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GENERALIZED NUMERICAL RANGES OF MATRIX POLYNOMIALS

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Abstract. In this paper, we introduce the notions of $C-$numerical range and $C-$spectrum of matrix polynomials. Some algebraic and geometrical properties are investigated. We also study the relationship between the $C-$numerical range of a matrix polynomial and the joint $C-$numerical range of its coefficients.

1. Introduction and preliminaries

Let M_n be the algebra of all $n \times n$ complex matrices. Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1.1}$$

is a matrix polynomial, where $A_i \in M_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred to as the *degree* and the *order* of $P(\lambda)$, respectively. Matrix polynomials arise in many applications and their spectral analysis is very important to study linear systems of ordinary differential equations with constant coefficients [8]. The matrix polynomial $P(\lambda)$, as in (1.1), is called a *monic matrix polynomial* if $A_m = I_n$, where I_n is the $n \times n$ identity matrix. It is said to be a *self-adjoint* matrix polynomial if all the coefficients A_i are Hermitian matrices. Also, $P(\lambda)$ is a *diagonal matrix polynomial* if all

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the coefficients A_i are diagonal matrices. A scalar $\lambda_0 \in \mathbf{C}$ is an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbf{C}^n$. The solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is said to be the *spectrum* of $P(\lambda)$, that is, $\sigma[P(\lambda)] = \{\mu \in \mathbf{C} : \det(P(\mu)) = 0\}$. The *(classical) numerical range* of $P(\lambda)$, as in (1.1), is defined as:

$$W[P(\lambda)] := \{\mu \in \mathbf{C} \ : \ x^*P(\mu)x = 0 \ \text{ for some nonzero } \ x \in \mathbf{C}^n\},$$

which is closed and contains $\sigma[P(\lambda)]$; see [15] for more information. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with a finite number of degrees of freedom, and it is also related to the stability theory; see e.g., [8] and [15]. Notice that the notion of $W[P(\lambda)]$ is a generalization of the *classical numerical range* of a matrix $A \in M_n$, namely:

$$W[\lambda I - A] = W(A) := \{x^*Ax \ : \ x \in \mathbf{C}^n, \ x^*x = 1\},$$

which has been studied extensively for many decades. It is useful in the study and to understand the matrices and operators, see [11, 12], and has many applications in numerical analysis, differential equations, system theory, etc; see e.g., [3, 7, 10, 22].
Another generalization of the classical numerical range of matrices, due to Goldberg and Straus [9], is the notion of $C-$numerical range of matrices. Let $A, C \in M_n$, and $\mathcal{U}_n$ be the group of $n \times n$ unitary matrices. The *$C-$numerical range*, the *$C-$numerical radius* and the *inner $C-$numerical radius* of A are defined, respectively, as:

$$W_C(A) = \{tr(CU^*AU) \ : \ U \in \mathcal{U}_n\}, \quad r_C(A) = \max_{z \in W_C(A)} |z|,$$

and $\tilde{r_C}(A) = \min_{z \in W_C(A)} |z|$, where $tr(X)$ denotes the *trace* of $X \in M_n$. The $C-$numerical range and the $C-$numerical radius of matrices are related to optimization problems, and have important applications in quantum control and quantum information; see e.g., [6, 21] and their references. Let C and A have eigenvalues $\gamma_1, \ldots, \gamma_n$, and $\alpha_1, \ldots, \alpha_n$, respectively. The $C-$spectrum of A is defined as:

$$\sigma_C(A) = \{ \sum_{j=1}^{n} \gamma_j \alpha_{i_j} \ : \ (i_1, \ldots, i_n) \textit{ is a permutation of } \{1, 2, \ldots, n\} \}.$$

The concept of $C-$spectrum of A is very useful in the study of $W_C(A)$. For a comprehensive survey of $W_C(A)$, $r_C(A)$ and $\sigma_C(A)$, see [13].
In the last few years, the generalization of the numerical range of matrix

polynomials has attracted much attention, many interesting results have been obtained; see e.g., [1, 5, 17, 19, 20]. In section 2 of this paper, we introduce $C-$spectrum and $C-$numerical range of matrix polynomials as a new generalization of the spectrum, and the numerical range of matrix polynomials and $C-$numerical range of matrices, respectively. We also study the boundedness, boundary points and some other geometric properties of the notion. In section 3, we consider the joint $C-$numerical range of a matrix polynomial as the joint $C-$numerical range of its coefficients, and we study some algebraic properties of this set.

At the end of this section, we list some properties of the $C-$numerical range and the $C-$spectrum of matrices which is useful in our discussin. For more details, see [4] and [13].

Proposition 1.1. *Let $A, C \in M_n$. Then the following assertions are true:*
(i) $W_C(A)$ *is a compact and connected set in* $\mathbb{C}$ *which contains* $\sigma_C(A)$;
(ii) If $\alpha, \beta \in \mathbb{C}$, *then* $W_C(\alpha A + \beta I) = \alpha W_C(A) + \beta tr(C)$ *and* $\sigma_C(\alpha A + \beta I) = \alpha\sigma_C(A) + \beta tr(C)$;
(iii) $W_{V^*CV}(U^*AU) = W_C(A) = W_A(C)$, *where* $U, V \in \mathcal{U}_n$;
(iv) $W_{\overline{C}}(\overline{A}) = \overline{W_C(A)}$;
(v) If $C = qE_{11} + \sqrt{1-|q|^2}E_{12}$, *where* $q \in \mathbb{C}$ *with* $|q| \leq 1$ *and* $E_{ij} \in M_n$ *has 1 in* $(i,j)-$*position and 0 elsewhere, then* $W_C(A) = W_q(A) := \{x^*Ay \ : \ x, y \in \mathbb{C}^n, \ x^*x = y^*y = 1, \ x^*y = q\}$ *and* $\sigma_C(A) = q\sigma(A)$;
(vi) $W_C(A)$ *is star-shaped with respect to star-center* $\frac{tr(A)\ tr(C)}{n}$, *here a nonempty subset* S *of a real linear space is said to be star-shaped with respect to star-center* $s \in S$ *if* $[s, x] \subseteq S$, *whenever* $x \in S$, *where* $[s, x]$ *denotes the line segment* $\{(1-t)s + tx \ : \ 0 \leq t \leq 1\}$.

The set $W_q(A)$ in Proposition 1.1(v), is called the *$q-$numerical range* of $A \in M_n$. It is a generalization of the classical numerical range of A; for more information, see [14].

2. Definitions and general properties

We begin by introducing the notions of $C-$spectrum and $C-$numerical range of a matrix polynomial.

Definition 2.1. *Let $P(\lambda)$ be a matrix polynomial as in (1.1), and $C \in M_n$ have eigenvalues $\gamma_1, \ldots, \gamma_n$. The $C-$spectrum of $P(\lambda)$ is defined as*

$$\sigma_C[P(\lambda)] = \{\mu \in \mathbb{C} \;\; : \;\; \sum_{j=1}^{n} \gamma_j \alpha_{i_j}^{(\mu)} = 0 \;\; for\ some\ permutation \\ (i_1, \ldots, i_n)\ of\ \{1, 2, \ldots, n\}\ \},$$

where, for $\mu \in \mathbb{C}$, $\alpha_1^{(\mu)}, \ldots, \alpha_n^{(\mu)}$ are eigenvalues of the matrix $P(\mu) \in M_n$.

Definition 2.2. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). For a given matrix $C \in M_n$, the $C-$numerical range of $P(\lambda)$ is defined and denoted by*

$$W_C[P(\lambda)] = \{\mu \in \mathbb{C} \; : \; tr(CU^*P(\mu)U) = 0 \;\; for\ some\ U \in \mathcal{U}_n\}.$$

Clearly for any fixed $\mu \in \mathbb{C}$, $P(\mu) \in M_n$. Hence, the $C-$spectrum and the $C-$numerical range of $P(\lambda)$ satisfy, respectively, the following relations:

$$\sigma_C[P(\lambda)] = \{\mu \in \mathbb{C} \; : \; 0 \in \sigma_C(P(\mu))\}, \tag{2.1}$$

$$W_C[P(\lambda)] = \{\mu \in \mathbb{C} \; : \; 0 \in W_C(P(\mu))\}. \tag{2.2}$$

If $tr(C) = 0$, then, by Proposition 1.1(vi), $W_C(P(\mu))$ is star-shaped with respect to star-center $0 = \frac{tr(P(\mu))\ tr(C)}{n}$ for all $\mu \in \mathbb{C}$. So, by (2.2), $W_C[P(\lambda)] = \mathbb{C}$. *Hence, to avoid trivial consideration, we shall assume that $tr(C) \neq 0$ in this paper.*

In view of relations (2.1) and (2.2), and Proposition 1.1(ii), for the special case $P(\lambda) = \lambda I - tr(C)A$, where $A \in M_n$, we have $\sigma_C[P(\lambda)] = \sigma_C(A)$ and $W_C[P(\lambda)] = W_C(A)$, and so, the notions of $C-$spectrum and $C-$numerical range of matrix polynomials are generalizations of $C-$spectrum and $C-$numerical range of matrices, respectively.
Let $q \in \mathbb{C}$ with $|q| \leq 1$. Assume that $P(\lambda)$ is a matrix polynomial as in (1.1). The $q-$numerical range of $P(\lambda)$ is defined, see [19], as

$$W_q[P(\lambda)] = \{\mu \in \mathbb{C} \;\; : \;\; x^*P(\mu)y = 0\ for\ some\ nonzero\ vectors \\ x, y \in \mathbb{C}^n\ with\ x^*y = q\},$$

which is a generalization of $W[P(\lambda)]$, namely, $W_1[P(\lambda)] = W[P(\lambda)]$. Now, set $C = qE_{11} + \sqrt{1 - |q|^2}E_{12} \in M_n$, where $q \in \mathbb{C}$ and $|q| \leq 1$. Then, by (2.2) and Proposition 1.1(v), we have $W_C[P(\lambda)] = W_q[P(\lambda)]$, and so, the $C-$numerical range of matrix polynomials is a new generalization of the $q-$numerical range (consequently, the numerical range) of matrix polynomials. Also, by (2.1) and Proposition 1.1(v), in the case $q = 0$, $\sigma_C[P(\lambda)] = \mathbb{C}$, and for $q \neq 0$, $\sigma_C[P(\lambda)] = \sigma[P(\lambda)]$. In the following theorem, which is a generalization of Theorem 2.1 in [15] and Proposition 1.1 in [19], we state some basic properties of the $C-$numerical range of matrix polynomials.

Theorem 2.3. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). Then the following assertions are true:*
(i) $W_C[P(\lambda)]$ is a closed set in $\mathbb{C}$ which contains $\sigma_C[P(\lambda)]$;
(ii) $W_C[P(\lambda + \alpha)] = W_C[P(\lambda)] - \alpha$, where $\alpha \in \mathbb{C}$;
(iii) $W_C[\alpha P(\lambda)] = W_C[P(\lambda)] = W_{\alpha C}[P(\lambda)]$, where $\alpha \in \mathbb{C}$ is nonzero;
(iv) $W_C[V^*P(\lambda)V] = W_{V^*CV}[P(\lambda)] = W_C[P(\lambda)]$, where $V \in \mathcal{U}_n$; and
(v) If $Q(\lambda) = \lambda^m P(\lambda^{-1}) := A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m$, then

$$W_C[Q(\lambda)] \setminus \{0\} = \{ \frac{1}{\mu} \ : \ \mu \in W_C[P(\lambda)], \ \mu \neq 0\};$$

(vi) If all the powers of λ in $P(\lambda)$ are even (or all of them are odd), then $W_C[P(\lambda)]$ is symmetric with respect to the origin;
(vii) If all entries of the matrices $C, A_0, A_1, \ldots, A_m$ lie on a line in the complex plain passing through origin, then $W_C[P(\lambda)]$ is symmetric with respect to the real axis.

Proof. (i); Let $\{\mu_k\}_{k=1}^{\infty} \subseteq W_C[P(\lambda)]$, and $\mu_k \longrightarrow \mu$ as $k \longrightarrow \infty$. By Definition 2.2, there exists a sequence $\{U_k\}_{k=1}^{\infty} \subseteq \mathcal{U}_n$ such that $tr(CU_k^*P(\mu_k)U_k) = 0$ for all $k \in \mathbb{N}$. We know that $\mathcal{U}_n$ is a compact set in M_n. So, to avoid reindexing, we assume, without loss of generality, that $U_k \longrightarrow U$ as $k \longrightarrow \infty$ for some $U \in \mathcal{U}_n$. Since the functions $tr(.)$ and $P(.)$ are continuous, $tr(CU^*P(\mu)U) = 0$. Therefore, $\mu \in W_C[P(\lambda)]$, and hence the result holds. Using relations (2.1), (2.2), and Proposition 1.1(i), we have $\sigma_C[P(\lambda)] \subseteq W_C[P(\lambda)]$.
By (2.2) and Proposition 1.1, the results in parts (ii), (iii), (iv) and (v) can be easily verified.
(vi); Clearly that $P(\lambda) = P(-\lambda)$ in the case that all the powers of λ in $P(\lambda)$ are even, and $P(\lambda) = -P(-\lambda)$ in the other case. So, the result follows from (2.2) and Proposition 1.1(ii).
(vii); By hypothesis, there exists a $\theta \in \mathbb{R}$ such that $e^{i\theta}C$ and all the coefficients of the matrix polynomial $e^{i\theta}P(\lambda)$ are real matrices. By part (iii), we have $W_C[P(\lambda)] = W_C[e^{i\theta}P(\lambda)]$. Then, we assume, without

loss of generality, that all matrices $C, A_0, A_1, \ldots, A_m$ are real. Now, the result can be easily follows from (2.2) and Proposition 1.1(iv). $\square$

Clearly $W_C[P(\lambda)]$ need not be bounded; see e.g., [15, Example 1] for $C = E_{11} \in M_n$. Here, for the boundedness of the $C-$numerical range of matrix polynomials, we state the following theorem. It is a generalization of the sufficient part of Theorem 1.2 in [19].

Theorem 2.4. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). If $0 \notin W_C(A_m)$, then $W_C[P(\lambda)]$ is bounded.*

Proof. Since $0 \notin W_C(A_m)$, $\tilde{r_C}(A_m) = \min_{z \in W_C(A_m)} \ |z| \ > 0$. Assume that
$N = \max\{r_C(A_0), r_C(A_1), \ldots, r_C(A_{m-1})\}$. By setting $M = \frac{N}{\tilde{r_C}(A_m)} + 1$, we will show that:

$$W_C[P(\lambda)] \subseteq \{\mu \in \mathbf{C} \ : \ |\mu| \leq M\}.$$

Let $\mu \in W_C[P(\lambda)]$, since $M \geq 1$, it is enough to assume that $|\mu| > 1$. By Definition 2.2, there exists a $U \in \mathcal{U}_n$ such that

$$tr(CU^*A_mU)\ \mu^m + tr(CU^*A_{m-1}U)\ \mu^{m-1} + \cdots + tr(CU^*A_0U) = 0.$$

We know that $tr(CU^*A_mU) \neq 0$. So, the above equation implies that $-\mu^m = \sum_{j=0}^{m-1} \frac{tr(CU^*A_jU)}{tr(CU^*A_mU)}\ \mu^j$, and hence, we have:

$$\begin{aligned} |\mu|^m &\leq \sum_{j=0}^{m-1} \frac{|tr(CU^*A_jU)|}{|tr(CU^*A_mU)|}\ |\mu|^j \\ &\leq \frac{N}{\tilde{r_C}(A_m)} \ \sum_{j=0}^{m-1} |\mu|^j \\ &= \frac{N}{\tilde{r_C}(A_m)}\ (\frac{|\mu|^m - 1}{|\mu| - 1}). \end{aligned}$$

Therefore, $|\mu| - 1 \leq \frac{N}{\tilde{r_C}(A_m)}\ (\frac{|\mu|^m-1}{|\mu|^m}) \leq \ \frac{N}{\tilde{r_C}(A_m)}$, and hence $|\mu| \leq M$. $\square$

For the case $C = qE_{11} + \sqrt{1-|q|^2}E_{12} \in M_n$, where $q \in \mathbf{C}$ and $|q| \leq 1$, the converse of Theorem 2.4 holds; see [19]. But, in general, the converse is not true; which is illustrated in the following example.

Example 2.5. *Let $C = I$, and $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial as in (1.1). Assume that $tr(A_m) = 0$, and there exists a $0 \le j \le m-1$ such that $tr(A_j) \ne 0$. By Definition 2.2, $W_C[P(\lambda)]$ has at most $m-1$ elements, and hence is bounded. However, $W_C(A_m) = \{tr(A_m)\} = \{0\}$.*

Now, we are going to study the boundary points. For this, we need the following lemma.

Lemma 2.6. [13, Section 3] *Let $C \in M_n$. Then $W_C(A)$ is convex for all $A \in M_n$ if one of the following conditions holds:*
(a) There exists $\beta \in \mathbf{C}$ such that $C - \beta I$ has rank one;
(b) There exist $\alpha, \beta \in \mathbf{C}$ with $\alpha \ne 0$ such that $\alpha C + \beta I$ is Hermitian, that is, C is essentially Hermitian;
(c) There exists $\beta \in \mathbf{C}$ such that $C - \beta I$ is similar to $[C_{ij}]$ unitarily in block form, where the diagonal blocks C_{ii} are square matrices and $C_{ij} = 0$ if $i \ne j+1$.

Theorem 2.7. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). Suppose that $C \in M_n$ satisfies one of the conditions in Lemma 2.6. If $\mu \in \mathbf{C}$ is a boundary point of $W_C[P(\lambda)]$, then the origin is a boundary point of $W_C(P(\mu))$.*

Proof. Since $W_C[P(\lambda)]$ is a closed set in $\mathbf{C}$ (Theorem 2.3(i)) and $\mu \in \mathbf{C}$ is a boundary point of $W_C[P(\lambda)]$, $\mu \in W_C[P(\lambda)]$ and $\mu \notin \mathrm{Int}(W_C[P(\lambda)])$, where $\mathrm{Int}(S)$ denotes the set of interior points of $S \subseteq \mathbf{C}$. Hence, by (2.2), $0 \in W_C(P(\mu))$, and in view of Proposition 1.1(i), it is enough to show that $0 \notin \mathrm{Int}(W_C(P(\mu)))$.
If $0 \in \mathrm{Int}(W_C(P(\mu)))$, then there exists a $\varepsilon > 0$ such that

$$B(0,\varepsilon) := \{z \in \mathbf{C} \ : \ |z| < \varepsilon\} \subseteq W_C(P(\mu)).$$

Now, let z_1, z_2, z_3 be three distinct points of $B(0,\varepsilon)$ such that $0 \in \mathrm{Int}(\mathrm{Conv}(\{z_1, z_2, z_3\})) \subseteq W_C(P(\mu))$, where $\mathrm{Conv}(S)$ denotes the convex hull of $S \subseteq \mathbf{C}$. Thus, there exist $U_1, U_2, U_3 \in \mathcal{U}_n$ such that

$$tr(CU_i^* P(\mu) U_i) = z_i \ ; \ i = 1,2,3.$$

Since $\mu \notin \mathrm{Int}(W_C[P(\lambda)])$, there exists a sequence $\{\mu_t\}_{t=1}^{\infty}$ of points in $\mathbf{C} \setminus W_C[P(\lambda)]$ converging to μ. We know that $tr(.)$ and $P(.)$ are continuous functions. So,

$$\lim_{t\to\infty} tr(CU_i^* P(\mu_t) U_i) = z_i \ ; \ i = 1,2,3.$$

Now, by taking an small enough neighborhood B_i of z_i for $i = 1, 2, 3$, there exists a $N > 0$ such that

$$tr(CU_i^* P(\mu_N)U_i) \in B_i;\ i = 1, 2, 3,\ and$$
$$0 \in \text{Conv}(\{tr(CU_i^* P(\mu_N)U_i)\ :\ i = 1, 2, 3\}).$$

By Lemma 2.6, $W_C(P(\mu_N))$ is convex. Hence, the last relation implies that $0 \in W_C(P(\mu_N))$. Consequently, $\mu_N \in W_C[P(\lambda)]$ which is a contradiction. □

Remark 2.8. *Let $q \in \mathbb{C}$ with $|q| \leq 1$ be given. It is clear that the matrix $C = qE_{11} + \sqrt{1-|q|^2}E_{12} \in M_n$ satisfies the condition (a) of Lemma 2.6. So, Theorem 2.7 is a generalization of Theorem* 2.2 *in* [19].

Since $0 \notin W_C(I)$, by Theorem 2.4, the $C-$numerical range of a monic matrix polynomial is bounded, and so, at the end of this section, we investigate a circular annulus for the location and an inclusion-exclusion methodology for the estimation of the $C-$numerical range of monic matrix polynomials. The following theorem is a generalization of Theorem 2.4 in [19].

Theorem 2.9. *Let $C \in M_n$, and $P(\lambda)$, as in (1.1), be a monic matrix polynomial. Then*

$$W_C[P(\lambda)] \subseteq \{z \in \mathbb{C}\ :\ r_1 \leq |z| \leq 1 + r_2\},$$

where $r_1 = \frac{\tilde{r_C}(A_0)}{\tilde{r_C}(A_0) + \max_{k=1,2,\dots,m} r_C(A_k)}$ *and* $r_2 = \max_{k=0,1,\dots,m-1} \frac{r_C(A_k)}{|tr(C)|}$.

Proof. Let $\mu \in W_C[P(\lambda)]$. Then, by Definition 2.2, there exists a $U \in \mathcal{U}_n$ such that

(2.3)
$$tr(C)\mu^m + tr(CU^*A_{m-1}U)\mu^{m-1} + \cdots + tr(CU^*A_1U)\mu + tr(CU^*A_0U) = 0.$$

We will show that $r_1 \leq |\mu| \leq 1 + r_2$.
For the left inequality, since $r_1 \leq 1$, it is enough to consider the case $|\mu| < 1$. Note that $r_C(A_m) = |tr(C)|$. So, in view of (2.3), we have:

$$\begin{aligned} \tilde{r_C}(A_0) &\leq |tr(CU^*A_0U)| \\ &\leq \left(\frac{|\mu|}{1-|\mu|}\right) \left(\max_{k=1,2,\dots,m} r_C(A_k)\right). \end{aligned}$$

Hence, $\tilde{r_C}(A_0) \leq |\mu|\ \tilde{r_C}(A_0) + |\mu| \max_{k=1,2,\dots,m} r_C(A_k)$, and so, the result holds.

For the right inequality, it is enough to consider the case $|\mu| > 1$. By (2.3), we have

$$\begin{aligned} |\mu|^m &\leq \sum_{k=0}^{m-1} \frac{|tr(CU^*A_kU)|}{|tr(C)|}|\mu|^k \\ &\leq r_2(\frac{|\mu|^m - 1}{|\mu| - 1}). \end{aligned}$$

Hence, the result holds. □

For a given matrix $C \in M_n$, the $C-$spectral norm of $A \in M_n$ is defined as

$$\|A\|_C = \max\{\ |\text{tr}(CUAV)|\ :\ U, V \in \mathcal{U}_n\}.$$

It is known, see e.g. [13] and its references, that the set $\{\ |\text{tr}(CUAV)|\ :\ U, V \in \mathcal{U}_n\}$ is a circular disk at the origin with radius $\sum_{i=1}^{n} s_i(C)s_i(A)$, where $s_1(C) \geq s_2(C) \geq \cdots \geq s_n(C)$ and $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of C and A, respectively. So, $\|A\|_C = \sum_{i=1}^{n} s_i(C)s_i(A)$. It is clear that $\|.\|_C$ is a unitarily invariant norm on M_n, and $r_C(A) \leq \|A\|_C$. For the case $C = E_{11} \in M_n$, $\|.\|_C$ coincides with the spectral matrix norm, $\|.\|_2$ (i.e. the matrix norm subordinate to the Euclidean vector norm).
Now, we are ready to state the following theorem which is a generalization of Theorem 2.1 in [16]. Note that, the open circular disk with center at $\mu \in \mathbf{C}$ and radius $\rho > 0$ is denoted by $S(\mu, \rho) = \{z \in \mathbf{C}\ :\ |z-\mu| < \rho\}$.

Theorem 2.10. *Let $C \in M_n$, and $P(\lambda)$, as in (1.1), be a monic matrix polynomial. If $\mu \notin W_C[P(\lambda)]$, then $S(\mu, \rho_\mu) \bigcap W_C[P(\lambda)] = \emptyset$, where*

$$\rho_\mu = \frac{\tilde{r_C}(P(\mu))}{\tilde{r_C}(P(\mu)) + \max_{j=1,2,\ldots,m} \|\frac{1}{j!}P^{(j)}(\mu)\|_C}.$$

Proof. Note that the relation $\mu \notin W_C[P(\lambda)]$ implies that $\rho_\mu > 0$. By setting $Q(\lambda) = P(\lambda + \mu) = B_m\lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_1\lambda + B_0$, we have $B_j = \frac{1}{j!}P^{(j)}(\mu)\ ;\ j = 0, 1, \ldots, m$. Now, let $z \in W_C[Q(\lambda)]$ be given. Since $Q(\lambda)$ is a monic matrix polynomial, Theorem 2.9 implies that

$$\begin{aligned} |z| &\geq \frac{\tilde{r_C}(B_0)}{\tilde{r_C}(B_0) + \max_{j=1,2,\ldots,m} r_C(B_j)} \\ &= \frac{\tilde{r_C}(P(\mu))}{\tilde{r_C}(P(\mu)) + \max_{j=1,2,\ldots,m} r_C(\frac{1}{j!}P^{(j)}(\mu))}. \end{aligned}$$

Since $r_C(\frac{1}{j!}P^{(j)}(\mu)) \leq \|\frac{1}{j!}P^{(j)}(\mu)\|_C$ for $j = 1, 2, \ldots, m$, the above inequality implies that $|z| \geq \rho_\mu$. Therefore, $W_C[Q(\lambda)] \bigcap S(0, \rho_\mu) = \emptyset$. By Theorem 2.3$(ii)$, $W_C[Q(\lambda)] = W_C[P(\lambda)] - \mu$, and hence the result holds. □

Remark 2.11. *Let $C \in M_n$, and $P(\lambda)$, as in (1.1), be a monic matrix polynomial. Since $r_C(A_j) \leq \|A_j\|_C$; $j = 0, 1, \ldots, m-1$, Theorem 2.9 implies that*

$$W_C[P(\lambda)] \subseteq S(0, 1 + \max_{j=0,1,\ldots,m-1} \frac{\sum_{i=1}^n s_i(C)s_i(A_j)}{|\mathrm{tr}(C)|}) =: \Omega.$$

By, using Theorem 2.10, we can give the following algorithm to approximate the shape of $W_C[P(\lambda)]$.

Algorithm:

Step i: *construct a gride G_Ω of Ω;*

Step ii: *For every gride point $\mu \in G_\Omega$, repeat the following:*

(a) If $\mu \notin W_C[P(\lambda)]$, or equivalently, if $0 \notin W_C(P(\mu))$, then compute $\tilde{r_C}(P(\mu))$ and the matrices $B_j = \frac{1}{j!}P^{(j)}(\mu)$; $j = 0, 1, \ldots, m$

(b) construct the open circular disk $S(\mu, \rho_\mu)$ with radius

$$\rho_\mu = \frac{\tilde{r_C}(P(\mu))}{\tilde{r_C}(P(\mu)) + \max_{j=1,2,\ldots,m} \Sigma_{i=1}^n s_i(C)s_i(\frac{1}{j!}P^{(j)}(\mu))};$$

Step iii: *The set $\Omega \setminus \bigcup_{\mu \in G_\Omega, 0 \notin W_C(P(\mu))} S(\mu, \rho_\mu)$ is an approximation for the shape of $W_C[P(\lambda)]$.*

3. Joint $C-$numerical range of matrix polynomials

Let $C \in M_n$, and $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial as in (1.1). The *joint $C-$numerical range* of $P(\lambda)$ is defined as the joint $C-$numerical range of $A_0, A_1, \ldots, A_m$, namely [2],

$$\begin{aligned} JW_C[P(\lambda)] &:= W_C(A_0, A_1, \ldots, A_m) \\ &= \{(\ tr(CU^*A_0U),\ \ldots,\ tr(CU^*A_mU)\ )\ :\ U \in \mathcal{U}_n\}. \end{aligned}$$

Since $JW_C[P(\lambda)]$ can be viewed as the range of the continuous function

$$U \longmapsto (\ tr(CU^*A_0U),\ tr(CU^*A_1U),\ \ldots,\ tr(CU^*A_mU)\ )$$

from the compact connected set $\mathcal{U}_n$ to $\mathbf{C}^{m+1}$, one easily gets that $JW_C[P(\lambda)]$ is a compact and connected set in $\mathbf{C}^{m+1}$. Also, for the case

$C = E_{11} \in M_n$, we have

$$JW_C[P(\lambda)] = \{\ (x^*A_0x, \ldots, x^*A_mx)\ :\ x \in \mathbb{C}^n,\ x^*x = 1\},$$

which is the joint numerical range of $P(\lambda)$; see [18] for more information. So, the joint $C-$numerical range of matrix polynomials is a generalization of the joint numerical range.

In the following theorem, the relationship between the $C-$numerical range of $P(\lambda)$ and the joint $C-$numerical range of its coefficients is stated. Also, using the $C-$numerical range of diagonal matrix polynomials, we can approximate the shape of the $C-$numerical range of any matrix polynomial. For the case $C = E_{11} \in M_n$, see [18].

Theorem 3.1. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). Then the following assertions are true:*
(i) $W_C[P(\lambda)] = \{\mu \in \mathbb{C}\ :\ a_m\mu^m + \cdots + a_1\mu + a_0 = 0,\ (a_0, a_1, \ldots, a_m) \in W_C(A_0, A_1, \ldots, A_m)\}$;
(ii) $W_C[P(\lambda)] = \bigcup W_C[D(\lambda)]$, *where the union is taken over all diagonal matrix polynomials $D(\lambda)$ of degree m and order n such that $JW_C[D(\lambda)] \subseteq JW_C[P(\lambda)]$.*

Proof. The result in part (i) follows easily from Definition 2.2 and the definition of joint $C-$numerical range of $A_0, A_1, \ldots, A_m$.
To prove (ii), by (i), $\supseteq$ is clear. Let now $\mu \in W_C[P(\lambda)]$ be given. By (i), there exists a $(a_0, a_1, \ldots, a_m) \in JW_C[P(\lambda)]$ such that $a_m\mu^m + \cdots + a_1\mu + a_0 = 0$. Let $D(\lambda) = \frac{a_m}{tr(C)} I\lambda^m + \cdots + \frac{a_1}{tr(C)} I\lambda + \frac{a_0}{tr(C)} I$, then we have $JW_C[D(\lambda)] = \{\ (a_0, a_1, \ldots, a_m)\ \} \subseteq JW_C[P(\lambda)]$, and $\mu \in W_C[D(\lambda)]$. Hence, the proof of $\subseteq$ is complete. $\square$

Corollary 3.2. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). If $(0, 0, \ldots, 0) \in JW_C[P(\lambda)]$, then $W_C[P(\lambda)] = \mathbb{C}$.*

Theorem 3.3. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). Suppose that $C \in M_n$ satisfies one of the conditions in Lemma 2.6. Then*

$$\begin{aligned} W_C[P(\lambda)] = \{\mu \in \mathbb{C}\ \ :\ \ & a_m\mu^m + \cdots + a_1\mu + a_0 = 0, \\ & (a_0, a_1, \ldots, a_m) \in \mathrm{Conv}(W_C(A_0, A_1, \ldots, A_m))\}, \end{aligned}$$

where Conv(.) *denotes the convex hull.*

Proof. By Theorem 3.1(i), $\subseteq$ is clear.
For the opposite inclusion, let $\mu \in \mathbb{C}$ be such that $a_m\mu^m + \cdots + a_1\mu + a_0 = 0$ for some $(a_0, a_1, \ldots, a_m) \in \mathrm{Conv}(W_C(A_0, A_1, \ldots, A_m))$. So, there

are nonnegative real numbers $t_1, t_2, \ldots, t_k$ summing to 1, and unitary matrices $U_1, U_2, \ldots, U_k \in \mathcal{U}_n$ such that

$$(a_0, a_1, \ldots, a_m) = \sum_{j=1}^{k} t_j (tr(CU_j^* A_0 U_j), \ldots, tr(CU_j^* A_m U_j)).$$

So, we have:

$$\begin{aligned} 0 = \sum_{i=0}^{m} a_i \mu^i &= \sum_{i=0}^{m} (\sum_{j=1}^{k} t_j \ tr(CU_j^* A_i U_j)) \mu^i \\ &= \sum_{j=1}^{k} t_j (\sum_{i=0}^{m} tr(CU_j^* A_i U_j) \mu^i) \\ &= \sum_{j=1}^{k} t_j \ tr(CU_j^* P(\mu) U_j) \\ &\in \operatorname{Conv}(W_C(P(\mu))). \end{aligned}$$

By Lemma 2.6, $W_C(P(\mu))$ is convex, and hence $\operatorname{Conv}(W_C(P(\mu))) = W_C(P(\mu))$. Thus, the above relations show that $0 \in W_C(P(\mu))$. Therefore, $\mu \in W_C[P(\lambda)]$, and the proof is complete. □

Finally, we show that every interior point of $JW_C[P(\lambda)]$ produces an interior point of $W_C[P(\lambda)]$.

Theorem 3.4. *Let $C \in M_n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). If $a_m\mu^m + \cdots + a_1\mu + a_0 = 0$, where $\mu \in \mathbb{C}$ and $(a_0, a_1, \ldots, a_m) \in \operatorname{Int}(JW_C[P(\lambda)])$, then $\mu \in \operatorname{Int}(W_C[P(\lambda)])$. Here, $\operatorname{Int}(S)$ denotes the set of all interior points of $S \subseteq \mathbb{C}$.*

Proof. By hypothesis and Theorem 3.1(i), $\mu \in W_C[P(\lambda)]$. Also, there exist complex numbers $b_0, b_1, \ldots, b_{m-1}$ such that for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} a_m\lambda^m + \cdots a_1\lambda + a_0 &= (\lambda - \mu)(b_{m-1}\lambda^{m-1} + \cdots + b_1\lambda + b_0) \\ &= b_{m-1}\lambda^m + (b_{m-2} - \mu b_{m-1})\lambda^{m-1} + \cdots \\ &+ (b_0 - \mu b_1)\lambda + (-b_0\mu) \\ &= c_m(\mu)\lambda^m + c_{m-1}(\mu)\lambda^{m-1} + \cdots \\ &+ c_1(\mu)\lambda + c_0(\mu), \qquad (*) \end{aligned}$$

where by setting $b_{-1} = b_m = 0$, $c_j(\mu) := b_{j-1} - \mu b_j = a_j$ for $j = 0, 1, \ldots, m$.

Now, we will show that $\mu \in \text{Int}(W_C[P(\lambda)])$.
If $\mu \notin \text{Int}(W_C[P(\lambda)])$, then there exists a sequence

$$\{\mu_t\}_{t=1}^{\infty} \subseteq \mathbf{C} \setminus W_C[P(\lambda)],$$

such that $\mu_t \longrightarrow \mu$ as $t \longrightarrow \infty$. Hence

$$\lim_{t\longrightarrow\infty}(c_0(\mu_t), \ldots, c_m(\mu_t)) = (a_0, \ldots, a_m). \qquad (**)$$

In view of $(*)$, we have

$$c_m(\mu_t)\lambda^m + \cdots + c_1(\mu_t)\lambda + c_0(\mu_t) = (\lambda - \mu_t)(b_{m-1}\lambda^{m-1} + \cdots + b_1\lambda + b_0),$$

for all $\lambda \in \mathbf{C}$ and $t \in \mathbb{N}$. So,

$$c_m(\mu_t)\mu_t^m + c_{m-1}(\mu_t)\mu_t^{m-1} + \cdots + c_1(\mu_t)\mu_t + c_0(\mu_t) = 0, \quad for \ all \ t \in \mathbb{N}.$$

Since $\mu_t \notin W_C[P(\lambda)]$ for all $t \in \mathbb{N}$, by Theorem 3.1(i),

$$(c_0(\mu_t), \ldots, c_m(\mu_t)) \notin JW_C[P(\lambda)] \ for \ all \ t \in \mathbb{N}.$$

Therefore, relation $(**)$ shows that $(a_0, a_1, \ldots, a_m) \notin \text{Int}(JW_C[P(\lambda)])$, which is a contradiction. □

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