backslash Y \to \sigma X \backslash X$ also is a homeomorphism, as above, it induces an order-homomorphism

$$\psi : \big(\Theta_Y\big(\mathscr{E}^C_{\mathcal{P}}(Y)\big), \subseteq\big) \to \big(\Theta_X\big(\mathscr{E}^C_{\mathcal{P}}(X)\big), \subseteq\big)$$

which is defined by

$$\psi(D) = f^{-1}\big(D \cap (\sigma Y \backslash Y)\big) \cup (\beta X \backslash \sigma X),$$

for $D \in \Theta_Y(\mathscr{E}^C_{\mathcal{P}}(Y))$. It is now easy to see that $\psi = \phi^{-1}$, which shows that ϕ is an order-isomorphism.

(2) *implies* (1). Suppose that (2) holds. Suppose that either X or Y, say X, is σ-compact (and non-compact). Then, $\sigma X = \beta X$, and thus, by Lemmas 3.8 and 3.10, we have $\mathscr{E}^C_{\mathcal{P}}(X) = \mathscr{E}^C(X)$. Suppose that Y is non-σ-compact. Note that X, being paracompact and non-compact, is non-pseudocompact (see Theorems 3.10.21, 5.1.5 and 5.1.20 of [3]) and therefore, X^* contains at least two elements, as almost compact spaces are pseudocompact (see Problem 5U (1) of [16]; recall that a Tychonoff space T is called *almost compact* if $\beta T \backslash T$ has at most one element). Thus, there exist two disjoint non-empty zero-sets of X^* corresponding to two elements in $\mathscr{E}^C(X)$ with no common upper bound in $\mathscr{E}^C(X)$. But, this is not true, as $\mathscr{E}^C(X)$ is order-isomorphic to $\mathscr{E}^C_{\mathcal{P}}(Y)$, and any two elements in the latter have a common upper bound in $\mathscr{E}^C_{\mathcal{P}}(Y)$. (Note that since Y is non-σ-compact, the set $\beta Y \backslash \sigma Y$ is non-empty, and by Lemma 3.10, the image of any element in $\mathscr{E}^C_{\mathcal{P}}(Y)$ under Θ_Y contains $\beta Y \backslash \sigma Y$.) Therefore, Y also is σ-compact and by Lemmas 3.8 and 3.10, we have $\mathscr{E}^C_{\mathcal{P}}(Y) = \mathscr{E}^C(Y)$. Now, since $\sigma Y = \beta Y$, the result follows from Lemma 3.15.

Next, suppose that X and Y are both non-σ-compact. We show that the two compact spaces $\omega\sigma X\backslash X$ and $\omega\sigma Y\backslash Y$ are homeomorphic, by showing that their corresponding sets of zero-sets (partially ordered with $\subseteq$) are order-isomorphic. Since Θ_X and Θ_Y are anti-order-isomorphisms, condition (2) implies the existence of an order-isomorphism

$$\phi : \big(\Theta_X\big(\mathscr{E}^C_\mathcal{P}(X)\big), \subseteq\big) \to \big(\Theta_Y\big(\mathscr{E}^C_\mathcal{P}(Y)\big), \subseteq\big).$$

We define an order-isomorphism

$$\psi : \big(\mathscr{Z}(\omega\sigma X\backslash X), \subseteq\big) \to \big(\mathscr{Z}(\omega\sigma Y\backslash Y), \subseteq\big)$$

as follows. Let $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$. Suppose that $\Omega \in Z$. Then, since $(\omega\sigma X\backslash X)\backslash Z$ is a cozero-set in (the compact space) $\omega\sigma X\backslash X$, it is σ-compact. Thus, $(\omega\sigma X\backslash X)\backslash Z \subseteq X^*_G$, for some countable $G \subseteq I$. Since X^*_G is clopen in X^*, we have

$$\big(Z\backslash\{\Omega\}\big) \cup (\beta X\backslash\sigma X) = (Z \cap X^*_G) \cup (X^*\backslash X^*_G) \in \mathscr{Z}(X^*).$$

In this case, we let

$$\psi(Z) = \big(\phi\big(\big(Z\backslash\{\Omega\}\big) \cup (\beta X\backslash\sigma X)\big)\backslash(\beta Y\backslash\sigma Y)\big) \cup \{\Omega'\}.$$

Now, suppose that $\Omega \notin Z$. Then, $Z \subseteq \sigma X\backslash X$, and therefore $Z \subseteq X^*_G$, for some countable $G \subseteq I$, and thus, using this, one can write

$$Z = X^*\backslash \bigcup_{n=1}^\infty Z_n \text{ where } \beta X\backslash\sigma X \subseteq Z_n \in \mathscr{Z}(X^*) \text{ for } n \in \mathbf{N}. \tag{3.2}$$

In this case, we let

$$\psi(Z) = Y^*\backslash \bigcup_{n=1}^\infty \phi(Z_n).$$

We check that ψ is well-defined. Assume the representation given in (3.2). Since $Y^*\backslash\phi(Z_n) \subseteq \sigma Y$, for $n \in \mathbf{N}$, there exists a countable $H \subseteq J$ such that $Y^*\backslash\phi(Z_n) \subseteq Y^*_H$, for all $n \in \mathbf{N}$.

Claim. *For $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ with $\Omega \notin Z$ assume the representation given in (3.2). Let $H \subseteq J$ be countable and such that $Y^*\backslash\phi(Z_n) \subseteq Y^*_H$, for all $n \in \mathbf{N}$. Let A be such that $\phi(A) = Y^*\backslash Y^*_H$. Then,*

$$Y^*\backslash \bigcup_{n=1}^\infty \phi(Z_n) = \phi(A \cup Z)\backslash\phi(A).$$

Proof of the claim. Suppose that $y \in Y^*$ and $y \notin \phi(Z_n)$, for each $n \in \mathbf{N}$. If $y \notin \phi(A \cup Z)\backslash\phi(A)$, then since $y \notin \phi(Z_1) \supseteq \phi(A)$ we have $y \notin \phi(A \cup Z)$. Therefore, there exists some $B \in \mathscr{Z}(Y^*)$ containing y such that $B \cap \phi(A \cup Z) = \emptyset$ and $B \cap \phi(Z_n) = \emptyset$, for $n \in \mathbf{N}$. Let C be such that $\phi(C) = B \cup \phi(A \cup Z)$, and let S_n, for $n \in \mathbf{N}$, be such that

$$\begin{aligned} \phi(S_n) &= \phi(C) \cap \phi(Z_n) \\ &= \big(B \cup \phi(A \cup Z)\big) \cap \phi(Z_n) \\ &= \big(B \cap \phi(Z_n)\big) \cup \big(\phi(A \cup Z) \cap \phi(Z_n)\big) = \phi(A \cup Z) \cap \phi(Z_n). \end{aligned}$$

Since $A \subseteq Z_n$, as $\phi(A) \subseteq \phi(Z_n)$ and $Z \cap Z_n = \emptyset$, we have $A \cap Z = \emptyset$, which implies that

$$(A \cup Z) \cap Z_n = (A \cap Z_n) \cup (Z \cap Z_n) = A,$$

for $n \in \mathbf{N}$. Clearly, $S_n \subseteq (A \cup Z) \cap Z_n$, as by above $\phi(S_n) \subseteq \phi(A \cup Z)$ and $\phi(S_n) \subseteq \phi(Z_n)$, for $n \in \mathbf{N}$. Thus, $\phi(S_n) \subseteq \phi(A)$, for $n \in \mathbf{N}$. But, since $\phi(A) \subseteq \phi(Z_n)$, we have $\phi(A) \subseteq \phi(S_n)$, and therefore

$$\phi(C \cap Z_n) \subseteq \phi(C) \cap \phi(Z_n) = \phi(S_n) = \phi(A),$$

for $n \in \mathbf{N}$. This implies that $C \cap Z_n \subseteq A$, for $n \in \mathbf{N}$. Thus,

$$C\backslash Z = C \cap \bigcup_{n=1}^{\infty} Z_n = \bigcup_{n=1}^{\infty} (C \cap Z_n) \subseteq A.$$

Therefore, $C \subseteq A \cup Z$ and we have $B \subseteq \phi(C) \subseteq \phi(A \cup Z)$, which is a contradiction, as $B \cap \phi(A \cup Z) = \emptyset$. This shows that

$$Y^*\backslash \bigcup_{n=1}^{\infty} \phi(Z_n) \subseteq \phi(A \cup Z)\backslash\phi(A).$$

Now, suppose that $y \in \phi(A \cup Z)\backslash\phi(A)$. Suppose to the contrary that $y \in \phi(Z_n)$, for some $n \in \mathbf{N}$. Then,

$$y \in \phi(Z_n) \cap \phi(A \cup Z) = \phi(D),$$

for some D. Clearly, $D \subseteq Z_n$ and $D \subseteq A \cup Z$, as $\phi(D) \subseteq \phi(Z_n)$ and $\phi(D) \subseteq \phi(A \cup Z)$. This implies that

$$D \subseteq Z_n \cap (A \cup Z) = (Z_n \cap A) \cup (Z_n \cap Z) = Z_n \cap A \subseteq A$$

and thus $y \in \phi(A)$, as $\phi(D) \subseteq \phi(A)$, which is a contradiction. This proves the claim.

Now, suppose that

$$Z = X^*\backslash \bigcup_{n=1}^{\infty} S_n \text{ and } Z = X^*\backslash \bigcup_{n=1}^{\infty} Z_n$$

are two representations for $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ with $\Omega \notin Z$ such that each $S_n, Z_n \in \mathscr{Z}(X^*)$ contains $\beta X\backslash\sigma X$, for $n \in \mathbf{N}$. Choose a countable $H \subseteq J$ such that

$$Y^*\backslash\phi(S_n) \subseteq Y^*_H \text{ and } Y^*\backslash\phi(Z_n) \subseteq Y^*_H,$$

for $n \in \mathbf{N}$. Then, by the claim, we have

$$Y^*\backslash \bigcup_{n=1}^{\infty} \phi(S_n) = \phi(A\cup Z)\backslash\phi(A) = Y^*\backslash \bigcup_{n=1}^{\infty} \phi(Z_n)$$

where A is such that $\phi(A) = Y^*\backslash Y^*_H$. This shows that ψ is well-defined. Next, we show that ψ is an order-isomorphism. Suppose that $S, Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ and $S \subseteq Z$. We consider the following cases.

Case 1: Suppose that $\Omega \in S$. Then, $\Omega \in Z$, and clearly,

$$\begin{aligned}\psi(S) &= \big(\phi\big(\big(S\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X)\big)\backslash(\beta Y\backslash\sigma Y)\big)\cup\{\Omega'\}\\ &\subseteq \big(\phi\big(\big(Z\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X)\big)\backslash(\beta Y\backslash\sigma Y)\big)\cup\{\Omega'\} = \psi(Z).\end{aligned}$$

Case 2: Suppose that $\Omega \notin S$ but $\Omega \in Z$. Let

$$E = \phi\big(\big(Z\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X)\big)$$

and let

$$S = X^*\backslash \bigcup_{n=1}^{\infty} S_n$$

where each $S_n \in \mathscr{Z}(X^*)$ contains $\beta X\backslash\sigma X$, for $n \in \mathbf{N}$. Clearly, $Y^*\backslash E \subseteq \sigma Y$. Let $H \subseteq J$ be countable and such that $Y^*\backslash\phi(S_n) \subseteq Y^*_H$, for all $n \in \mathbf{N}$ and $Y^*\backslash E \subseteq Y^*_H$. By the claim, we have $\psi(S) = \phi(A\cup S)\backslash\phi(A)$, where $\phi(A) = Y^*\backslash Y^*_H$. Since $Y^*\backslash Y^*_H \subseteq E$, we have

$$A \subseteq \big(Z\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X).$$

Now,

$$\psi(S) = \phi(A\cup S)\backslash\phi(A) \subseteq \phi(A\cup S) \subseteq \phi\big(\big(Z\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X)\big)$$

which implies that

$$\psi(S) \subseteq \big(\phi\big(\big(Z\backslash\{\Omega\}\big)\cup(\beta X\backslash\sigma X)\big)\backslash(\beta Y\backslash\sigma Y)\big)\cup\{\Omega'\} = \psi(Z).$$

Case 3: Suppose that $\Omega \notin Z$. Then, $\Omega \notin S$. Let

$$S = X^*\backslash \bigcup_{n=1}^{\infty} S_n \text{ and } Z = X^*\backslash \bigcup_{n=1}^{\infty} Z_n$$

where each $S_n, Z_n \in \mathscr{Z}(X^*)$ contains $\beta X\backslash\sigma X$, for $n \in \mathbf{N}$. Clearly,

$$S = S \cap Z = \Big(X^*\backslash \bigcup_{n=1}^{\infty} S_n\Big) \cap \Big(X^*\backslash \bigcup_{n=1}^{\infty} Z_n\Big) = X^*\backslash \bigcup_{n=1}^{\infty} (S_n \cup Z_n)$$

and thus, since $\phi(Z_n) \subseteq \phi(S_n \cup Z_n)$, for $n \in \mathbf{N}$, it follows that

$$\psi(S) = Y^*\backslash \bigcup_{n=1}^{\infty} \phi(S_n \cup Z_n) \subseteq Y^*\backslash \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z).$$

Note that since

$$\phi^{-1} : \big(\Theta_Y\big(\mathscr{E}^C_{\mathcal{P}}(Y)\big), \subseteq\big) \to \big(\Theta_X\big(\mathscr{E}^C_{\mathcal{P}}(X)\big), \subseteq\big)$$

also is an order-isomorphism, as above, it induces an order-isomorphism

$$\gamma : \big(\mathscr{Z}(\omega\sigma Y\backslash Y), \subseteq\big) \to \big(\mathscr{Z}(\omega\sigma X\backslash X), \subseteq\big)$$

which is easy to see that $\gamma = \psi^{-1}$. Therefore, ψ is an order-isomorphism. It then follows that there exists a homeomorphism $f : \omega\sigma X\backslash X \to \omega\sigma Y\backslash Y$ such that $f(Z) = \psi(Z)$, for any $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$. Now, since for each countable $G \subseteq I$ we have

$$f(X^*_G) = \psi(X^*_G) \subseteq \sigma Y\backslash Y$$

it follows that $f(\sigma X\backslash X) = \sigma Y\backslash Y$. Thus, $\sigma X\backslash X$ and $\sigma Y\backslash Y$ are homeomorphic.

(1) *implies* (3). Suppose that (1) holds. Suppose that either X or Y, say X, is σ-compact. Then, $\sigma X = \beta X$ and thus, arguing as in part (1)$\Rightarrow$(2), it follows that Y also is σ-compact. Therefore, $\sigma Y = \beta Y$. Note that by Lemmas 3.7 and 3.11 we have $\mathscr{E}^*_{local-\mathcal{P}}(X) = \mathscr{E}^*(X)$ and since $X^* \in \mathscr{Z}(\beta X)$ (as X is σ-compact and locally compact, see 1B of [19]) by Lemmas 3.7 and 3.8 we have $\mathscr{E}^*(X) = \mathscr{E}^C(X)$. Thus, $\mathscr{E}^*_{local-\mathcal{P}}(X) = \mathscr{E}^C(X)$ and similarly $\mathscr{E}^*_{local-\mathcal{P}}(Y) = \mathscr{E}^C(Y)$. The result now follows from Lemma 3.15.

Suppose that X and Y are non-σ-compact. Let $f : \sigma X\backslash X \to \sigma Y\backslash Y$ be a homeomorphism. We define an order-isomorphism

$$\phi : \big(\Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big), \subseteq\big) \to \big(\Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big), \subseteq\big),$$

as follows. Let $Z \in \Theta_X(\mathscr{E}^*_{local-\mathcal{P}}(X))$. By Lemma 3.11 we have $Z \in \mathscr{Z}(\beta X)$ and $Z \subseteq \sigma X \backslash X$. Thus, $Z \subseteq X^*_G$, for some countable $G \subseteq I$. Now, $f(Z) \in \mathscr{Z}(\sigma Y \backslash Y)$ and since $f(Z)$ is compact, as it is a continuous image of a compact space, it follows that $f(Z) \subseteq Y^*_H$, for some countable $H \subseteq J$. Therefore, $f(Z) \in \mathscr{Z}(Y^*_H)$ and then $f(Z) \in \mathscr{Z}(\mathrm{cl}_{\beta Y} Y_H)$. Since $\mathrm{cl}_{\beta Y} Y_H$ is clopen in βY we have $f(Z) \in \mathscr{Z}(\beta Y)$. Define

$$\phi(Z) = f(Z).$$

It is obvious that ϕ is an order-homomorphism. If we let

$$\psi : \big(\Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big), \subseteq\big) \to \big(\Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big), \subseteq\big)$$

be defined by

$$\psi(Z) = f^{-1}(Z),$$

then $\psi = \phi^{-1}$ which shows that ϕ is an order-isomorphism.

(3) *implies* (1). Suppose that (3) holds. Suppose that either X or Y, say X, is σ-compact (and non-compact). Then, $\sigma X = \beta X$, and thus, by Lemmas 3.7 and 3.11, we have $\mathscr{E}^*_{local-\mathcal{P}}(X) = \mathscr{E}^*(X)$. Therefore, since $X^* \in \mathscr{Z}(\beta X)$ the set $\mathscr{E}^*_{local-\mathcal{P}}(X)$ has a smallest element (namely, its one-point compactification ωX). Thus, $\mathscr{E}^*_{local-\mathcal{P}}(Y)$ also has a smallest element; denote this element by T. Then, for each countable $H \subseteq J$ we have

$$Y^*_H \in \Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big)$$

and therefore $\sigma Y \backslash Y \subseteq \Theta_Y(T)$. By Lemma 3.14 (with $\Theta_Y(T)$ and Y^* as the zero-sets in its statement) we have $Y^* \subseteq \Theta_Y(T)$. This implies that $Y^* \in \mathscr{Z}(\beta Y)$ which shows that Y is σ-compact. Thus, $\sigma Y = \beta Y$, and by Lemmas 3.7 and 3.11, we have $\mathscr{E}^*_{local-\mathcal{P}}(Y) = \mathscr{E}^*(Y)$. Therefore, in this case (and since by Lemmas 3.7 and 3.8 we have $\mathscr{E}^*(X) = \mathscr{E}^C(X)$ and $\mathscr{E}^*(Y) = \mathscr{E}^C(Y)$) the result follows from Lemma 3.15.

Next, suppose that X and Y are both non-σ-compact. Since Θ_X and Θ_Y are both anti-order-isomorphisms, there exists an order-isomorphism

$$\phi : \big(\Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big), \subseteq\big) \to \big(\Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big), \subseteq\big).$$

We extend ϕ by letting $\phi(\emptyset) = \emptyset$. We define a function

$$\psi : \big(\mathscr{Z}(\omega\sigma X \backslash X), \subseteq\big) \to \big(\mathscr{Z}(\omega\sigma Y \backslash Y), \subseteq\big)$$

and verify that it is an order-isomorphism. Let $Z \in \mathscr{Z}(\omega\sigma X \backslash X)$ with $\Omega \notin Z$. Since $Z \subseteq X^*_G$, for some countable $G \subseteq I$, we have $Z \in \mathscr{Z}(\beta X)$, and therefore,

$$Z \in \Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big) \cup \{\emptyset\}.$$

In this case, let

$$\psi(Z) = \phi(Z).$$

Now, suppose that $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ and $\Omega \in Z$. Then, $(\omega\sigma X\backslash X)\backslash Z$ is a cozero-set in $\omega\sigma X\backslash X$, and we have

$$Z = (\omega\sigma X\backslash X)\backslash \bigcup_{n=1}^{\infty} Z_n \text{ where } Z_n \in \mathscr{Z}(\omega\sigma X\backslash X) \text{ for } n \in \mathbf{N}. \tag{3.3}$$

Thus, as above, it follows that

$$Z_n \in \Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big) \cup \{\emptyset\},$$

for $n \in \mathbf{N}$. We verify that

$$\bigcup_{n=1}^{\infty} \phi(Z_n) \in Coz(\omega\sigma Y\backslash Y). \tag{3.4}$$

To show this, note that since $\phi(Z_n) \subseteq \sigma Y\backslash Y$ there exists a countable $H \subseteq J$ such that $\phi(Z_n) \subseteq Y^*_H$, for $n \in \mathbf{N}$.

Claim. *For $Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ with $\Omega \in Z$ assume the representation given in (3.3). Let $H \subseteq J$ be countable and such that $\phi(Z_n) \subseteq Y^*_H$, for all $n \in \mathbf{N}$. Let A be such that $\phi(A) = Y^*_H$. Then,*

$$\phi(A \cap Z) = \phi(A)\backslash \bigcup_{n=1}^{\infty} \phi(Z_n).$$

Proof of the claim. For each $n \in \mathbf{N}$, since $A \cap Z \cap Z_n = \emptyset$, we have $\phi(A \cap Z) \cap \phi(Z_n) = \emptyset$, as otherwise, $\phi(A \cap Z)$ and $\phi(Z_n)$ will have a common lower bound in $\Theta_Y(\mathscr{E}^*_{local-\mathcal{P}}(Y))$, that is, $\phi(A \cap Z) \cap \phi(Z_n)$, whereas $A \cap Z$ and Z_n do not have. Also, $\phi(A \cap Z) \subseteq \phi(A)$. Therefore,

$$\phi(A \cap Z) \subseteq \phi(A)\backslash \bigcup_{n=1}^{\infty} \phi(Z_n).$$

To show the reverse inclusion, let $y \in \phi(A)$ be such that $y \notin \phi(Z_n)$, for $n \in \mathbf{N}$. There exists $B \in \mathscr{Z}(\beta Y)$ such that $y \in B$ and $B \cap \phi(Z_n) = \emptyset$, for all $n \in \mathbf{N}$. If $y \notin \phi(A \cap Z)$, then there exists some $C \in \mathscr{Z}(\beta Y)$ such that $y \in C$ and $C \cap \phi(A \cap Z) = \emptyset$. Let $D = \phi(A) \cap B \cap C$ and let E be such that $\phi(E) = D$. For each $n \in \mathbf{N}$, since $\phi(E) \cap \phi(Z_n) = \emptyset$, we have $E \cap Z_n = \emptyset$, and thus $E \subseteq Z$. On the other hand, since $\phi(E) \subseteq \phi(A)$ we have $E \subseteq A$, and therefore $E \subseteq A \cap Z$. Thus, $\phi(E) \subseteq \phi(A \cap Z)$, which implies that $\phi(E) = \emptyset$, as $\phi(E) \subseteq C$. This contradiction shows that $y \in \phi(A \cap Z)$, which proves the claim.

Let A be such that $\phi(A) = Y^*_H$. Now, $\phi(A \cap Z) \in \mathscr{Z}(\omega\sigma Y\backslash Y)$, as $\phi(A \cap Z) \subseteq \phi(A)$. By the claim we have

$$\begin{aligned}(\omega\sigma Y\backslash Y)\backslash \bigcup_{n=1}^{\infty} \phi(Z_n) &= \Big(\phi(A)\backslash \bigcup_{n=1}^{\infty} \phi(Z_n)\Big) \cup \big((\omega\sigma Y\backslash Y)\backslash \phi(A)\big)\\ &= \phi(A\cap Z) \cup \big((\omega\sigma Y\backslash Y)\backslash \phi(A)\big) \in \mathscr{Z}(\omega\sigma Y\backslash Y)\end{aligned}$$

and (3.4) is verified. In this case, we let

$$\psi(Z) = (\omega\sigma Y\backslash Y)\backslash \bigcup_{n=1}^{\infty} \phi(Z_n).$$

Next, we show that ψ is well-defined. Assume that

$$Z = (\omega\sigma X\backslash X)\backslash \bigcup_{n=1}^{\infty} S_n$$

with $S_n \in \mathscr{Z}(\omega\sigma X\backslash X)$, for $n \in \mathbf{N}$, is another representation of Z. We need to show that

$$\bigcup_{n=1}^{\infty} \phi(Z_n) = \bigcup_{n=1}^{\infty} \phi(S_n). \tag{3.5}$$

Without any loss of generality, suppose to the contrary that there exists some $m \in \mathbf{N}$ and $y \in \phi(Z_m)$ such that $y \notin \phi(S_n)$, for all $n \in \mathbf{N}$. Then, there exists some $A \in \mathscr{Z}(\beta Y)$ such that $y \in A$ and $A \cap \phi(S_n) = \emptyset$, for $n \in \mathbf{N}$. Consider

$$A \cap \phi(Z_m) \in \Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big).$$

Let B be such that $\phi(B) = A \cap \phi(Z_m)$. Since $\phi(B) \subseteq A$ we have $\phi(B) \cap \phi(S_n) = \emptyset$ from which it follows that $B \cap S_n = \emptyset$, for $n \in \mathbf{N}$. But, $B \subseteq Z_m$, as $\phi(B) \subseteq \phi(Z_m)$, and we have

$$B \subseteq \bigcup_{n=1}^{\infty} Z_n = \bigcup_{n=1}^{\infty} S_n$$

which implies that $B = \emptyset$. But, this is a contradiction, as $\phi(B) \neq \emptyset$. Therefore, (3.5) holds, and thus ψ is well-defined. To prove that ψ is an order-isomorphism, let $S, Z \in \mathscr{Z}(\omega\sigma X\backslash X)$ and $S \subseteq Z$. The case when $S = \emptyset$ holds trivially. Assume that $S \neq \emptyset$. We consider the following cases.

Case 1: Suppose that $\Omega \notin Z$. Then, $\Omega \notin S$ and we have

$$\psi(S) = \phi(S) \subseteq \phi(Z) = \psi(Z).$$

Case 2: Suppose that $\Omega \notin S$ but $\Omega \in Z$. Let

$$Z = (\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} Z_n$$

with $Z_n \in \mathscr{Z}(\omega\sigma X \backslash X)$, for $n \in \mathbf{N}$. Then, since $S \subseteq Z$ we have $S \cap Z_n = \emptyset$, and therefore $\phi(S) \cap \phi(Z_n) = \emptyset$, for $n \in \mathbf{N}$. Thus,

$$\psi(S) = \phi(S) \subseteq (\omega\sigma Y \backslash Y) \backslash \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z).$$

Case 3: Suppose that $\Omega \in S$. Then, $\Omega \in Z$. Let

$$S = (\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} S_n \text{ and } Z = (\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} Z_n$$

where $S_n, Z_n \in \mathscr{Z}(\omega\sigma X \backslash X)$, for $n \in \mathbf{N}$. Therefore,

$$\begin{aligned} S = S \cap Z &= \Big((\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} S_n\Big) \cap \Big((\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} Z_n\Big) \\ &= (\omega\sigma X \backslash X) \backslash \bigcup_{n=1}^{\infty} (S_n \cup Z_n). \end{aligned}$$

Thus, since $\phi(Z_n) \subseteq \phi(S_n \cup Z_n)$, for $n \in \mathbf{N}$, we have

$$\psi(S) = (\omega\sigma Y \backslash Y) \backslash \bigcup_{n=1}^{\infty} \phi(S_n \cup Z_n) \subseteq (\omega\sigma Y \backslash Y) \backslash \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z).$$

This shows that ψ is an order-homomorphism. To show that ψ is an order-isomorphism, we note that

$$\phi^{-1} : \big(\Theta_Y\big(\mathscr{E}^*_{local-\mathcal{P}}(Y)\big), \subseteq\big) \to \big(\Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big), \subseteq\big)$$

is an order-isomorphism. Let

$$\gamma : \big(\mathscr{Z}(\omega\sigma Y \backslash Y), \subseteq\big) \to \big(\mathscr{Z}(\omega\sigma X \backslash X), \subseteq\big)$$

be the induced order-homomorphism which is defined as above. Then, it is straightforward to see that $\gamma = \psi^{-1}$, that is, ψ is an order-isomorphism. This implies the existence of a homeomorphism $f : \omega\sigma X \backslash X \to \omega\sigma Y \backslash Y$

such that $f(Z) = \psi(Z)$, for every $Z \in \mathscr{Z}(\omega\sigma X \backslash X)$. Therefore, for any countable $G \subseteq I$, since $X_G^* \in \mathscr{Z}(\omega\sigma X \backslash X)$, we have

$$f(X_G^*) = \psi(X_G^*) = \phi(X_G^*) \subseteq \sigma Y \backslash Y.$$

Thus, $f(\sigma X \backslash X) \subseteq \sigma Y \backslash Y$, which shows that $f(\Omega) = \Omega'$. Therefore, $\sigma X \backslash X$ and $\sigma Y \backslash Y$ are homeomorphic.

Example 3.17. The Lindelöf property and the linearly Lindelöf property (besides σ-compactness itself) are examples of topological properties $\mathcal{P}$ satisfying the assumption of Theorem 3.16. To see this, let X be a locally compact paracompact space. Assume a representation for X as in Notation 2.9. Recall that a Hausdorff space X is said to be *linearly Lindelöf* [6] provided that every linearly ordered (by set inclusion $\subseteq$) open cover of X has a countable subcover, equivalently, if every uncountable subset of X has a complete accumulation point in X. (Recall that a point $x \in X$ is called a *complete accumulation point* of a set $A \subseteq X$ if for every neighborhood U of x in X we have $|U \cap A| = |A|$.) Note that if X is non-σ-compact, then (using the notation of Notation 2.9) the set I is uncountable. Let $A = \{x_i : i \in I\}$ where $x_i \in X_i$, for $i \in I$. Then, A is an uncountable subset of X without (even) accumulation points. Thus, X cannot be linearly Lindelöf as well. For the converse, note that if X is not linearly Lindelöf, then, obviously, X is not Lindelöf, and therefore, is non-σ-compact, as it is well-known that σ-compactness and the Lindelöf property coincide in the realm of locally compact paracompact spaces (this fact is evident from the representation given for X in Notation 2.9).

Theorem 3.16 above might leave the impression that $(\mathscr{E}_{\mathcal{P}}^{C}(X), \leq)$ and $(\mathscr{E}_{local-\mathcal{P}}^{*}(X), \leq)$ are order-isomorphic. The following is to settle this, showing that in most cases this is indeed not going to be the case.

Theorem 3.18. *Let X be a locally compact paracompact (non-compact) space and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. Then, the following are equivalent:*

(1) *X is σ-compact.*
(2) *$(\mathscr{E}_{\mathcal{P}}^{C}(X), \leq)$ and $(\mathscr{E}_{local-\mathcal{P}}^{*}(X), \leq)$ are order-isomorphic.*

Proof. Since X is locally compact, the set X^* is closed in (the normal space) βX and thus, using the Tietze-Urysohn Theorem, every zero-set of X^* is extendible to a zero-set of βX. Now, if X is σ-compact (since X is also locally compact) we have $X^* \in \mathscr{Z}(\beta X)$ and therefore every zero-set of X^* is a zero-set of βX. Note that $\lambda_{\mathcal{P}} X = \sigma X = \beta X$. Thus, using Lemmas 3.10 and 3.11 we have

$$\Theta_X\big(\mathscr{E}^C_{\mathcal{P}}(X)\big) = \mathscr{Z}(X^*)\backslash\{\emptyset\} = \Theta_X\big(\mathscr{E}^*_{local-\mathcal{P}}(X)\big)$$

from which it follows that

$$\mathscr{E}^C_{\mathcal{P}}(X) = \mathscr{E}^*_{local-\mathcal{P}}(X).$$

If X is non-σ-compact, then any two elements of $\mathscr{E}^C_{\mathcal{P}}(X)$ have a common upper bound while this is not the case for $\mathscr{E}^*_{local-\mathcal{P}}(X)$. To see this, note that by Lemma 3.10 the set $\Theta_X(\mathscr{E}^C_{\mathcal{P}}(X))$ is closed under finite intersections (note that the finite intersections are non-empty, as they contain $\beta X\backslash\sigma X$ and the latter is non-empty, as X is non-σ-compact) while there exist (at least) two elements in $\Theta_X(\mathscr{E}^*_{local-\mathcal{P}}(X))$ with empty intersection; simply consider X_i^* and X_j^*, for some distinct $i, j \in I$ (we are assuming the representation for X given in Notation 2.9).

Project 3.19. Let X be a (locally compact paracompact) space and let $\mathcal{P}$ be a (closed hereditary) topological property (of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces). Explore the relationship between the order structures of $(\mathscr{E}^C_{\mathcal{P}}(X), \leq)$ and $(\mathscr{E}^*_{local-\mathcal{P}}(X), \leq)$.

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