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# ON *n*-COHERENT RINGS, *n*-HEREDITARY RINGS AND *n*-REGULAR RINGS

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ABSTRACT. We observe some new characterizations of *n*-presented modules. Using the concepts of (n, 0)-injectivity and (n, 0)-flatness of modules, we also present some characterizations of right *n*-coherent rings, right *n*-hereditary rings, and right *n*-regular rings.

# 1. Introduction

Throughout this paper, n is a positive integer unless a special note, R denotes an associative ring with identity and all modules considered are unitary. For any R-module M,  $M^+ = Hom(M, \mathbb{Q}/\mathbb{Z})$  will be the character module of M.

B. Stenström [10] defined and studied FP-injective modules. Following [10], a right *R*-module *M* is said to be FP-injective, if  $\operatorname{Ext}_R^1(A, M) =$ 0, for every finitely presented right *R*-module *A*. A right *R*-module *A* is said to be finitely presented, if there is an exact sequence  $F_1 \to F_0 \to$  $A \to 0$  in which  $F_1, F_0$  are finitely generated free right *R*-modules, or equivalently, if there is an exact sequence  $P_1 \to P_0 \to A \to 0$  in which  $P_1, P_0$  are finitely generated projective right *R*-modules. *FP*-injective modules are also called absolutely pure modules in [8], these modules

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<sup>251</sup> 

have been studied by many authors. In papers [8] and [10], right Noetherian rings, right coherent rings, right semihereditary rings and regular rings are characterized by FP-injective right R-modules. It is well known that a left *R*-module *M* is flat if and only if  $\operatorname{Tor}_{1}^{R}(A, M) = 0$ , for every finitely presented right R-module A. Costa [2] introduced the concept of n-presented modules. Let n be a non-negative integer. According to [2], a right *R*-module M is called *n*-presented in case there is an exact sequence of right *R*-modules  $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$  in which every  $F_i$  is a finitely generated free, equivalently projective right *R*-module; And a ring *R* is called right *n*-coherent [2] in case every *n*presented right R-module is (n+1)-presented. Clearly, a ring R is right coherent if and only if it is right 1-coherent. We remark that the terminology of "*n*-coherence" in this paper is Costa's "*n*-coherence" but is not the same as that of [3]. Let n, d be non-negative integers. According to [12], a right *R*- module *M* is called (n, d)-injective, if  $\operatorname{Ext}_{R}^{d+1}(A, M)=0$ , for every *n*-presented right *R*-module *A*; A left *R*- module *M* is called (n, d)-flat, if  $\operatorname{Tor}_{d+1}^{R}(A, M) = 0$ , for every *n*-presented right *R*-module *A*; A ring R is called a right (n, d)-ring, if every n-presented right R-module has the projective dimension at most d. Recall that a commutative right (n, d)-ring is called an (n, d)-ring [2], (n, d)-rings have been studied by several authors [2, 5, 6, 7, 12]. An (n, 0)-ring is called an *n*-von Neumann regular ring in papers [6] and [7].

In this paper, We will give Some characterizations and properties of n-presented modules and (n, 0)-injective modules as well as (n, 0)-flat modules. Moreover, we will generalize the concept of right semihereditary rings to right n-hereditary rings, and then we will generalize the concepts of regular rings and n-von Neumann regular rings to right nregular rings. Right n-coherent rings, right n-hereditary rings and right n-regular rings will be characterized by (n, 0)-injective right R-modules and (n, 0)-flat left R-modules. (n, 0)-injective dimensions of right R-modules over right n-coherent rings and (n, 0)-flat dimensions of right R-modules over right n-coherent rings will be discussed.

First of all, we give some characterizations of n-presented modules.

**Proposition 1.1.** The following statements are equivalent for a right *R*-module *M*:

- (1) M is *n*-presented.
- (2) There exists an exact sequence of right R-modules

 $0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ 

such that  $F_0, \dots, F_{n-1}$  are finitely generated free right R-modules and  $K_n$  is finitely generated.

(3) There exists an exact sequence of right R-modules

 $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ 

such that  $P_0, \dots, P_{n-1}$  are finitely generated projective right *R*-modules and  $K_n$  is finitely generated.

(4) There exists an exact sequence of right R-modules

$$P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$$

such that  $P_0, \dots, P_{n-1}, P_n$  are finitely generated projective right *R*-modules.

(5) There exists an exact sequence of right R-modules

$$0 \to K \to F \to M \to 0$$

such that F is finitely generated free right R-module and K is (n-1)-presented.

(6) There exists an exact sequence of right R-modules

$$0 \to K \to P \to M \to 0$$

such that P is finitely generated projective right R-module and K is (n-1)-presented.

- (7) *M* is finitely generated and, if the sequence of right *R*-modules  $0 \to L \to F \to M \to 0$  is exact with *F* finitely generated free, then *L* is (n-1)-presented.
- (8) M is finitely generated and, if the sequence of right R-modules  $0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0$  is exact with P finitely generated projective, then L is (n-1)-presented.

**Proof.** (1)  $\Leftrightarrow$  (2)  $\Rightarrow$  (3)  $\Leftrightarrow$  (4), (7)  $\Rightarrow$  (5), (8)  $\Rightarrow$  (6) are obvious.

 $(3) \Rightarrow (2)$ . Use induction on n. In case of n = 1, suppose there exists an exact sequence of right R-modules  $0 \to K_1 \to P_0 \to M \to 0$ , where  $P_0$  is finitely generated projective module and  $K_1$  is finitely generated. Then, there exists an exact sequence of right R-modules  $0 \to K \to F_0 \to M \to 0$  with  $F_0$  finitely generated free module. By Schanuel's Lemma,  $K_1 \oplus F_0 \cong K \oplus P_0$ , so K is finitely generated and the result follows. Now, suppose (3) implies (2), for n - 1. Then, if there exists an exact sequence of right R-modules

$$0 \to K_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that  $P_0, \dots, P_{n-1}$  are finitely generated projective right *R*-modules and  $K_n$  is finitely generated. Then, we have an exact sequence

$$0 \to im(d_{n-1}) \to P_{n-2} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

. By induction hypothesis, There exists an exact sequence of right  $R\operatorname{\!-modules}$ 

$$0 \to K_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$$

such that  $F_0, \dots, F_{n-2}$  are finitely generated free right R-modules and  $K_{n-1}$  is finitely generated. Let  $F_{n-1} \xrightarrow{\pi} K_{n-1}$  be epic with  $F_{n-1}$  finitely generated free module, then we obtain an exact sequence

$$0 \to Ker(\pi) \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

. By the generalization of Schanuel's Lemma [9, Exercise 3.37],  $Ker(\pi)$  is finitely generated, and (2) follows.

 $(2) \Rightarrow (5)$ . Suppose there exists an exact sequence of right *R*-modules

$$0 \to K_n \to F_{n-1} \to \dots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

such that  $F_0, \dots, F_{n-1}$  are finitely generated free right *R*-modules and  $K_n$  is finitely generated. Take  $K = im(d_1)$ , then K is (n-1)-presented and the sequence  $0 \to K \to F_0 \to M \to 0$  is exact.

 $(5) \Rightarrow (2)$ . Let  $0 \to K \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$  be exact, where K is (n-1)-presented. Then, there exists an exact sequence

$$0 \to K_n \to F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} K \to 0$$

such that  $F_1, \dots, F_{n-1}$  are finitely generated free right *R*-modules and  $K_n$  is finitely generated, and thus we have an exact sequence of right *R*-modules

$$0 \to K_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \stackrel{\alpha d_1}{\to} F \stackrel{\beta}{\to} M \to 0,$$

and (2) follows.

 $(5) \Rightarrow (7)$ . Assume (5), then there exists an exact sequence of right R-modules  $0 \to K \to F' \to M \to 0$ . Clearly, M is finitely generated. If the sequence of right R-modules  $0 \to L \to F \to M \to 0$  is exact with F finitely generated free, then by Schanuel's Lemma,  $K \oplus F \cong L \oplus F'$ , and so L is (n-1)-presented by [11, Theorem 1].

 $(3) \Rightarrow (6)$  is similar to  $(2) \Rightarrow (5)$ ,  $(6) \Rightarrow (8)$  is similar to  $(5) \Rightarrow (7).\square$ 

From Proposition 1.1(5), it is easy to see that right *n*-coherent ring is right (n + 1)-coherent.

### 2. *n*-coherent rings

We begin this section with some characterizations of right n-coherent rings.

**Theorem 2.1.** The following statements are equivalent for a ring R:

- (1) R is right n-coherent.
- (2) If the sequence

(\*) 
$$F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

is exact, where each  $F_i$  is a finitely generated free right R-module, then there exists an exact sequence of right R-modules

$$(**) F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where each  $F_i$  is a finitely generated free right R-module.

(3) Every (n-1)-presented submodule of a projective right R-module is n-presented.

**Proof.**  $(1) \Rightarrow (2)$ . By the exactness of (\*), we have an exact sequence

$$0 \to Ker(d_n) \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \dots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

Since R is right n-coherent, M is (n+1)-presented, so there exists an exact sequence of right R-modules

$$0 \to L_{n+1} \to F'_n \to F'_{n-1} \to \cdots \to F'_1 \to F'_0 \to M \to 0,$$

where each  $F'_i$  is finitely generated free,  $L_{n+1}$  is finitely generated. By the generalization of Schanuel's Lemma [9, Exercise 3.37],  $Ker(d_n)$  is finitely generated, and then there exists a finitely generated free module  $F_{n+1}$  such that (\*\*) holds.

 $(2) \Rightarrow (1)$  is clear.

 $(1) \Leftrightarrow (3)$  by Proposition 1.1.

Recall that a right *R*-module *M* is *FP*-injective if and only if it is pure in every module containing it as a submodule. A submodule *A* of the right *R*-module *B* is said to be a pure submodule if for all left *R*-module *M*, the induced map  $A \otimes_R M \to B \otimes_R M$  is monic, or equivalently, every finitely presented module is projective with respect to the exact sequence  $0 \to A \to B \to B/A \to 0$ . In this case, the exact sequence  $0 \to A \to B \to B/A \to 0$  is called pure. We call a short exact sequence of right *R*-modules  $0 \to A \to B \to C \to 0$  *n*-pure, if every *n*-presented right *R*- module is projective with respect to this sequence.

Next, we give some characterizations of (n, 0)-injective modules.

**Theorem 2.2.** Let M be a right R-module, then the following statements are equivalent:

- (1) M is (n, 0)-injective.
- (2) *M* is injective with respect to every exact sequence  $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$  of right *R*-modules with *A* n-presented.
- (3) If K is an (n-1)-presented submodule of a projective right R-module P, then every right R-homomorphism f from K to M extends to a homomorphism from P to M.
- (4) Every exact sequence  $0 \to M \to M' \to M'' \to 0$  is n-pure.
- (5) There exists an n-pure exact sequence  $0 \to M \to M' \to M'' \to 0$ of right R-modules with M' injective.
- (6) There exists an n-pure exact sequence  $0 \to M \to M' \to M'' \to 0$ of right R-modules with M'(n,0)-injective.

**Proof.** (1)  $\Rightarrow$  (2). By the exact sequence  $Hom(B, M) \rightarrow Hom(C, M) \rightarrow Ext^{1}_{B}(A, M) = 0.$ 

 $(2) \Rightarrow (3)$ . Let  $F = P \oplus P'$ , where F is a free right R-module. Since K is finitely generated, there exists a finitely generated free module  $F_1$  such that  $K \leq F_1 \leq^{\oplus} F$ . But, K is (n-1)-presented, so  $F_1/K$  is n-presented, and thus the induced map  $Hom(F_1, M) \to Hom(K, M)$  is surjective by (2), and (3) follows.

 $(3) \Rightarrow (1)$ . For any *n*-presented module A, there exists an exact sequence  $0 \to K \to P \to A \to 0$ , where P is finitely generated projective, K is (n-1)-presented. Hence, we get an exact sequence  $Hom(P, M) \to Hom(K, M) \to Ext_R^1(A, M) \to Ext_R^1(P, M) = 0$ , and thus  $Ext_R^1(A, M) = 0$  by (3). Therefore, M is (n, 0)-injective.

 $(1) \Rightarrow (4)$ . Assume (1). Then, we have an exact sequence  $Hom(A, M') \rightarrow Hom(A, M'') \rightarrow Ext_R^1(A, M) = 0$ , for every *n*-presented module *A*, and so (4) follows.

 $(4) \Rightarrow (5) \Rightarrow (6)$  are obvious.

 $(6) \Rightarrow (1)$ . By (6), we have an *n*-pure exact sequence  $0 \to M \to M' \xrightarrow{J} M'' \to 0$  of right *R*-modules where M' is (n, 0)-injective, and so, for

each *n*-presented module A, we have an exact sequence  $Hom(A, M') \xrightarrow{J_*}$  $Hom(A, M'') \to Ext^1_B(A, M) \to Ext^1_B(A, M') = 0$  with  $f_*$  epic. Which implies that  $Ext_R^1(A, M) = 0$ , and (1) follows. 

Lemma 2.9(2) in [1] and Theorem 2.2(3) immediately yield the next two results.

**Proposition 2.3.** Let  $n \geq 2$ , then every direct limit of (n, 0)-injective right R-modules is (n, 0)-injective.

**Proposition 2.4.** Let  $\{M_i \mid i \in I\}$  be a family of right *R*-modules, then the following statements are equivalent:

(1) Each  $M_i$  is (n, 0)-injective.

(2)  $\prod_{i \in I} M_i$  is (n, 0)-injective.

(3)  $\oplus_{i \in I} M_i$  is (n, 0)-injective.

**Lemma 2.5.** Let E be an injective right R-module and N its (k, 0)injective submodule, then E/N is (k+1,0)-injective.

**Proof.** Let A be any (k + 1)-presented right R-module. Then, there exists an exact sequence  $0 \to B \to P \to A \to 0$ , where P is a finitely generated projective module and B is k-presented. So we get two exact sequences

sequences  $0 = Ext_R^1(A, E) \to Ext_R^1(A, E/N) \to Ext_R^2(A, N) \to Ext_R^2(A, E) = 0$ and

and

$$0 = Ext^1_R(P,N) \to Ext^1_R(B,N) \to Ext^2_R(A,N) \to Ext^2_R(P,N) = 0$$

Hence,  $Ext_R^1(A, E/N) \cong Ext_R^1(B, N) = 0$ , this follows that E/N is  $\pm 1.0$  -injective. (k+1, 0)-injective. 

**Theorem 2.6.** Let A be an (n-1)-presented right R-module. Then, A is n-presented if and only if  $Ext^{1}_{R}(A, M) = 0$ , for any (n, 0)-injective module M.

**Proof.**  $\Rightarrow$  . It is obvious.

 $\Leftarrow$ . Use induction on n. In case n = 1, then the implication holds by [4]. Suppose the implication holds when n = k. Then, when n = k + 1, assume A is an k-presented right R-module and  $Ext^{1}_{R}(A, M) = 0$ , for every (k+1, 0)-injective module M. Since A is k-presented, there exists an exact sequence  $0 \to L \to F \to A \to 0$  with F finitely generated

free and L(k-1)-presented. So, for any (k, 0)-injective module N, we have  $Ext_R^1(L, N) \cong Ext_R^2(A, N) \cong Ext_R^1(A, E(N)/N)$ . By Lemma 2.5, E(N)/N is (k+1, 0)-injective, so  $Ext_R^1(A, E(N)/N) = 0$  by conditions, and whence  $Ext_R^1(L, N) = 0$ . Therefore, L is k-presented by hypothesis, which shows that A is (k+1)-presented.

**Theorem 2.7.** The following statements are equivalent for a ring R.

- (1) R is right n-coherent.
- (2)  $Ext_R^1(A, N) = 0$ , for any n-presented right R-module A and any (n+1, 0)-injective right R-module N.
- (3)  $Ext_R^2(A, N) = 0$ , for any n-presented right R-module A and any (n, 0)-injective right R-module N.
- (4) If N is an (n,0)-injective right R-module, N<sub>1</sub> is an (n,0)-injective submodule of N, then N/N<sub>1</sub> is (n,0)-injective.
- (5) For any (n,0)-injective right R-module N, E(N)/N is (n,0)-injective.

**Proof.**  $(1) \Rightarrow (2)$  and  $(4) \Rightarrow (5)$  are obvious.

 $(2) \Rightarrow (1)$  by Theorem 2.6.

 $(1) \Rightarrow (3)$ . Since A is n-presented, by Proposition 1.1(5), there exists an exact sequence of right R-modules  $0 \to K \to F \to A \to 0$ , where F is finitely generated free, K is (n-1)-presented, and we get an induced exact sequence

$$0 = Ext^1_R(F, N) \to Ext^1_R(K, N) \to Ext^2_R(A, N) \to Ext^2_R(F, N) = 0.$$

Hence,  $Ext_R^2(A, N) \cong Ext_R^1(K, N)$ . Since R is right n-coherent, by Theorem 2.1, K is n-presented, so  $Ext_R^1(K, N) = 0$ , and thus  $Ext_R^2(A, N) = 0$ .

(3)  $\Rightarrow$  (4). For any *n*-presented right *R*-module *A*. The exact sequence  $0 \rightarrow N_1 \rightarrow N \rightarrow N/N_1 \rightarrow 0$  induces the exactness of the sequence

$$0 = Ext^{1}(A, N) \to Ext^{1}(A, N/N_{1}) \to Ext^{2}(A, N_{1}) = 0.$$

Therefore,  $Ext^1(A, N/N_1) = 0$ , as desired.

 $(5) \Rightarrow (1)$ . Let A be any n-presented right R-module. Then, by Proposition 1.1(5), there is an exact sequence of right R-modules  $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ , where F is finitely generated free, K is (n-1)-presented. Then, for any (n, 0)-injective module N, E(N)/N is (n, 0)-injective by (5). From the exactness of the two sequences

$$0 = Ext^{1}(F, N) \to Ext^{1}(K, N) \to Ext^{2}(A, N) \to Ext^{2}(F, N) = 0$$

$$0 = Ext^{1}(A, E(N)) \rightarrow Ext^{1}(A, E(N)/N) \rightarrow Ext^{2}(A, N) \rightarrow Ext^{2}(A, E(N)) = 0,$$

we have  $Ext^{1}(K, N) \cong Ext^{2}(A, N) \cong Ext^{1}(A, E(N)/N) = 0$ , so  $Ext^{1}(K, N) = 0$ . By Theorem 2.6, K is n-presented, hence A is (n + 1)-presented. Therefore, R is right n-coherent.

## Definition 2.8.

- (1). The (n,0)-injective dimension of a module  $M_R$  is defined by (n,0)-id $(M_R) = \inf\{k : Ext_R^{k+1}(A,M) = 0, \text{ for every}$ n-presented module  $A\}$
- (2). The right (n,0)-injective global dimension of a ring R is defined by

$$r.(n,0)-ID(R)=\sup\{(n,0)-id(M): M \text{ is a right } R\text{-module}\}$$

**Lemma 2.9.** Let R be a right n-coherent ring and let M be a right R-module, then the following statements are equivalent:

(1) (n, 0)- $id(M) \le k$ .

(2)  $Ext_{R}^{k+1}(A, M) = 0$ , for every n-presented right R-module A.

**Proof.** (1)  $\Rightarrow$  (2). Use induction on k. Clearly, if (n, 0)-id(M) = k. If (n, 0)- $id(M) \leq k - 1$ . Since A is n-presented, there exists an exact sequence  $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$ , where P is a finitely generated projective module and N is (n - 1)-presented. But, R is right n-coherent, N is n-presented by Theorem 2.1, and so  $Ext_R^{k+1}(A, M) \cong Ext_R^k(N, M) = 0$  by induction hypothesis.

 $(2) \Rightarrow (1)$  is clear.

**Corollary 2.10.** Let R be a right n-coherent ring and let  $M_R$  be (n,0)injective, then  $Ext_R^k(A, M) = 0$ , for all n-presented modules A and all
positive integers k.

**Corollary 2.11.** Let R be a right n-coherent ring and let M be a right R-module. If the sequence  $0 \to M \stackrel{\varepsilon}{\to} E_0 \stackrel{d_0}{\to} \cdots \to E_{k-1} \stackrel{d_{k-1}}{\to} E_k \to 0$  is exact with  $E_0, \cdots, E_{k-1}$  (n,0)-injective, then  $Ext_R^{k+1}(A, M) \cong Ext_R^1(A, E_k)$ , for any n-presented right R-module A.

**Proof.** Since R is right n-coherent and  $E_0, E_1, \dots, E_{k-1}$  are (n, 0)injective, by Corollary 2.10, we have  $Ext_R^{k+1}(A, M) \cong Ext_R^k(A, im(d_0))$   $\cong Ext_R^{k-1}(A, im(d_1)) \cong \dots \cong Ext_R^1(A, im(d_{k-1})) = Ext_R^1(A, E_k). \square$ 

**Theorem 2.12.** Let R be a right n-coherent ring, M a right R-module and k a non-negative integer, then the following statements are equivalent:

- (1) (n,0)- $id(M_R) \le k$ .
- (2)  $Ext_{R}^{k+l}(A, M) = 0$ , for all n-presented modules A and all positive integers l.
- (3)  $Ext_{R}^{k+1}(A, M) = 0$ , for all n-presented modules A.
- (4) If the sequence  $0 \to M \to E_0 \to \cdots \to E_{k-1} \to E_k \to 0$  is exact with  $E_0, \cdots, E_{k-1}$  (n, 0)-injective, then  $E_k$  is also (n, 0)-injective.
- (5) There exists an exact sequence  $0 \to M \to E_0 \to \cdots \to E_{k-1} \to E_k \to 0$  of right *R*-modules with  $E_0, \cdots, E_{k-1}, E_k$  (n,0)-injective.

**Proof.** (1)  $\Rightarrow$  (2). Assume (1), then (n, 0)- $id(M_R) \leq k + l - 1$ , and so (2) follows from Lemma 2.9.

 $(2) \Rightarrow (3)$  and  $(4) \Rightarrow (5)$  are obvious.  $(3) \Rightarrow (4)$  and  $(5) \Rightarrow (1)$  by Corollary 2.11.

**Theorem 2.13.** A right *R*-module *M* is (n,0)-flat if and only if the canonical map  $M \otimes K \to M \otimes P$  is monic for every finitely generated projective left *R*-module *P* and any (n-1)-presented submodule *K* of *P*.

**Proof.** It follows from the exact sequence

$$0 = Tor_1^R(M, P) \to Tor_1^R(M, P/K) \to M \otimes K \to M \otimes P.$$

**Theorem 2.14.** Let  $\{M_i \mid i \in I\}$  be a family of right *R*-modules, consider the following conditions:

(1) Each  $M_i$  is (n, 0)-flat.

- (2)  $\oplus_{i \in I} M_i$  is (n, 0)-flat.
- (3)  $\prod_{i \in I} M_i$  is (n, 0)-flat.

Then, we always have  $(3) \Rightarrow (1) \Leftrightarrow (2)$ . If  $n \ge 2$ , then these conditions are equivalent.

**Proof.** (1)  $\Leftrightarrow$  (2) by the isomorphism  $Tor_1^R(\prod_{i \in I} M_i, A) \cong \prod_{i \in I} Tor_1^R(M_i, A)$ . (3)  $\Rightarrow$  (1) is obvious. If  $n \ge 2$ , then by [1, Lemma 2.10], there is an isomorphism  $Tor_1^R(\prod_{i \in I} M_i, A) \cong \prod_{i \in I} Tor_1^R(M_i, A)$ , for every *n*-presented left *R*-module *A*, so in this case, the conditions (1), (2) and (3) are equivalent.

### **Theorem 2.15.** Let M be a right R-module, then

(1) M is (n,0)-flat if and only if  $M^+$  is (n,0)-injective.

(2) If  $n \ge 2$ , then M is (n, 0)-injective if and only if  $M^+$  is (n, 0)-flat.

**Proof.** (1) follows from the isomorphism  $Tor_1^R(M, A)^+ \cong Ext_R^1(A, M^+)$ . (2). Since  $n \ge 2$ , we have an isomorphism  $Tor_1^R(A, M^+) \cong Ext_R^1(A, M)^+$ , for every *n*-presented right *R*-module *A* by [1, Lemma 2.7(2)], and so (2) holds.

**Corollary 2.16.** If R is right coherent, then a right R-module M is FP-injective if and only if  $M^+$  is flat.

**Proof.** Since R is right coherent, a right R-module is finitely presented if and only if it is 2-presented. And so the result follows from Theorem 2.15(2).

**Corollary 2.17.** Pure submodules of (n, 0)-flat modules is (n, 0)-flat.

**Proof.** Let M be an (n, 0)-flat module and  $M_1$  a pure submodule of M, then the pure exact sequence  $0 \to M_1 \to M \to M/M_1 \to 0$  induces a split exact sequence  $0 \to (M/M_1)^+ \to M^+ \to M_1^+ \to 0$ . By Theorem 2.15(1),  $M^+$  is (n, 0)-injective, so  $M_1^+$  is (n, 0)-injective by Theorem 2.4, and hence  $M_1$  is (n, 0)-flat by Theorem 2.15(1).

**Definition 2.18.** The (n, 0)-flat dimension of a module  $M_R$  is defined by

$$(n,0)-fd(M_R) = \inf\{k : Tor_{k+1}^R(M,A) = 0, \text{ for all } n\text{-}presented \ left \\ R\text{-}modules \ A.\}$$

**Lemma 2.19.** Let R be a left n-coherent ring and let M be a right R-module, then the following statements are equivalent:

(1) (n, 0)- $fd(M_R) \le k$ .

(2)  $Tor_{k+1}^{\dot{R}}(M, A) = 0$ , for every n-presented left R-module A.

**Proof.** (1)  $\Rightarrow$  (2). Use induction on k. Clear, if (n,0)-fd(M) = k. If (n,0)- $fd(M) \leq k-1$ . Since A is n-presented, there exists an exact

sequence  $0 \to N \to P \to A \to 0$ , where P is a finitely generated projective module and N is (n-1)-presented. But, R is left n-coherent, N is *n*-presented, and hence  $Tor_{k+1}^{R}(M, A) \cong Tor_{k}^{R}(M, N) = 0$  by induction hypothesis. 

 $(2) \Rightarrow (1)$  is clear.

positive integers k.

**Corollary 2.20.** Let R be a left n-coherent ring and  $M_R$  be (n,0)flat, then  $Tor_k^R(M, A) = 0$ , for all n-presented left R-modules A and all

Corollary 2.21. Let R be a left n-coherent ring and M be a right *R*-module. If the sequence of right *R*-modules  $0 \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}}$  $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0 \text{ is exact with } F_0, \cdots, F_{k-1} (n, 0) \text{-flat, then} \\ Tor_1^R(F_k, A) \cong Tor_{k+1}^R(M, A), \text{ for any n-presented left } R \text{ module } A.$ 

**Proof.** Since R is left n-coherent and  $F_0, F_1, \cdots, F_{k-1}$  are (n, 0)-flat, by Corollary 2.20, we have

$$Tor_{k+1}^{R}(M,A) \cong Tor_{k}^{R}(Ker(d_{0}),A) \cong Tor_{k-1}^{R}(Ker(d_{1}),A) \cong \cdots$$
$$\cong Tor_{1}^{R}(Ker(d_{k-1}),A) \cong Tor_{1}^{R}(F_{k},A).$$

**Theorem 2.22.** Let R be a left n-coherent ring, M be a right R-module and  $k \geq 0$ , then the following statements are equivalent:

- (1)  $(n,0)-fd(M_R) \leq k$ . (2)  $Tor_{k+l}^R(M,A) = 0$ , for all n-presented left R-modules A and all positive integers l.
- (3)  $Tor_{k+1}^{R}(M, A) = 0$ , for all n-presented left R-modules A.
- (4) If the sequence  $0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M \to 0$  is exact with  $F_0, \dots, F_{k-1}$  (n, 0)-flat, then also  $F_k$  is (n, 0)-flat.
- (5) There exists an exact sequence  $0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M \to 0$  of right *R*-modules with  $F_0, \cdots, F_{k-1}, F_k$  (n, 0)-flat.

**Proof.** (1)  $\Rightarrow$  (2). Assume (1), then (n, 0)- $fd(M_R) \leq k + l - 1$ , and so (2) follows from Lemma 2.19.

 $(2) \Rightarrow (3)$  and  $(4) \Rightarrow (5)$  are obvious.  $(3) \Rightarrow (4)$  and  $(5) \Rightarrow (1)$  by Corollary 2.21 and Lemma 2.19.

#### 3. *n*-hereditary rings and *n*-regular rings

Recall that a ring R is called right semihereditary, if every finitely generated right ideal of R is projective, or equivalently, if every finitely generated submodule of a projective right R-modules is projective. Next, we define n-hereditary rings as follows.

**Definition 3.1.** A ring R is called right n-hereditary, if every (n-1)-presented submodule of projective right R-module is projective.

Clearly, a ring R is right semihereditary if and only if it is right 1-hereditary. Right *n*-hereditary ring is right (n + 1)-hereditary.

**Theorem 3.2.** The following statements are equivalent for a ring R:

- (1) R is right n-hereditary.
- (2) R is right n-coherent and r.(n,0)- $ID(R) \leq 1$ .
- (3) Factor module of (n,0)-injective right R-module is (n,0)-injective.
- (4) Factor module of injective right R-module is (n,0)-injective.
- (5) R is a right (n,1)-ring.

**Proof.** (1)  $\Rightarrow$  (2). Since *R* is right *n*-hereditary, every (n-1)-presented submodule of a projective right *R*-module is finitely generated projective, and hence *n*-presented, so *R* is right *n*-coherent. Now, let *M* be any right *R*-module. Then, for any *n*-presented right *R*-module *A*, we have an exact sequence  $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$  of right *R*-modules, where *P* is finitely generated and projective, *N* is (n-1)-presented and projective. Thus, the exact sequence  $0 = Ext_R^1(N, M) \rightarrow Ext_R^2(A, M) \rightarrow Ext_R^2(P, M) = 0$  implies that  $Ext_R^2(A, M) = 0$ . This follows that r.(n, 0)- $ID(R) \leq 1$  by Definition 2.8.

 $(2) \Rightarrow (3)$ . Let M be an (n, 0)-injective right R-module and K its submodule. Then, for any n-presented module A, we have an exact sequence  $0 = Ext_R^1(A, M) \rightarrow Ext_R^1(A, M/K) \rightarrow Ext_R^2(A, K) = 0$  by (2) and Lemma 2.9, and so  $Ext_R^1(A, M/K) = 0$ , as required.

 $(3) \Rightarrow (4)$ . It is obvious.

 $(4) \Rightarrow (5)$ . Since  $Ext_R^2(A, B) \cong Ext_R^1(A, E(B)/B)$  holds for any right *R*-modules *A* and *B*, so (5) follows from (4).

 $(5) \Rightarrow (1)$ . Let N be an (n-1)-presented submodule of a projective right R-module P. Then, there exists a finitely generated free module F such that N is a submodule of F. Now, for any injective right R-module E and every submodule K of E, since F/N is

*n*-presented,  $Ext_R^2(F/N, K) = 0$  by (5), and so  $Ext_R^1(N, K) = 0$  as the sequence  $0 = Ext_R^1(F, K) \to Ext_R^1(N, K) \to Ext_R^2(F/N, K) = 0$  is exact. This shows that N is E-projective because of the exact sequence  $Hom(N, E) \to Hom(N, E/K) \to Ext_R^1(N, K) = 0$ . Therefore, N is projective.

**Example 3.3.** Let R be a non-coherent commutative ring of weak dimension one, then R is a (2,1)-ring but not a (1,1)-ring by [2, Example (6.5)], and so R is a 2-hereditary ring which is not 1-hereditary by Theorem 3.2.

**Theorem 3.4.** A domain R is n-hereditary if and only if every (n-1)-presented torsion-free R-module is projective.

**Proof** Since R is a domain, every finitely generated torsion-free R-module may be imbedded in a free module and every submodule of a free R-module is torsion-free. Hence, the results follows.

**Theorem 3.5.** If  $n \ge 2$ , then the following statements are equivalent for a ring R:

- (1) R is a right n-hereditary ring.
- (2) Every submodule of an (n, 0)-flat left R-module is (n, 0)-flat.

**Proof** (1)  $\Rightarrow$  (2). Let M be an (n, 0)-flat left R-module and let K be its submodule. Then, for any n-presented right R-module A, there exists an exact sequence  $0 \rightarrow N \rightarrow P \rightarrow A \rightarrow 0$ , where P is a finitely generated projective module and N is (n-1)-presented. Since R is a right n-hereditary ring, N is projective, hence we have an exact sequence  $0 = Tor_2^R(P, M/K) \rightarrow Tor_2^R(A, M/K) \rightarrow Tor_1^R(N, M/K) =$ 0, it shows that  $Tor_2^R(A, M/K) = 0$ . Therefore, by the exact sequence  $0 = Tor_2^R(A, M/K) \rightarrow Tor_1^R(A, K) \rightarrow Tor_1^R(A, M) = 0$ , we get  $Tor_1^R(A, K) = 0$ , i.e., K is (n, 0)-flat.

 $(2) \Rightarrow (1)$ . Suppose B is an (n, 0)-injective right R-module with an exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . Then,  $B^+$  is an (n, 0)-flat left R-module by Theorem 2.15(2), and the sequence  $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$  is exact. By (2),  $C^+$  is (n, 0)-flat, so C is (n, 0)-injective again by Theorem 2.15(2). Hence, R is right n-hereditary by Theorem 3.2(3).

**Corollary 3.6.** If  $n \ge 2$  and the weak dimension of  $R wD(R) \le 1$ , then R is left and right n-hereditary.

**Proof.** Assume M is an (n, 0)-flat right R-module and K is a submodule of M. Then, for any n-presented left R-module A, since  $wD(R) \leq 1$ ,  $Tor_2^R(M/K, A) = 0$ , this follows that  $Tor_1^R(K, A) = 0$  because M is (n, 0)-flat, and thus K is (n, 0)-flat. By Theorem 3.5, R is left n-hereditary. Similarly, one can prove that R is right n-hereditary.  $\Box$ 

Next, we generalize the concepts of regular rings and n-von Neumann rings to right n-regular rings.

**Definition 3.7.** A ring R is called right n-regular, if it is a right (n, 0)-ring.

Clearly, R is regular if and only if it is right 1-regular, R is n-von Neumann ring, if it is a commutative right n-regular ring. Right nregular ring is right (n + 1)-regular.

**Example 3.8.** Let K be a field and E be a K-vector space with infinite rank. Set  $B = K \propto E$  the trivial extension of K by E. Then, by [6, Theorem 3.4], R is a commutative 2-regular rings which is not regular. So, in general, right 2-regular ring need not be regular.

**Theorem 3.9.** The following conditions are equivalent for a ring R.

- (1) R is a right n-regular ring.
- (2) Every right R-module is (n, 0)-injective.
- (3) Every finitely generated right R-module is (n, 0)-injective.
- (4) R is right n-hereditary and  $R_R$  is (n,0)-injective.
- (5) R is right n-coherent and every n-presented right R-module is (n, 0)-injective.
- (6) Every (n-1)-presented submodule of a projective right R-module is a direct summand.
- (7) Every n-presented right R-module is flat.
- (8) Every left R-module is (n, 0)-flat.

**Proof.**  $(1) \Rightarrow (2) \Rightarrow (3)$  are obvious.

 $(3) \Rightarrow (4)$ . Assume (3). Then, clearly  $R_R$  is (n, 0)-injective. Let P be a projective module and let K be an (n-1)-presented submodule of P. By (3), K is (n, 0)-injective, so by Theorem 2.2(3), we have that K is a direct summand of P and hence K is projective. Therefore, R is right n-hereditary.

 $(4) \Rightarrow (5)$ . Assume (4), then every (n-1)-presented submodule of a projective module is projective and finitely generated, and then it is *n*-presented, so *R* is right *n*-coherent by Theorem 2.1(3). Now, let *M* be an *n*-presented right *R*-module, then there exists an exact sequence of right *R*-modules  $F \rightarrow M \rightarrow 0$ , where *F* is finitely generated free. Since  $R_R$  is (n, 0)-injective, by Proposition 2.4, *F* is (n, 0)-injective. Observing that *R* is right *n*-hereditary, by Theorem 3.2(3), *M* is (n, 0)-injective.

 $(5) \Rightarrow (6)$ . Let M be an (n-1)-presented submodule of a projective right R-module P. Then, M is a submodule of a finitely generated free right R-module F. By Proposition 1.1(5), F/M is n-presented. Since Ris right n-coherent, F/M is (n + 1)-presented. So, M is n-presented by Proposition 1.1(7), and hence M is (n, 0)-injective by (5). This follows that M is a direct summand of P by Theorem 2.2(3).

 $(6) \Rightarrow (1)$ . Let M be an n-presented right R-module, then there exists an exact sequence of right R-modules  $0 \to K \to P \to M \to 0$ , where P is finitely generated projective and K is (n-1)-presented. By hypothesis, K is a direct summand of P. Hence, M is isomorphic to a direct summand of P, and so M is projective.

 $(1) \Leftrightarrow (7)$  and  $(7) \Leftrightarrow (8)$  are obvious.

266

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