Bulletin of the Iranian Mathematical Society Vol. 37 *No. 4* (*2011*), *pp 251-267.*

ON n-COHERENT RINGS, n-HEREDITARY RINGS AND n-REGULAR RINGS

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Communicated by Fariborz Azarpanah

Abstract. We observe some new characterizations of n-presented modules. Using the concepts of $(n, 0)$-injectivity and $(n, 0)$-flatness of modules, we also present some characterizations of right n-coherent rings, right n-hereditary rings, and right n-regular rings.

1. **Introduction**

Throughout this paper, n is a positive integer unless a special note, R denotes an associative ring with identity and all modules considered are unitary. For any R-module M, $M^+ = Hom(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M.

B. Stenström [10] defined and studied *FP-injective modules*. Following [10], a right R-module M is said to be FP-injective, if $\mathrm{Ext}^1_R(A, M) = 0$, for every finitely presented right R-module A. A right R-module A is said to be finitely presented, if there is an exact sequence $F_1 \to F_0 \to A \to 0$ in which F_1, F_0 are finitely generated free right R-modules, or equivalently, if there is an exact sequence $P_1 \to P_0 \to A \to 0$ in which P_1, P_0 are finitely generated projective right R-modules. FP-injective modules are also called absolutely pure modules in [8], these modules

MSC(2000): Primary: 16D10; Secondary: 16E40.

Keywords: $(n, 0)$-injective modules, $(n, 0)$-flat modules, n-coherent rings, n-hereditary rings n-regular rings.

Received: 13 February 2009, Accepted: 16 August 2010.

have been studied by many authors. In papers [8] and [10], right Noetherian rings, right coherent rings, right semihereditary rings and regular rings are characterized by FP-injective right R-modules. It is well known that a left R-module M is flat if and only if $\mathrm{Tor}_1^R(A, M) = 0$, for every finitely presented right R-module A. Costa [2] introduced the concept of *n-presented* modules. Let n be a non-negative integer. According to [2], a right R-module M is called n-presented in case there is an exact sequence of right R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is a finitely generated free , equivalently projective right R-module; And a ring R is called right *n-coherent* [2] in case every n-presented right R-module is $(n+1)$-presented. Clearly, a ring R is right coherent if and only if it is right 1-coherent. We remark that the terminology of "n-coherence" in this paper is Costa's "n-coherence" but is not the same as that of [3]. Let n, d be non-negative integers. According to [12], a right R- module M is called (n, d)-injective, if $\mathrm{Ext}_R^{d+1}(A, M)$=0, for every n-presented right R-module A; A left R- module M is called (n, d)-flat, if $\mathrm{Tor}_{d+1}^R(A, M)$=0, for every n-presented right R-module A; A ring R is called a right (n, d)-ring, if every n-presented right R-module has the projective dimension at most d. Recall that a commutative right (n, d)-ring is called an (n, d)-ring [2], (n, d)-rings have been studied by several authors [2, 5, 6, 7, 12]. An $(n, 0)$-ring is called an n-von Neumann regular ring in papers [6] and [7].

In this paper, We will give Some characterizations and properties of n-presented modules and $(n, 0)$-injective modules as well as $(n, 0)$-flat modules. Moreover, we will generalize the concept of right semihereditary rings to right n-hereditary rings, and then we will generalize the concepts of regular rings and n-von Neumann regular rings to right n-regular rings. Right n-coherent rings, right n-hereditary rings and right n-regular rings will be characterized by $(n, 0)$-injective right R-modules and $(n, 0)$-flat left R-modules. $(n, 0)$-injective dimensions of right R-modules over right n-coherent rings and $(n, 0)$-flat dimensions of right R-modules over left n-coherent rings will be discussed.

First of all, we give some characterizations of n-presented modules.

Proposition 1.1. *The following statements are equivalent for a right R-module M:*

(1) *M is n-presented.*
(2) *There exists an exact sequence of right R-modules*

$$0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

such that $F_0, \cdots, F_{n-1}$ are finitely generated free right R-modules and K_n is finitely generated.

(3) *There exists an exact sequence of right R-modules*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that $P_0, \cdots, P_{n-1}$ are finitely generated projective right R-modules and K_n is finitely generated.

(4) *There exists an exact sequence of right R-modules*

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that $P_0, \cdots, P_{n-1}, P_n$ are finitely generated projective right R-modules.

(5) *There exists an exact sequence of right R-modules*

$$0 \to K \to F \to M \to 0$$

such that F is finitely generated free right R-module and K is $(n-1)$-presented.

(6) *There exists an exact sequence of right R-modules*

$$0 \to K \to P \to M \to 0$$

such that P is finitely generated projective right R-module and K is $(n-1)$-presented.

(7) *M is finitely generated and, if the sequence of right R-modules $0 \to L \to F \to M \to 0$ is exact with F finitely generated free, then L is $(n-1)$-presented.*

(8) *M is finitely generated and, if the sequence of right R-modules $0 \to L \to P \to M \to 0$ is exact with P finitely generated projective, then L is $(n-1)$-presented.*

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) , (7) $\Rightarrow$ (5), (8) $\Rightarrow$ (6) are obvious.

(3) $\Rightarrow$ (2). Use induction on n. In case of $n = 1$, suppose there exists an exact sequence of right R-modules $0 \to K_1 \to P_0 \to M \to 0$, where P_0 is finitely generated projective module and K_1 is finitely generated. Then, there exists an exact sequence of right R-modules $0 \to K \to F_0 \to M \to 0$ with F_0 finitely generated free module. By Schanuel's Lemma, $K_1 \oplus F_0 \cong K \oplus P_0$, so K is finitely generated and the result follows. Now, suppose (3) implies (2), for $n - 1$. Then, if there exists an exact sequence of right R-modules

$$0 \to K_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that $P_0, \cdots, P_{n-1}$ are finitely generated projective right R-modules and K_n is finitely generated. Then, we have an exact sequence

$$0 \to im(d_{n-1}) \to P_{n-2} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

. By induction hypothesis, There exists an exact sequence of right R-modules

$$0 \to K_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$$

such that $F_0, \cdots, F_{n-2}$ are finitely generated free right R-modules and K_{n-1} is finitely generated. Let $F_{n-1} \xrightarrow{\pi} K_{n-1}$ be epic with F_{n-1} finitely generated free module, then we obtain an exact sequence

$$0 \to Ker(\pi) \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

. By the generalization of Schanuel's Lemma [9, Exercise 3.37], $Ker(\pi)$ is finitely generated, and (2) follows.

(2) $\Rightarrow$ (5). Suppose there exists an exact sequence of right R-modules

$$0 \to K_n \to F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

such that $F_0, \cdots, F_{n-1}$ are finitely generated free right R-modules and K_n is finitely generated. Take $K = im(d_1)$, then K is $(n-1)$-presented and the sequence $0 \to K \to F_0 \to M \to 0$ is exact.

(5) $\Rightarrow$ (2). Let $0 \to K \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$ be exact, where K is $(n-1)$-presented. Then, there exists an exact sequence

$$0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} K \to 0$$

such that $F_1, \cdots, F_{n-1}$ are finitely generated free right R-modules and K_n is finitely generated, and thus we have an exact sequence of right R-modules

$$0 \to K_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \xrightarrow{\alpha d_1} F \xrightarrow{\beta} M \to 0,$$

and (2) follows.

(5) $\Rightarrow$ (7). Assume (5), then there exists an exact sequence of right R-modules $0 \to K \to F' \to M \to 0$. Clearly, M is finitely generated. If the sequence of right R-modules $0 \to L \to F \to M \to 0$ is exact with F finitely generated free, then by Schanuel's Lemma, $K \oplus F \cong L \oplus F'$, and so L is $(n-1)$-presented by [11, Theorem 1].

(3) $\Rightarrow$ (6) is similar to (2) $\Rightarrow$ (5) , (6) $\Rightarrow$ (8) is similar to (5) $\Rightarrow$ (7).□

From Proposition 1.1(5), it is easy to see that right n-coherent ring is right $(n+1)$-coherent.

2. n-coherent rings

We begin this section with some characterizations of right n-coherent rings.

Theorem 2.1. *The following statements are equivalent for a ring R:*

(1) *R is right n-coherent.*

(2) *If the sequence*

$$(*) \qquad F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

is exact, where each F_i is a finitely generated free right R-module, then there exists an exact sequence of right R-modules

$$(**) \qquad F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where each F_i is a finitely generated free right R-module.

(3) *Every $(n-1)$-presented submodule of a projective right R-module is n-presented.*

Proof. (1) $\Rightarrow$ (2). By the exactness of (*), we have an exact sequence

$$0 \to Ker(d_n) \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

Since R is right n-coherent, M is $(n+1)$-presented, so there exists an exact sequence of right R-modules

$$0 \to L_{n+1} \to F'_n \to F'_{n-1} \to \cdots \to F'_1 \to F'_0 \to M \to 0,$$

where each F'_i is finitely generated free, L_{n+1} is finitely generated. By the generalization of Schanuel's Lemma [9, Exercise 3.37], $Ker(d_n)$ is finitely generated, and then there exists a finitely generated free module F_{n+1} such that (**) holds.

(2) $\Rightarrow$ (1) is clear.

(1) $\Leftrightarrow$ (3) by Proposition 1.1. □

Recall that a right R-module M is FP-injective if and only if it is pure in every module containing it as a submodule. A submodule A of the right R-module B is said to be a pure submodule if for all left R-module M, the induced map $A \otimes_R M \to B \otimes_R M$ is monic, or equivalently, every finitely presented module is projective with respect to the exact sequence $0 \to A \to B \to B/A \to 0$. In this case, the exact sequence $0 \to A \to B \to B/A \to 0$ is called pure. We call a short exact sequence of right R-modules $0 \to A \to B \to C \to 0$ *n-pure*, if every n-presented right R- module is projective with respect to this sequence.

Next, we give some characterizations of $(n, 0)$-injective modules.

Theorem 2.2. *Let M be a right R-module, then the following statements are equivalent:*

(1) *M is $(n, 0)$-injective.*
(2) *M is injective with respect to every exact sequence $0 \to C \to B \to A \to 0$ of right R-modules with A n-presented.*
(3) *If K is an $(n-1)$-presented submodule of a projective right R-module P, then every right R-homomorphism f from K to M extends to a homomorphism from P to M.*
(4) *Every exact sequence $0 \to M \to M' \to M'' \to 0$ is n-pure.*
(5) *There exists an n-pure exact sequence $0 \to M \to M' \to M'' \to 0$ of right R-modules with M' injective.*
(6) *There exists an n-pure exact sequence $0 \to M \to M' \to M'' \to 0$ of right R-modules with M' $(n, 0)$-injective.*

Proof. (1) ⇒ (2). By the exact sequence $Hom(B, M) \to Hom(C, M) \to Ext^1_R(A, M) = 0$.

(2) ⇒ (3). Let $F = P \oplus P'$, where F is a free right R-module. Since K is finitely generated, there exists a finitely generated free module F_1 such that $K \le F_1 \le^{\oplus} F$. But, K is $(n-1)$-presented, so F_1/K is n-presented, and thus the induced map $Hom(F_1, M) \to Hom(K, M)$ is surjective by (2), and (3) follows.

(3) ⇒ (1). For any n-presented module A, there exists an exact sequence $0 \to K \to P \to A \to 0$, where P is finitely generated projective, K is $(n-1)$-presented. Hence, we get an exact sequence $Hom(P, M) \to Hom(K, M) \to Ext^1_R(A, M) \to Ext^1_R(P, M) = 0$, and thus $Ext^1_R(A, M) = 0$ by (3). Therefore, M is $(n, 0)$-injective.

(1) ⇒ (4). Assume (1). Then, we have an exact sequence $Hom(A, M') \to Hom(A, M'') \to Ext^1_R(A, M) = 0$, for every n-presented module A, and so (4) follows.

(4) ⇒ (5) ⇒ (6) are obvious.

(6) ⇒ (1). By (6), we have an n-pure exact sequence $0 \to M \to M' \xrightarrow{f} M'' \to 0$ of right R-modules where M' is $(n, 0)$-injective, and so, for

each n-presented module A, we have an exact sequence $Hom(A, M') \xrightarrow{f_*} Hom(A, M'') \to Ext^1_R(A, M) \to Ext^1_R(A, M') = 0$ with f_* epic. Which implies that $Ext^1_R(A, M) = 0$, and (1) follows. $\square$

Lemma 2.9(2) in [1] and Theorem 2.2(3) immediately yield the next two results.

Proposition 2.3. *Let $n \geq 2$, then every direct limit of $(n, 0)$-injective right R-modules is $(n, 0)$-injective.*

Proposition 2.4. *Let $\{M_i \mid i \in I\}$ be a family of right R-modules, then the following statements are equivalent:*
(1) Each M_i is $(n, 0)$-injective.
(2) $\prod_{i \in I} M_i$ is $(n, 0)$-injective.
(3) $\oplus_{i \in I} M_i$ is $(n, 0)$-injective.

Lemma 2.5. *Let E be an injective right R-module and N its $(k, 0)$-injective submodule, then E/N is $(k+1, 0)$-injective.*

Proof. Let A be any $(k+1)$-presented right R-module. Then, there exists an exact sequence $0 \to B \to P \to A \to 0$, where P is a finitely generated projective module and B is k-presented. So we get two exact sequences

$$0 = Ext^1_R(A, E) \to Ext^1_R(A, E/N) \to Ext^2_R(A, N) \to Ext^2_R(A, E) = 0$$

and

$$0 = Ext^1_R(P, N) \to Ext^1_R(B, N) \to Ext^2_R(A, N) \to Ext^2_R(P, N) = 0$$

Hence, $Ext^1_R(A, E/N) \cong Ext^1_R(B, N) = 0$, this follows that E/N is $(k+1, 0)$-injective. $\square$

Theorem 2.6. *Let A be an $(n-1)$-presented right R-module. Then, A is n-presented if and only if $Ext^1_R(A, M) = 0$, for any $(n, 0)$-injective module M.*

Proof. $\Rightarrow$. It is obvious.

$\Leftarrow$. Use induction on n. In case $n = 1$, then the implication holds by [4]. Suppose the implication holds when $n = k$. Then, when $n = k+1$, assume A is an k-presented right R-module and $Ext^1_R(A, M) = 0$, for every $(k+1, 0)$-injective module M. Since A is k-presented, there exists an exact sequence $0 \to L \to F \to A \to 0$ with F finitely generated free and L $(k-1)$-presented. So, for any $(k,0)$-injective module N, we have $Ext^1_R(L,N) \cong Ext^2_R(A,N) \cong Ext^1_R(A,E(N)/N)$. By Lemma 2.5, $E(N)/N$ is $(k+1,0)$-injective, so $Ext^1_R(A,E(N)/N)=0$ by conditions, and whence $Ext^1_R(L,N)=0$. Therefore, L is k-presented by hypothesis, which shows that A is $(k+1)$-presented. □

Theorem 2.7. *The following statements are equivalent for a ring R.*

(1) *R is right n-coherent.*
(2) *$Ext^1_R(A,N)=0$, for any n-presented right R-module A and any $(n+1,0)$-injective right R-module N.*
(3) *$Ext^2_R(A,N)=0$, for any n-presented right R-module A and any $(n,0)$-injective right R-module N.*
(4) *If N is an $(n,0)$-injective right R-module, N_1 is an $(n,0)$-injective submodule of N, then N/N_1 is $(n,0)$-injective.*
(5) *For any $(n,0)$-injective right R-module N, $E(N)/N$ is $(n,0)$-injective.*

Proof. (1) ⇒ (2) and (4) ⇒ (5) are obvious.

(2) ⇒ (1) by Theorem 2.6.

(1) ⇒ (3). Since A is n-presented, by Proposition 1.1(5), there exists an exact sequence of right R-modules $0 \to K \to F \to A \to 0$, where F is finitely generated free, K is $(n-1)$-presented, and we get an induced exact sequence

$$0 = Ext^1_R(F,N) \to Ext^1_R(K,N) \to Ext^2_R(A,N) \to Ext^2_R(F,N) = 0.$$

Hence, $Ext^2_R(A,N) \cong Ext^1_R(K,N)$. Since R is right n-coherent, by Theorem 2.1, K is n-presented, so $Ext^1_R(K,N)=0$, and thus $Ext^2_R(A,N) = 0$.

(3) ⇒ (4). For any n-presented right R-module A. The exact sequence $0 \to N_1 \to N \to N/N_1 \to 0$ induces the exactness of the sequence

$$0 = Ext^1(A,N) \to Ext^1(A,N/N_1) \to Ext^2(A,N_1) = 0.$$

Therefore, $Ext^1(A,N/N_1) = 0$, as desired.

(5) ⇒ (1). Let A be any n-presented right R-module. Then, by Proposition 1.1(5), there is an exact sequence of right R-modules $0 \to K \to F \to A \to 0$, where F is finitely generated free, K is $(n-1)$-presented. Then, for any $(n,0)$-injective module N, $E(N)/N$ is $(n,0)$-injective by (5). From the exactness of the two sequences

$$0 = Ext^1(F,N) \to Ext^1(K,N) \to Ext^2(A,N) \to Ext^2(F,N) = 0$$

and

$$0 = Ext^1(A, E(N)) \to Ext^1(A, E(N)/N) \to Ext^2(A, N) \to Ext^2(A, E(N)) = 0,$$

we have $Ext^1(K, N) \cong Ext^2(A, N) \cong Ext^1(A, E(N)/N) = 0$, so $Ext^1(K, N) = 0$. By Theorem 2.6, K is n-presented, hence A is $(n+1)$-presented. Therefore, R is right n-coherent. □

Definition 2.8.

(1). *The (n,0)-injective dimension of a module M_R is defined by*

$$(n,0)\text{-}id(M_R) = \inf\{k : Ext_R^{k+1}(A, M) = 0, \textit{ for every } n\textit{-presented module } A\}$$

(2). *The right (n,0)-injective global dimension of a ring R is defined by*

$$r.(n,0)\text{-}ID(R) = \sup\{(n,0)\text{-}id(M) : M \textit{ is a right } R\textit{-module}\}$$

Lemma 2.9. *Let R be a right n-coherent ring and let M be a right R-module, then the following statements are equivalent:*
(1) $(n,0)$-$id(M) \leq k$.
(2) $Ext_R^{k+1}(A, M) = 0$, for every n-presented right R-module A.

Proof. (1) ⇒ (2). Use induction on k. Clearly, if $(n,0)$-$id(M) = k$. If $(n,0)$-$id(M) \leq k-1$. Since A is n-presented, there exists an exact sequence $0 \to N \to P \to A \to 0$, where P is a finitely generated projective module and N is $(n-1)$-presented. But, R is right n-coherent, N is n-presented by Theorem 2.1, and so $Ext_R^{k+1}(A, M) \cong Ext_R^k(N, M) = 0$ by induction hypothesis.

(2) ⇒ (1) is clear. □

Corollary 2.10. *Let R be a right n-coherent ring and let M_R be (n,0)-injective, then $Ext_R^k(A, M) = 0$, for all n-presented modules A and all positive integers k.*

Corollary 2.11. *Let R be a right n-coherent ring and let M be a right R-module. If the sequence $0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} \cdots \to E_{k-1} \xrightarrow{d_{k-1}} E_k \to 0$ is exact with $E_0, \cdots, E_{k-1}$ (n,0)-injective, then $Ext_R^{k+1}(A, M) \cong Ext_R^1(A, E_k)$, for any n-presented right R-module A .*

Proof. Since R is right n-coherent and $E_0, E_1, \cdots, E_{k-1}$ are $(n,0)$-injective, by Corollary 2.10, we have $Ext_R^{k+1}(A,M) \cong Ext_R^k(A, im(d_0)) \cong Ext_R^{k-1}(A, im(d_1)) \cong \cdots \cong Ext_R^1(A, im(d_{k-1})) = Ext_R^1(A, E_k)$. □

Theorem 2.12. *Let R be a right n-coherent ring, M a right R-module and k a non-negative integer, then the following statements are equivalent:*

(1) $(n,0)$-$id(M_R) \leq k$.

(2) $Ext_R^{k+l}(A,M) = 0$, *for all n-presented modules A and all positive integers l.*

(3) $Ext_R^{k+1}(A,M) = 0$, *for all n-presented modules A.*

(4) *If the sequence $0 \to M \to E_0 \to \cdots \to E_{k-1} \to E_k \to 0$ is exact with $E_0, \cdots, E_{k-1}$ (n,0)-injective, then E_k is also (n,0)-injective.*

(5) *There exists an exact sequence $0 \to M \to E_0 \to \cdots \to E_{k-1} \to E_k \to 0$ of right R-modules with $E_0, \cdots, E_{k-1}, E_k$ (n,0)-injective.*

Proof. (1) ⇒ (2). Assume (1), then $(n,0)$-$id(M_R) \leq k+l-1$, and so (2) follows from Lemma 2.9.

(2) ⇒ (3) and (4) ⇒ (5) are obvious. (3) ⇒ (4) and (5) ⇒ (1) by Corollary 2.11. □

Theorem 2.13. *A right R-module M is (n,0)-flat if and only if the canonical map $M \otimes K \to M \otimes P$ is monic for every finitely generated projective left R-module P and any $(n-1)$-presented submodule K of P.*

Proof. It follows from the exact sequence

$$0 = Tor_1^R(M,P) \to Tor_1^R(M, P/K) \to M \otimes K \to M \otimes P.$$

□

Theorem 2.14. *Let $\{M_i \mid i \in I\}$ be a family of right R-modules, consider the following conditions:*

(1) Each M_i is $(n,0)$-flat.

(2) $\oplus_{i\in I} M_i$ is $(n,0)$-flat.

(3) $\prod_{i\in I} M_i$ is $(n,0)$-flat.

Then, we always have (3) ⇒ (1) ⇔ (2). *If $n \geq 2$, then these conditions are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) by the isomorphism $Tor_1^R(\prod_{i\in I} M_i, A) \cong \prod_{i\in I} Tor_1^R(M_i, A)$. (3) $\Rightarrow$ (1) is obvious. If $n \geq 2$, then by [1, Lemma 2.10], there is an isomorphism $Tor_1^R(\prod_{i\in I} M_i, A) \cong \prod_{i\in I} Tor_1^R(M_i, A)$, for every n-presented left R-module A, so in this case, the conditions (1), (2) and (3) are equivalent. □

Theorem 2.15. *Let M be a right R-module, then*

(1) M is $(n, 0)$-flat if and only if M^+ is $(n, 0)$-injective.

(2) If $n \geq 2$, then M is $(n, 0)$-injective if and only if M^+ is $(n, 0)$-flat.

Proof. (1) follows from the isomorphism $Tor_1^R(M, A)^+ \cong Ext_R^1(A, M^+)$. (2). Since $n \geq 2$, we have an isomorphism $Tor_1^R(A, M^+) \cong Ext_R^1(A, M)^+$, for every n-presented right R-module A by [1, Lemma 2.7(2)], and so (2) holds. □

Corollary 2.16. *If R is right coherent, then a right R-module M is FP-injective if and only if M^+ is flat.*

Proof. Since R is right coherent, a right R-module is finitely presented if and only if it is 2-presented. And so the result follows from Theorem 2.15(2). □

Corollary 2.17. *Pure submodules of $(n, 0)$-flat modules is $(n, 0)$-flat.*

Proof. Let M be an $(n, 0)$-flat module and M_1 a pure submodule of M, then the pure exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ induces a split exact sequence $0 \to (M/M_1)^+ \to M^+ \to M_1^+ \to 0$. By Theorem 2.15(1), M^+ is $(n, 0)$-injective, so M_1^+ is $(n, 0)$-injective by Theorem 2.4, and hence M_1 is $(n, 0)$-flat by Theorem 2.15(1). □

Definition 2.18. *The $(n, 0)$-flat dimension of a module M_R is defined by*

$$(n, 0)\text{-}fd(M_R) = inf\{k : Tor_{k+1}^R(M, A) = 0, \text{ for all } n\text{-presented left } R\text{-modules } A.\}$$

Lemma 2.19. *Let R be a left n-coherent ring and let M be a right R-module, then the following statements are equivalent:*

(1) $(n, 0)$-$fd(M_R) \leq k$.

(2) $Tor_{k+1}^R(M, A) = 0$, for every n-presented left R-module A.

Proof. (1) $\Rightarrow$ (2). Use induction on k. Clear, if $(n, 0)$-$fd(M) = k$. If $(n, 0)$-$fd(M) \leq k - 1$. Since A is n-presented, there exists an exact sequence $0 \to N \to P \to A \to 0$, where P is a finitely generated projective module and N is $(n-1)$-presented. But, R is left n-coherent, N is n-presented, and hence $Tor_{k+1}^R(M,A) \cong Tor_k^R(M,N) = 0$ by induction hypothesis.

(2) ⇒ (1) is clear. □

Corollary 2.20. *Let R be a left n-coherent ring and M_R be $(n,0)$-flat, then $Tor_k^R(M,A) = 0$, for all n-presented left R-modules A and all positive integers k.*

Corollary 2.21. *Let R be a left n-coherent ring and M be a right R-module. If the sequence of right R-modules $0 \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ is exact with $F_0, \cdots, F_{k-1}$ $(n,0)$-flat, then $Tor_1^R(F_k, A) \cong Tor_{k+1}^R(M,A)$, for any n-presented left R module A.*

Proof. Since R is left n-coherent and $F_0, F_1, \cdots, F_{k-1}$ are $(n,0)$-flat, by Corollary 2.20, we have

$$Tor_{k+1}^R(M,A) \cong Tor_k^R(Ker(d_0),A) \cong Tor_{k-1}^R(Ker(d_1),A) \cong \cdots$$
$$\cong Tor_1^R(Ker(d_{k-1}),A) \cong Tor_1^R(F_k,A).$$

□

Theorem 2.22. *Let R be a left n-coherent ring , M be a right R-module and $k \geq 0$, then the following statements are equivalent:*

1. *(1) $(n,0)$-$fd(M_R) \leq k$.*
2. *(2) $Tor_{k+l}^R(M,A) = 0$, for all n-presented left R-modules A and all positive integers l.*
3. *(3) $Tor_{k+1}^R(M,A) = 0$, for all n-presented left R-modules A.*
4. *(4) If the sequence $0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M \to 0$ is exact with $F_0, \cdots, F_{k-1}$ $(n,0)$-flat, then also F_k is $(n,0)$-flat.*
5. *(5) There exists an exact sequence $0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M \to 0$ of right R-modules with $F_0, \cdots, F_{k-1}, F_k$ $(n,0)$-flat.*

Proof. (1) ⇒ (2). Assume (1), then $(n,0)$-$fd(M_R) \leq k+l-1$, and so (2) follows from Lemma 2.19.

(2) ⇒ (3) and (4) ⇒ (5) are obvious. (3) ⇒ (4) and (5) ⇒ (1) by Corollary 2.21 and Lemma 2.19. □

3. n-hereditary rings and n-regular rings

Recall that a ring R is called right semihereditary, if every finitely generated right ideal of R is projective, or equivalently, if every finitely generated submodule of a projective right R-modules is projective. Next, we define n-hereditary rings as follows.

Definition 3.1. *A ring R is called right n-hereditary, if every $(n-1)$-presented submodule of projective right R-module is projective.*

Clearly, a ring R is right semihereditary if and only if it is right 1-hereditary. Right n-hereditary ring is right $(n+1)$-hereditary.

Theorem 3.2. *The following statements are equivalent for a ring R:*
(1) R is right n-hereditary.
(2) R is right n-coherent and $r.(n,0)$-$ID(R) \leq 1$.
(3) Factor module of (n,0)-injective right R-module is (n,0)-injective.
(4) Factor module of injective right R-module is (n,0)-injective.
(5) R is a right (n,1)-ring.

Proof. (1) $\Rightarrow$ (2). Since R is right n-hereditary, every $(n-1)$-presented submodule of a projective right R-module is finitely generated projective, and hence n-presented, so R is right n-coherent. Now, let M be any right R-module. Then, for any n-presented right R-module A, we have an exact sequence $0 \to N \to P \to A \to 0$ of right R-modules, where P is finitely generated and projective, N is $(n-1)$-presented and projective. Thus, the exact sequence $0 = Ext_R^1(N, M) \to Ext_R^2(A, M) \to Ext_R^2(P, M) = 0$ implies that $Ext_R^2(A, M) = 0$. This follows that $r.(n, 0)$-$ID(R) \leq 1$ by Definition 2.8.

(2) $\Rightarrow$ (3). Let M be an $(n, 0)$-injective right R-module and K its submodule. Then, for any n-presented module A, we have an exact sequence $0 = Ext_R^1(A, M) \to Ext_R^1(A, M/K) \to Ext_R^2(A, K) = 0$ by (2) and Lemma 2.9, and so $Ext_R^1(A, M/K) = 0$, as required.

(3) $\Rightarrow$ (4). It is obvious.

(4) $\Rightarrow$ (5). Since $Ext_R^2(A, B) \cong Ext_R^1(A, E(B)/B)$ holds for any right R-modules A and B, so (5) follows from (4).

(5) $\Rightarrow$ (1). Let N be an $(n-1)$-presented submodule of a projective right R-module P. Then, there exists a finitely generated free module F such that N is a submodule of F. Now, for any injective right R-module E and every submodule K of E, since F/N is

n-presented, $Ext_R^2(F/N,K)=0$ by (5), and so $Ext_R^1(N,K)=0$ as the sequence $0=Ext_R^1(F,K)\to Ext_R^1(N,K)\to Ext_R^2(F/N,K)=0$ is exact. This shows that N is E-projective because of the exact sequence $Hom(N,E)\to Hom(N,E/K)\to Ext_R^1(N,K)=0$. Therefore, N is projective. □

Example 3.3. *Let R be a non-coherent commutative ring of weak dimension one, then R is a (2,1)-ring but not a (1,1)-ring by [2, Example (6.5)], and so R is a 2-hereditary ring which is not 1-hereditary by Theorem 3.2.*

Theorem 3.4. *A domain R is n-hereditary if and only if every $(n-1)$-presented torsion-free R-module is projective.*

Proof Since R is a domain, every finitely generated torsion-free R-module may be imbedded in a free module and every submodule of a free R-module is torsion-free. Hence, the results follows. □

Theorem 3.5. *If $n\geq 2$, then the following statements are equivalent for a ring R:*

(1) R is a right n-hereditary ring.

(2) Every submodule of an $(n,0)$-flat left R-module is $(n,0)$-flat.

Proof (1) $\Rightarrow$ (2). Let M be an $(n,0)$-flat left R-module and let K be its submodule. Then, for any n-presented right R-module A, there exists an exact sequence $0\to N\to P\to A\to 0$, where P is a finitely generated projective module and N is $(n-1)$-presented. Since R is a right n-hereditary ring, N is projective, hence we have an exact sequence $0=Tor_2^R(P,M/K)\to Tor_2^R(A,M/K)\to Tor_1^R(N,M/K)=0$, it shows that $Tor_2^R(A,M/K)=0$. Therefore, by the exact sequence $0=Tor_2^R(A,M/K)\to Tor_1^R(A,K)\to Tor_1^R(A,M)=0$, we get $Tor_1^R(A,K)=0$, i.e., K is $(n,0)$-flat.

(2) $\Rightarrow$ (1). Suppose B is an $(n,0)$-injective right R-module with an exact sequence $0\to A\to B\to C\to 0$. Then, B^+ is an $(n,0)$-flat left R-module by Theorem 2.15(2), and the sequence $0\to C^+\to B^+\to A^+\to 0$ is exact. By (2), C^+ is $(n,0)$-flat, so C is $(n,0)$-injective again by Theorem 2.15(2). Hence, R is right n-hereditary by Theorem 3.2(3). □

Corollary 3.6. *If $n \geq 2$ and the weak dimension of R $wD(R) \leq 1$, then R is left and right n-hereditary.*

Proof. Assume M is an $(n, 0)$-flat right R-module and K is a submodule of M. Then, for any n-presented left R-module A, since $wD(R) \leq 1$, $Tor_2^R(M/K, A) = 0$, this follows that $Tor_1^R(K, A) = 0$ because M is $(n, 0)$-flat, and thus K is $(n, 0)$-flat. By Theorem 3.5, R is left n-hereditary. Similarly, one can prove that R is right n-hereditary. □

Next, we generalize the concepts of regular rings and n-von Neumann rings to right n-regular rings.

Definition 3.7. *A ring R is called right n-regular, if it is a right $(n,0)$-ring.*

Clearly, R is regular if and only if it is right 1-regular, R is n-von Neumann ring, if it is a commutative right n-regular ring. Right n-regular ring is right $(n + 1)$-regular.

Example 3.8. *Let K be a field and E be a K-vector space with infinite rank. Set $B = K \propto E$ the trivial extension of K by E. Then, by [6, Theorem 3.4], R is a commutative 2-regular rings which is not regular. So, in general, right 2-regular ring need not be regular.*

Theorem 3.9. *The following conditions are equivalent for a ring R.*

(1) *R is a right n-regular ring.*
(2) *Every right R-module is $(n, 0)$-injective.*
(3) *Every finitely generated right R-module is $(n, 0)$-injective.*
(4) *R is right n-hereditary and R_R is $(n,0)$-injective.*
(5) *R is right n-coherent and every n-presented right R-module is $(n, 0)$-injective.*
(6) *Every $(n-1)$-presented submodule of a projective right R-module is a direct summand.*
(7) *Every n-presented right R-module is flat.*
(8) *Every left R-module is $(n, 0)$-flat.*

Proof. (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Assume (3). Then, clearly R_R is $(n, 0)$-injective. Let P be a projective module and let K be an $(n - 1)$-presented submodule of P. By (3), K is $(n, 0)$-injective, so by Theorem 2.2(3), we have that K is a direct summand of P and hence K is projective. Therefore, R is right n-hereditary .

(4) $\Rightarrow$ (5). Assume (4), then every $(n-1)$-presented submodule of a projective module is projective and finitely generated, and then it is n-presented, so R is right n-coherent by Theorem 2.1(3). Now, let M be an n-presented right R-module, then there exists an exact sequence of right R-modules $F \to M \to 0$, where F is finitely generated free. Since R_R is $(n, 0)$-injective, by Proposition 2.4, F is $(n, 0)$-injective. Observing that R is right n-hereditary, by Theorem 3.2(3), M is $(n, 0)$-injective.

(5) $\Rightarrow$ (6). Let M be an $(n-1)$-presented submodule of a projective right R-module P. Then, M is a submodule of a finitely generated free right R-module F. By Proposition 1.1(5), F/M is n-presented. Since R is right n-coherent, F/M is $(n+1)$-presented. So, M is n-presented by Proposition 1.1(7), and hence M is $(n, 0)$-injective by (5). This follows that M is a direct summand of P by Theorem 2.2(3).

(6) $\Rightarrow$ (1). Let M be an n-presented right R-module, then there exists an exact sequence of right R-modules $0 \to K \to P \to M \to 0$, where P is finitely generated projective and K is $(n-1)$-presented. By hypothesis, K is a direct summand of P. Hence, M is isomorphic to a direct summand of P, and so M is projective.

(1) $\Leftrightarrow$ (7) and (7) $\Leftrightarrow$ (8) are obvious. □

Acknowledgments

The author would like to thank the referee for a careful reading of the article and giving a detailed report.

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