

## ON A CONJECTURE OF A BOUND FOR THE EXPONENT OF THE SCHUR MULTIPLIER OF A FINITE $p$ -GROUP

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**ABSTRACT.** Let  $G$  be a  $p$ -group of nilpotency class  $k$  with finite exponent  $\exp(G)$  and let  $m = \lfloor \log_p k \rfloor$ . We show that  $\exp(M^{(c)}(G))$  divides  $\exp(G)p^{m(k-1)}$ , for all  $c \geq 1$ , where  $M^{(c)}(G)$  denotes the  $c$ -nilpotent multiplier of  $G$ . This implies that  $\exp(M(G))$  divides  $\exp(G)$ , for all finite  $p$ -groups of class at most  $p-1$ . Moreover, we show that our result is an improvement of some previous bounds for the exponent of  $M^{(c)}(G)$  given by M. R. Jones, G. Ellis and P. Moravec in some cases.

### 1. Introduction and motivation

Let a group  $G$  be presented as a quotient of a free group  $F$  by a normal subgroup  $R$ . Then, the  $c$ -nilpotent multiplier of  $G$  (the Baer invariant of  $G$  with respect to the variety of nilpotent group of class at most  $c$ ) is defined to be

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]},$$

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where  $[R, \underbrace{cF}_{c\text{-times}}]$  denotes the commutator subgroup  $[R, \underbrace{F, \dots, F}_{c\text{-times}}]$  and  $c \geq 1$ .

The case  $c = 1$  which has been much studied is the Schur multiplier of  $G$ , denoted by  $M(G)$ . When  $G$  is finite,  $M(G)$  is isomorphic to the second cohomology group  $H^2(G, \mathbb{C}^*)$  (see G. Karpilovsky [6] and C. R. Leedham-Green and S. McKay [8] for further details).

It has been interested to find a relation between the exponent of  $M^{(c)}(G)$  and the exponent of  $G$ . Let  $G$  be a finite  $p$ -group of nilpotency class  $k \geq 2$  with exponent  $\exp(G)$ . M. R. Jones [5] proved that  $\exp(M(G))$  divides  $\exp(G)^{k-1}$ . This has been improved by G. Ellis [3] who showed that  $\exp(M^{(c)}(G))$  divides  $\exp(G)^{\lceil k/2 \rceil}$ , where  $\lceil k/2 \rceil$  denotes the smallest integer  $n$  such that  $n \geq k/2$ . For  $c = 1$ , P. Moravec [11] showed that  $\lceil k/2 \rceil$  can be replaced by  $2\lfloor \log_2 k \rfloor$  which is an improvement, if  $k \geq 11$ .

In this paper, we will show that if  $G$  is a finite exponent  $p$ -group of class  $k \geq 1$ , then  $\exp(M^{(c)}(G))$  divides  $\exp(G)p^{m(k-1)}$ , for all  $c \geq 1$ , where  $m = \lfloor \log_p k \rfloor$ . Note that this result is an improvement of the results of Jones, Ellis and Moravec, if  $\lfloor \log_p k \rfloor(k-1)/k < e$ ,  $\lfloor \log_p k \rfloor(k-1)/\lceil k/2 \rceil - 1 < e$ ,  $\lfloor \log_p k \rfloor(k-1)/2\lfloor \log_2 k \rfloor - 1 < e$ , respectively, where  $\exp(G) = p^e$ .

It was a longstanding open problem as to whether  $\exp(M(G))$  divides  $\exp(G)$ , for every finite group  $G$ . In fact, it was conjectured that the exponent of the Schur multiplier of a finite  $p$ -group is a divisor of the exponent of the group itself. I. D. Macdonld and J. W. Wamsley [1] constructed an example of a group of order  $2^{21}$  which has exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. Also, Moravec [12] gave an example of a group of order 2048 and nilpotency class 6 which has exponent 4 and multiplier of exponent 8. He also proved that if  $G$  is a group of exponent 4, then  $\exp(M(G))$  divides 8. Nevertheless, Jones [5] has shown that the conjecture is true for  $p$ -groups of class 2 and emphasized that it is true for some  $p$ -groups of class 3. S. Kayvanfar and M. A. Sanati [7] have proved the conjecture for  $p$ -groups of class 4 and 5, with some conditions. A. Lubotzky and A. Mann [9] showed that the conjecture is true for powerful  $p$ -groups. The first and the third authors [10] showed that the conjecture is true for nilpotent multipliers of powerful  $p$ -groups. Finally, Moravec [11, 12] showed that the conjecture is true for metabelian groups of exponent  $p$ ,  $p$ -groups with potent filtration and  $p$ -groups of maximal

class. Note that a consequence of our result shows that the conjecture is true for all finite  $p$ -groups of class at most  $p - 1$ .

## 2. Preliminaries

In this section, we are going to recall some notions we will use in the next section.

**Definition 2.1.** (*M. Hall [4]*). Let  $X$  be an independent subset of a free group, and select an arbitrary total order for  $X$ . We define the basic commutators on  $X$ , their weight  $wt$ , and the ordering among them as follows:

- (1) The elements of  $X$  are basic commutators of weight one, ordered according to the total order previously chosen.
- (2) Having defined the basic commutators of weight less than  $n$ , the basic commutators of weight  $n$  are the  $c_k = [c_i, c_j]$ , where:
  - (a)  $c_i$  and  $c_j$  are basic commutators and  $wt(c_i) + wt(c_j) = n$ , and
  - (b)  $c_i > c_j$ , and if  $c_i = [c_s, c_t]$ , then  $c_j \geq c_t$ .
- (3) The basic commutators of weight  $n$  follow those of weight less than  $n$ . The basic commutators of weight  $n$  are ordered among themselves lexicographically; that is, if  $[b_1, a_1]$  and  $[b_2, a_2]$  are basic commutators of weight  $n$ , then  $[b_1, a_1] \leq [b_2, a_2]$  if and only if  $b_1 < b_2$  or  $b_1 = b_2$  and  $a_1 < a_2$ .

**Lemma 2.2.** (*R. R. Struik [13]*). Let  $x_1, x_2, \dots, x_r$  be any elements of a group and let  $v_1, v_2, \dots$  be the sequence of basic commutators of weight at least two in the  $x_i$ 's, in ascending order. Then,

$$(x_1 x_2 \dots x_r)^\alpha = x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_r}^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \dots v_i^{f_i(\alpha)} \dots,$$

where  $\{i_1, i_2, \dots, i_r\} = \{1, 2, \dots, r\}$ ,  $\alpha$  is a nonnegative integer and

$$f_i(\alpha) = a_1 \binom{\alpha}{1} + a_2 \binom{\alpha}{2} + \dots + a_{w_i} \binom{\alpha}{w_i}, \quad (I)$$

with  $a_1, \dots, a_{w_i} \in \mathbf{Z}$  and  $w_i$  is the weight of  $v_i$  in the  $x_i$ 's.

**Lemma 2.3.** (Struik [13]). Let  $\alpha$  be a fixed integer and  $G$  be a nilpotent group of class at most  $k$ . If  $b_1, \dots, b_r \in G$  and  $r < k$ , then

$$[b_1, \dots, b_{i-1}, b_i^\alpha, b_{i+1}, \dots, b_r] = [b_1, \dots, b_r]^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \dots,$$

where  $v_i$ 's are commutators in  $b_1, \dots, b_r$  of weight strictly greater than  $r$ , and every  $b_j$ ,  $1 \leq j \leq r$ , appears in each commutator  $v_i$ , the  $v_i$ 's listed in ascending order. The  $f_i(\alpha)$ 's are of the form (I), with  $a_1, \dots, a_{w_i} \in \mathbf{Z}$  and  $w_i$  is the weight of  $v_i$  (in the  $b_j$ 's) minus  $(r-1)$ .

**Remark 2.4.** Outer commutators on the letters  $x_1, x_2, \dots, x_n, \dots$  are defined inductively as follows:

The letter  $x_i$  is an outer commutator word of weight one. If  $u = u(x_1, \dots, x_s)$  and  $v = v(x_{s+1}, \dots, x_{s+t})$  are outer commutator words of weights  $s$  and  $t$ , then  $w(x_1, \dots, x_{s+t}) = [u(x_1, \dots, x_s), v(x_{s+1}, \dots, x_{s+t})]$  is an outer commutator word of weight  $s+t$  and will be written  $w = [u, v]$ .

It is noted by Struik [13] that Lemma 2.3 can be proved by a similar method, if  $[b_1, \dots, b_{i-1}, b_i^\alpha, b_{i+1}, \dots, b_r]$  and  $[b_1, \dots, b_r]$  are replaced with outer commutators.

By a routine calculation we have the following useful fact.

**Lemma 2.5.** Let  $p$  be a prime number and  $k$  be a nonnegative integer. If  $m = \lfloor \log_p k \rfloor$ , then  $p^t$  divides  $\binom{m+t}{k}$ , for all integers  $t \geq 1$ .

### 3. Main results

In order to prove the main result we need the following lemma.

**Lemma 3.1.** Let  $G$  be a  $p$ -group of class  $k$  and exponent  $p^e$  with a free presentation  $F/R$ . Then, for any  $c \geq 1$ , every outer commutator of weight  $w > c$  in  $F/[R, {}_c F]$  has an order dividing  $p^{e+m(c+k-w)}$ , where  $m = \lfloor \log_p k \rfloor$ .

**Proof.** Since  $\gamma_{k+1}(F) \subseteq R$ , we have  $\gamma_{c+k+1}(F) \subseteq [R, {}_c F]$ . Also, for all  $x$  in  $F$  and  $t \geq 0$  we have  $x^{p^{e+t}} \in R$  and hence every outer commutator of weight  $w > c$  in  $F$ , in which  $x^{p^{e+t}}$  appears, belongs to  $[R, {}_c F]$ . Now, we use inverse induction on  $w$  to prove the lemma. For the first step,  $w = c+k$ , the result follows by the above argument and Lemma 2.3.

Now, assume that the result is true, for all  $l > w$ . Put  $\alpha = p^{e+m(c+k-w)}$  and let  $u = [x_1, \dots, x_w]$  be an outer commutator of weight  $w$ . Then, by Lemma 2.3 and Remark 2.4 we have

$$[x_1^\alpha, \dots, x_w] = [x_1, \dots, x_w]^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \dots,$$

where the  $v_i^{f_i(\alpha)}$  are as in Lemma 2.3. Note that  $w < w_i = wt(v_i) \leq c+k$  modulo  $[R, {}_cF]$  and hence  $f_i(\alpha) = a_1 \binom{\alpha}{1} + a_2 \binom{\alpha}{2} + \dots + a_{w_i} \binom{\alpha}{k_i}$ , where  $k_i = w_i - w + 1 \leq c + k - w + 1 \leq k$ , for all  $i \geq 1$ . Thus, Lemma 2.5 implies that  $p^{e+m(c+k-w-1)}$  divides the  $f_i(\alpha)$ 's. Now, by induction hypothesis  $v_i^{f_i(\alpha)} \in [R, {}_cF]$ , for all  $i \geq 1$ . On the other hand, since  $x_1^\alpha \in R$  and  $w > c$ ,  $[x_1^\alpha, \dots, x_w] \in [R, {}_cF]$ . Therefore,  $u^\alpha \in [R, {}_cF]$  and this completes the proof.

**Theorem 3.2.** *Let  $G$  be a  $p$ -group of class  $k$  and exponent  $p^e$ . Let  $G = F/R$  be any free presentation of  $G$ . Then, the exponent of  $\gamma_{c+1}(F)/[R, {}_cF]$  divides  $p^{e+m(k-1)}$ , where  $m = \lfloor \log_p k \rfloor$ , for all  $c \geq 1$ .*

**Proof.** It is easy to see that every element  $g$  of  $\gamma_{c+1}(F)$  can be expressed as  $g = y_1 y_2 \dots y_n$ , where  $y_i$ 's are commutators of weight at least  $c+1$ . Put  $\alpha = p^{e+m(k-1)}$ . Now, Lemma 2.2 implies the identity

$$g^\alpha = y_{i_1}^\alpha y_{i_2}^\alpha \dots y_{i_n}^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \dots,$$

where  $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$  and  $v_i^{f_i(\alpha)}$ 's are as in Lemma 2.2. Then, the  $v_i$ 's are basic commutators of weight at least two and at most  $k$  in the  $y_i$ 's modulo  $[R, {}_cF]$  (note that  $\gamma_{c+k+1}(F) \subseteq [R, {}_cF]$ ). Thus, Lemma 2.5 yields that  $p^{e+m(k-2)}$  divides the  $f_i(\alpha)$ 's. Hence,  $v_i^{f_i(\alpha)} \in [R, {}_cF]$ , for all  $i \geq 1$  and  $y_j^\alpha \in [R, {}_cF]$ , for all  $1 \leq j \leq n$ , by Lemma 3.1. Therefore, we have  $g^\alpha \in [R, {}_cF]$  and the desired result now follows.

Now, we are in a position to state and prove the main result of the paper.

**Theorem 3.3.** *Let  $G$  be a  $p$ -group of class  $k$  and exponent  $p^e$ . Then,  $\exp(M^{(c)}(G))$  divides  $\exp(G)p^{m(k-1)}$ , where  $m = \lfloor \log_p k \rfloor$ , for all  $c \geq 1$ .*

**Proof.** Let  $G = F/R$  be any free presentation of  $G$ . Then,  $M^{(c)}(G) \leq \gamma_{c+1}(F)/[R, {}_cF]$ . Therefore,  $\exp(M^{(c)}(G))$  divides  $\exp(\gamma_{c+1}(F)/[R, {}_cF])$ . Now, the result follows by Theorem 2.3.

Note that the above result improves some previous bounds for the exponent of  $M(G)$  and  $M^{(c)}(G)$  as follows.

Let  $G$  be a  $p$ -group of class  $k$  and exponent  $p^e$ , then we have the following improvements.

(i) If  $\lfloor \log_p k \rfloor (k-1)/k < e$ , then  $\exp(G)p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{k-1}$ . Hence, in this case our result is an improvement of Jones's result [5]. In particular, our result improves the Jones's one for every  $p$ -group of exponent  $p^e$  and of class at most  $p^e - 1$ .

(ii) If  $\lfloor \log_p k \rfloor (k-1)/\lceil k/2 \rceil - 1 < e$ , then  $\exp(G)p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{\lceil k/2 \rceil}$  which shows that in this case our result is an improvement of Ellis's result [3]. In particular, our result improves the Ellis's one for every  $p$ -group of exponent  $p^e$  and of class  $k < p^{e/3}$ , for all  $k \geq 3$ , or of class  $k < p^{e/4}$ , for all  $k \geq 4$ .

(iii) If  $\lfloor \log_p k \rfloor (k-1)/2 \lfloor \log_2 k \rfloor - 1 < e$ , then  $\exp(G)p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{2 \lfloor \log_2 k \rfloor}$ . Thus, in this case our result is an improvement of Moravec's result [11]. In particular, our result improves the Moravec's one for every  $p$ -group of exponent  $p^e$  and of class  $k < e$ , for all  $k \geq 2$ .

**Corollary 3.4.** *Let  $G$  be a finite  $p$ -group of class at most  $p-1$ , then  $\exp(M^{(c)}(G))$  divides  $\exp(G)$ , for all  $c \geq 1$ . In particular,  $\exp(M(G))$  divides  $\exp(G)$ .*

Note that the above corollary shows that the mentioned conjecture on the exponent of the Schur multiplier of a finite  $p$ -group holds for all finite  $p$ -group of class at most  $p-1$ .

**Remark 3.5.** *Let  $G$  be a finite nilpotent group of class  $k$ . Then,  $G$  is the direct product of its Sylow subgroups,  $G = S_{p_1} \times \cdots \times S_{p_n}$ . Clearly,*

$$\exp(G) = \prod_{i=1}^n \exp(S_{p_i}).$$

By a result of G. Ellis [2, Theorem 5] we have

$$M^{(c)}(G) = M^{(c)}(S_{p_1}) \times \cdots \times M^{(c)}(S_{p_n}).$$

For all  $1 \leq i \leq n$ , put  $m_i = \lfloor \log_{p_i} k \rfloor$ . Then, by Theorem 3.3 we have

$$\exp(M^{(c)}(G)) \mid \exp(G) \prod_{i=1}^n p_i^{m_i(k-1)}.$$

Hence, the conjecture on the exponent of the Schur multiplier holds for all finite nilpotent group  $G$  of class at most  $\text{Max}\{p_1 - 1, \dots, p_n - 1\}$ , where  $p_1, \dots, p_n$  are all the distinct prime divisors of the order of  $G$ .

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