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A NEW PROOF FOR THE BANACH-ZARECKI THEOREM: A LIGHT ON INTEGRABILITY AND CONTINUITY

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Abstract. To demonstrate more visibly the close relation between the continuity and integrability, a new proof for the Banach-Zarecki theorem is presented on the basis of the Radon-Nikodym theorem which emphasizes on measure-type properties of the Lebesgue integral. The Banach-Zarecki theorem says that a real-valued function F is absolutely continuous on a finite closed interval if and only if it is continuous and of bounded variation when it satisfies Lusin's condition. In the present proof indeed a more general result is obtained for the Jordan decomposition of F.

1. Introduction

The original motivation for the present work concerns with the open debate of the regularity of hydrodynamical parameters of fluid flows. It is still not known that starting from a smooth initial condition in a three dimensional fluid, when and how any kind of blow up or singularity will happen. Much works consider this problem in various special cases and have been obtained many results. It was known that the type of singularity is so strong such that many kinds of integral norms of hydrodynamical quantities are also singular. However, almost all of these

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integral norms are obtained by the Lebesgue integration but we know that there are other types of integration that are generalizations of the usual Riemann integral and do not coincide the Lebesgue integral.

So, a natural question comes that how could we get information when our functions are not Lebesgue integrable? How should one replace (absolute) continuities and regularities in these new cases? As the first step it looks necessary to test and generalize a direct relation between the integration, continuity, and Banach–Zarecki Theorem provides perhaps the most visible case to observe such a relation. It was therefore needed to discover a more direct and closer relation between the absolute continuity and the Lebesgue integral to be an arrow for other works.

Banach–Zarecki Theorem is a classical theorem in real analysis with many applications mostly in geometric and functional analysis as well as in some physical and engineering subjects. The origin of this theorem was stated and proved by Banach and independently by Zarecki for a real–valued function on an interval [10]. For functions of a real variable with values in reflexive Banach spaces, the result is contained in [6], Theorem 2.10.13, where the codomain space has the Radon-Nikodym property. There also exists another version of the theorem initiated by an old result of Lusin [8], later extended for a function of a real variable with values in a metric space [3, 4].

It is not surprising that there is a variety of extensions for this theorem to more variables in many ways and also by natural changes in properties well-known in one dimensional case such as almost everywhere continuity and differentiability, integration by parts and so on [5, 12, 9]. In fact the theorem can be generalized to the concept of approximate continuity that plays an important role to understand the relationship between Riemann integrability (for almost everywhere continuous functions) and continuity on the one hand, and the relationship between approximate continuity and Lebesgue integrability (for almost everywhere approximately continuous functions), on the other hand [4].

There exist alternative proofs for this theorem; although these are of different appearance but they are constructed from a common root (see e.g. [1, 2, 11, 13]). In the present work the classical form of the theorem is considered, since it looks possible to a natural extension of the result to more general cases mentioned above. The most convenient statement of the Banach–Zarecki theorem is [1]:

Theorem 1.1. *Let F is a real–valued function defined on a real bounded closed interval $[a, b]$. A necessary and sufficient condition for F to be*

absolutely continuous is that
(i) *F is continuous and of bounded variation on $[a, b]$,*
(ii) *F satisfies Lusin's condition, i.e. it maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

The necessary condition is straightforward and will not be discussed here. Its proof is given in almost any text book of real analysis [1, 7]. However the sufficient condition is rather technical and requires some non–trivial efforts and may rarely be found in common references. Thus, we try to provide an alternative proof for the sufficient condition, that is, if a real–valued function is continuous and of bounded variation and also satisfies Lusin's condition, then it is absolutely continuous. In [1], there is another proof for the sufficient condition. The main tools of this approach are the almost everywhere differentiability and the Vitali covering theorem.

However the present proof is based on the close relation between the Lebesgue integral and the properties of a measure space which manifests itself essentially through the Radon-Nikodym theorem. Thus, the main used tools here are the Radon-Nikodym theorem and the properties of variations of functions. This new proof may however cost to be considered because of several reasons such as the following. Here a slightly more general result is proven, namely Lemma 2.2 while we need only Corollary 2.3 for our proof. The concept of almost everywhere differentiability and thus the Vitali covering lemma is not used. The methods and techniques used here seem to be applicable and naturally generalizable to a class of similar problems. There is a hope to generalize this method to obtain an analogue version for the absolute continuity in relation with other types of integration rather than the Lebesgue integral.

Finally it is seen that here some statements are proven employing only conditions (i) and (ii) mentioned in the Banach–Zarecki theorem and without using the absolute continuity condition, while these statements are usually proved through a direct application of the absolute continuity condition in the common literatures.

In order to prove Theorem 1.1, our strategy is to establish the following theorem which illustrates more clearly, the relation between the absolute continuity and the Lebesgue integral.

Theorem 1.2. *Suppose that $F : [a, b] \longrightarrow \mathbb{R}$ is a continuous and of bounded variation and satisfies Lusin's condition. Then there exists an integrable function and in fact a Borel–measurable function $f : [a, b] \longrightarrow$*

$\mathbb{R}$ *such that*

$$F(x) = F(a) + \int_{[a,x]} f\, d\lambda \ : \ \forall x \in [a,b],$$

where $d\lambda$ *in the integral comes from the Lebesgue measure* λ*.*

Theorem 1.2 will immediately yield Theorem 1.1 by the application of the well known statement [1, 7]:

> *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a Lebesgue integrable function and let* $F(x) = F(a) + \int_{[a,x]} f\, d\lambda$*, then* F *is absolute continuous on* $[a,b]$*.*

In the next section, we prove Theorem 1.2 in three steps, the first of which is well known in text books [7] while step 2 and especially step 3 are of our main interests.

Throughout the paper we assume that the notation λ implies the Lebesgue measure, unless otherwise stated.

2. The main result: A new proof of Theorem 1.2

The proof is divided into three interconnected steps.

Step 1. First, we prove the theorem assuming that F is strictly increasing. In this case, the proof coincides the standard proof given in common text books (see e.g. Theorem 4.3.8 of [7]) which uses the Radon–Nikodym theorem. To have a complete discussion, let us briefly review the proof here.

Since F is strictly increasing, F is a homeomorphism from $I = [a,b]$ to $J = F(I) = [F(a), F(b)]$ and so F preserves Borel sets between I and J. Let $\mathcal{B}$ be the collection of Borel measurable subsets of I, then we can define the new measure $\nu : \mathcal{B} \longrightarrow [0,\infty)$ as $\nu(E) = \lambda(F(E))$. It is clear that ν is a finite measure and is absolutely continuous relative to λ (since F satisfies Lusin's condition). Therefore, according to the Radon–Nikodym theorem, there exists a (Borel) measurable and Lebesgue integrable function $f : I \longrightarrow \mathbb{R}$ such that

$$\nu(E) = \int_E f\, d\lambda, \qquad E \in \mathcal{B}. \tag{2.1}$$

Especially if $E = [a, x]$ for $x \in I$, then $F(E) = [F(a), F(x)]$ and Eq. (2.1) immediately implies that

$$F(x) = F(a) + \int_{[a,x]} f \, d\lambda, \quad x \in I.$$

This completes the proof of step 1.

Step 2. Let F is non–decreasing (i.e., increasing but not strictly increasing). So the function $G(x) = F(x) + x$ is continuous, of bounded variation and strictly increasing. The proof will be complete if we prove that Lusin's property is fulfilled by G, i.e. for $N \subset [a, b]$ if $\lambda(N) = 0$ then $\lambda(G(N)) = 0$. Since F is non–decreasing, one easily observes that the constant values of F make sense in disjoint intervals S_k and the continuity of F implies that S_ks are closed intervals, say $[a_k, b_k]$. Hence, in general, on $S = \bigcup_{k=1}^{+\infty} S_k$, F takes the values $F(S) = \Big\{\mu_k\Big\}_{k=1}^{+\infty}$ where μ_k is the value of F on S_k.

The intervals S_k may be so small and their union S is not necessarily closed. Now, since S_k s are disjoint, we can write

$$N_1 = N \cap S, \qquad N_2 = N - N_1.$$

Therefore, we have

$$\lambda(G(N)) \leq \lambda(G(N_1)) + \lambda(G(N_2)),$$

while

$$\lambda(G(N_1)) = \lambda\Big(\bigcup_{k=1}^{+\infty} G(N \cap S_k)\Big) \leq \sum_{k=1}^{+\infty} \lambda(G(N \cap S_k)).$$

On the other hand, $G(N \cap S_k) = \{\mu_k + x \mid x \in N \cap S_k\}$ and thus $\lambda(G(N \cap S_k)) = \lambda(N \cap S_k)$, so

$$\begin{aligned}\lambda(G(N_1)) &\leq \sum_{k=1}^{+\infty} \lambda(G(N \cap S_k)) \\ &= \lambda(\bigcup_{k=1}^{+\infty} (N \cap S_k)) = \lambda(N_1) \leq \lambda(N) = 0.\end{aligned}$$

Therefore $\lambda(G(N_1)) = 0$. To prove $\lambda(G(N_2)) = 0$, we notice that F satisfies Lusin's condition i.e. $\lambda(N_2) = 0$ results in $\lambda(F(N_2)) = 0$, so for each $\epsilon > 0$, we can find an open set U such that $F(N_2) \subset U$ with $\lambda(U) < \epsilon$. In addition, since $\lambda(N_2) = 0$, one can find an open set U' including

N_2 such that $\lambda(U') < \epsilon$. The open set $V := U' \cap F^{-1}(U)$ contains N_2 such that $\lambda(V) < \epsilon$ and $\lambda(F(V)) < \epsilon$. Suppose $V = \bigcup_{k=1}^{+\infty} I_k$ where I_ks are disjoint open intervals. For each I_k, consider two closed intervals (if exist) S_i and S_j intersecting I_k from the left and right containing the left and right boundary points of I_k resp. Define $I'_k = I_k - (S_i \cup S_j)$ (it is possible that I'_k is empty). Thus $I'_k \subset I_k$ and I'_k s are mutually disjoint.

Let $V' := \bigcup_{k=1}^{+\infty} I'_k$ and since S_k s are all out of N_2, $N_2 \subset V'$ and thus $F(N_2) \subset F(V')$. It is important to attend that for each l and k, S_l is either completely contained in I'_k or is disjoint from it. According to the conditions on F, i.e. non–increasing and continuity, one can deduce that $F(I'_k)$ is an interval (not necessarily closed or open) which we denote it by J_k. Now we acclaim that J_k s are mutually disjoint. If else, for example if $y \in J_k \cap J_l$ for some k and l, then there exist at least two points $x_k \in I'_k$ and $x_l \in I'_l$ such that $F(x_k) = F(x_l)$. Hence there exists an S_i so that $[x_k, x_l] \subset S_i$ but then S_i is not completely in I'_k or I'_l which is a contradiction. The relations

$$F(N_2) \subset F(V') = \bigcup_{k=1}^{+\infty} F(I'_k) \subset U,$$

imply that

$$\lambda\Big(\bigcup_{k=1}^{+\infty} J_k\Big) \leq \lambda(U) < \epsilon,$$

and since J_k s are disjoint sets,

$$\sum_{k=1}^{+\infty} \lambda(J_k) < \epsilon.$$

The remaining work is to determine $G(I'_k)$ and approximate their measure. For each k, we have

$$\begin{aligned} G(I'_k) &= \Big\{F(x) + x \mid x \in I'_k\Big\} \\ &\subseteq \Big\{y + x \mid y \in J_k,\ x \in I'_k\Big\} \\ &\subseteq \Big(\inf(I'_k) + \inf(J_k)\ ,\ \sup(I'_k) + \sup(J_k)\Big), \end{aligned}$$

which results in

$$\lambda(G(I'_k)) \leq \lambda(I'_k) + \lambda(J_k).$$

So we obtain

$$\lambda(G(N_2)) \leq \lambda\Big(\bigcup_{k=1}^{+\infty} G(I'_k)\Big) \leq \sum_{k=1}^{+\infty} \lambda(G(I'_k)).$$

The latter equations clarify that

$$\lambda(G(N_2)) \leq \sum_{k=1}^{+\infty} \lambda(I'_k) + \sum_{k=1}^{+\infty} \lambda(J_k) < \epsilon + \lambda(U) < 2\epsilon.$$

Thus $\lambda(G(N_2)) = 0$. This shows that Lusin's condition is fulfilled for $G(x) = F(x) + x$. Now pertaining to Step 1, there is an integrable and Borel-measurable function $f_1 : [a, b] \longrightarrow \mathbb{R}$ s.t.

$$G(x) - G(a) = \int_{[a,x]} f_1 \, d\lambda,$$

hence

$$F(x) - F(a) = \int_{[a,x]} f \, d\lambda,$$

thus if we let $f = f_1 - 1$, this completes the proof of Step 2.

Step 3. Finally, we assume that F is continuous and of bounded variation which satisfies Lusin's condition and show that the theorem holds. To accomplish the claim, we make use of the following two lemmas.

Lemma 2.1. *Let $F : [a, b] \longrightarrow \mathbb{R}$ be a continuous function of bounded variation. If $F = p - n$ is the Jordan decomposition for F, then p and n are continuous.*

Lemma 2.2. *With the hypothesis of Lemma 2.1, let $N \subset [a, b]$ be such that $F(N)$ has a zero Lebesgue measure. Then $p(N)$ and $n(N)$ are also of Lebesgue measure zero.*

Lemma 2.2 immediately yields the following result.

Corollary 2.3. *With the hypothesis of Lemma 2.1, let F satisfies Lusin's condition, then p and n also satisfy Lusin's condition.*

It is seen from Lemma 2.1 and Corollary 2.3 that both p and n are continuous and of bounded variation and Lusin's condition is valid for them. Then since they are non–decreasing, there exist integrable and Borel–measurable real–valued functions g and h on $[a, b]$ so that $p(x) = p(a) + \int_{[a,x]} g \, d\lambda$ and $n(x) = n(a) + \int_{[a,x]} h \, d\lambda$, therefore the proof is complete substituting $f = g - h$.

3. **Proof of Lemma 2.1**

It is sufficient to prove that p is continuous. First, we note that p is a right–continuous function. The continuity of p can be directly achieved by $(\epsilon - \delta)$ method. However an alternative proof is presented here because of its easier application in the proof of Lemma 2.2.

By definition,

$$p(x) = \bigvee_a^x (F) = \sup_P |F(P)| = \sup_P \sum_{k=1}^{n(P)} |F(x_k) - F(x_{k-1})| \tag{3.1}$$

is the variation of F from a to x where the supremum is taken over all partitions

$$P\,:\, a = x_0 < x_1 < \cdots < x_n = x$$

of $[a, x]$ and $n = n(P) = \#P - 1$. Therefore for arbitrary $\epsilon > 0$ there is a partition P such that

$$0 \leq \bigvee_a^x (F) - |F(P)| < \epsilon. \tag{3.2}$$

Definition 3.1. *For the given partition $P\,:\, a = x_0 < x_1 < \cdots < x_n = b$, let $x \in [x_{i-1}, x_i]$. Two adjacent partitions $P_1(x)$ and $P_2(x)$ are defined as*

$$P_1(x)\,:\, a = x_0 < \cdots < x_{i-1} \leq x,$$
$$P_2(x)\,:\, x \leq x_i < \cdots < x_n = b,$$

and the partition $P'(x)$ considered as a refinement of P is

$$P'(x):\;\; a = x_0 < \cdots < x_{i-1} \leq x \leq x_i < \cdots < x_n = b.$$

For $\epsilon > 0$ and its corresponding partition P considered in Eq. (3.2), one can define continuous functions $\mathrm{w}_i : [x_{i-1}, x_i] \longrightarrow \mathbb{R}$ as

$$\mathrm{w}_i(x) = |F(P_1(x))|. \tag{3.3}$$

Application of Lemma 2.2 implies the existence of a continuous function $\mathrm{u}_\epsilon : [a, b] \longrightarrow \mathbb{R}$ so that on each $[x_{i-1}, x_i]$, u_ϵ is equal to w_i. Therefore

$$\Big(\bigvee_a^x (F) - |F(P_1(x))|\Big) + \Big(\bigvee_x^b (F) - |F(P_2(x))|\Big) = \bigvee_a^b (F) - |F(P'(x))| < \epsilon.$$

The two terms on the left hand side of the above relation are nonnegative and especially considering the first term, one finds that

$$0 \leq p(x) - \mathrm{u}_\epsilon(x) < \epsilon, \tag{3.4}$$

in which Eqs. (3.2) and (3.3) were applied.

Now consider $\left\{\mathrm{u}_{2^{-k}}\right\}_{k=1}^{\infty}$ as a sequence of continuous functions. Equation (3.4) with $\epsilon = 2^{-k}$ shows that this sequence converges uniformly to p, thus p is continuous.

4. **Proof of Lemma 2.2**

Let $N \subset [a, b]$ be such that $\lambda(F(N)) = 0$. For arbitrary $\epsilon > 0$, consider its corresponding partition P as introduced in Eq. (3.2). It is sufficient to prove that $\lambda(p(N_i)) = \lambda(n(N_i)) = 0$ where $N_i = N \cap [x_{i-1}, x_i]$ $(1 \leq i \leq n)$. Since $F(N_i)$ has zero Lebesgue measure, there exists a sequence of disjoint open intervals $\{J_k\}_{k=1}^{\infty}$ such that $F(N_i) \subset \bigcup_{k=1}^{\infty} J_k$ and

$$\sum_{k=1}^{\infty} \lambda(J_k) < \epsilon. \tag{4.1}$$

At most one of J_ks contains the point $F(x_{i-1})$ and at most one of them contains $F(x_i)$. If so, we exclude these two points from J_ks and split the interval(s) containing the points into two adjacent open intervals. This process clearly leaves relation (4.1) unchanged. For each J_k we have $F^{-1}(J_k) = \bigcup_{l=1}^{\infty} I_{kl}$ where intervals $I_{kl} = (a_{kl}, b_{kl})$ are disjoint. By our hypothesis, one can easily observe that

$$\lambda(p(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})). \tag{4.2}$$

Choose any finite number of intervals I_{kl}s and call them (a_1, b_1), $\cdots$, (a_m, b_m) in such an order that we have the partition

$$Q: \; b_0 = x_{i-1} \leq a_1 < b_1 < a_2 < \cdots < a_m < b_m \leq a_{m+1} = x_i. \tag{4.3}$$

Thus

$$\bigvee_{x_{i-1}}^{x_i}(F) - |F(Q)| < \epsilon,$$

which means that

$$\sum_{j=1}^{m}\Big(\bigvee_{a_j}^{b_j}(F)-|F(b_j)-F(a_j)|\Big)+\sum_{j=0}^{m}\Big(\bigvee_{b_j}^{a_{j+1}}(F)-|F(a_{j+1})-F(b_j)|\Big)<\epsilon.$$

Each term in the left side is nonnegative, especially noting to the first term and recalling the definition of p by (3.1) and its non-decreasing property, one concludes that

$$\sum_{j=1}^{m}\lambda(p(a_j,b_j))<\epsilon+\sum_{j=1}^{m}|F(b_j)-F(a_j)|.$$

The above inequality holds for any finite number of I_{kl} thus

$$\sum_{k,l=1}^{\infty}\lambda(p(I_{kl}))<\epsilon+\sum_{k,l=1}^{\infty}|F(b_{kl})-F(a_{kl})|. \tag{4.4}$$

Our next task is to find an upper bound proportional to ϵ for the second term of the last equation. To do this we consider two separate cases. The first case is when $F(x_{i-1})=F(x_i)$. Choose again the finite number of I_{kl}, namely (a_j,b_j), for $1\le j\le m$ and construct the partition Q as introduced in Eq. (4.3). The partition is a refinement of $x_{i-1}<x_i$ and so $|F(Q)|-|F(x_i)-F(x_{i-1})|<\epsilon$ and thus

$$\sum_{j=1}^{m}|F(b_j)-F(a_j)|\le|F(Q)|<\epsilon.$$

The last inequality holds for any finite number of I_{kl} so it is also valid for all of them. Therefore when $F(x_{i-1})=F(x_i)$ by the use of (4.4) we have

$$\lambda(p(N_i))\le\sum_{k,l=1}^{\infty}\lambda(p(I_{kl}))<2\,\epsilon. \tag{4.5}$$

The second case is related to the condition $F(x_{i-1})<F(x_i)$ (the opposite case is similar). Recall that J_k where disjoint open intervals containing $F(N_i)$ (probably except for the two points $F(x_{i-1})$ and $F(x_i)$) with total measure less than ϵ. Thus we are able to divide them into three types: J_k^+ whose points are greater than $F(x_i)$, J_k^- whose points are less than $F(x_{i-1})$ and J_k° whose points are between $F(x_{i-1})$ and $F(x_i)$. At first attend to J_k^+. In this case, take any finite number of I_{kl}^+ (whose images are inside J_k^+ s) say (a_j^+,b_j^+) s for $1\le j\le m$ such that $x_{i-1}\le a_1^+<b_1^+<a_2^+<\cdots<a_m^+<b_m^+\le x_i$. Images of these intervals

lie inside a finite (say s) number of J_k^+ s, namely $J_{k_r}^+ = (c_r^+, d_r^+)$ for $1 \leq r \leq s$ where obviously $s \leq m$.

Suppose that in addition, (c_r^+, d_r^+),s are arranged increasingly such that $F(x_i) \leq c_1^+ < d_1^+ < c_2^+ < \cdots < c_s^+ < d_s^+$. The compact set $[x_{i-1}, x_i] \cap F^{-1}(c_1^+)$ has a minimum and maximum respectively α^+ and β^+. Since the images of all (a_j^+, b_j^+) are greater than $c_1^+ \geq F(x_i)$, the intermediate value theorem implies that they all lie between α^+ and β^+. Thus there exist partition $R_1: \ x_{i-1} < \alpha^+ < \beta^+ < x_i$ and its refinement $R_2: \ x_{i-1} < \alpha^+ < a_1^+ < b_1^+ < \cdots a_m^+ < b_m^+ < \beta^+ < x_i$. The relation $|F(R_2)| - |F(R_1)| < \epsilon$ regarding the fact that $F(\alpha^+) = F(\beta^+) = c_1^+$ implies that

$$\sum_{j=1}^{m} |F(b_j^+) - F(a_j^+)| < \epsilon,$$

but since this is true for any finite number of considered intervals, so for $I_{kl}^+ = (a_{kl}^+, b_{kl}^+)$ s we have

$$\sum_{k,l} |F(b_{kl}^+) - F(a_{kl}^+)| < \epsilon. \tag{4.6}$$

Similarly, for $I_{kl}^- = (a_{kl}^-, b_{kl}^-)$ we have

$$\sum_{k,l} |F(b_{kl}^-) - F(a_{kl}^-)| < \epsilon.$$

Finally consider J_k° whose points are between $F(x_{i-1})$ and $F(x_i)$ where for each k, $F^{-1}(J_k^\circ) = \bigcup_{l=1}^{\infty} I_{kl}^\circ$. Similar to the previous case choose a finite number of I_{kl}° s such as (a_j°, b_j°) s for $1 \leq j \leq m$ such that $x_{i-1} \leq a_1^\circ < b_1^\circ < a_2^\circ < \cdots < a_m^\circ < b_m^\circ \leq x_i$ and assume their images lie in $J_{k_r}^\circ = (c_r^\circ, d_r^\circ)$ for $1 \leq r \leq s$ where clearly $s \leq m$. Again suppose (c_r°, d_r°) s are arranged increasingly such that

$$F(x_{i-1}) \leq c_1^\circ < d_1^\circ < c_2^\circ < \cdots < c_s^\circ < d_s^\circ \leq F(x_i). \tag{4.7}$$

Now define $\alpha_r^\circ = \min\Big([x_{i-1}, x_i] \cap F^{-1}(c_r^\circ)\Big)$ for $1 \leq r \leq s$. The relation (4.7) and the intermediate value theorem imply that

$$x_{i-1} \leq \alpha_1^\circ < \alpha_2^\circ < \cdots < \alpha_s^\circ < \alpha_{s+1}^\circ = x_i. \tag{4.8}$$

Note that in the above relation α_{s+1}° is defined to be x_i. In addition, define $\beta_r^\circ = \max\Big([x_{i-1}, \alpha_{r+1}^\circ] \cap F^{-1}(d_r^\circ)\Big)$ for $1 \leq r \leq s$ and also define $\beta_0^\circ = x_{i-1}$. This definition immediately yields that for each $r = 1, \cdots s-$

1 we have $\alpha_r^\circ < \beta_r^\circ < \alpha_{r+1}^\circ$ while for $r = 0$ we have $x_{i-1} = \beta_0^\circ \leq \alpha_1^\circ$ and for $r = s$ we have $\alpha_s^\circ < \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i$. Thus, relation (4.8) is finally improved to admit to define the partition

$$S_1: \ x_{i-1} = \beta_0^\circ \leq \alpha_1^\circ < \beta_1^\circ < \alpha_2^\circ < \cdots < \alpha_s^\circ < \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i. \tag{4.9}$$

In this position we claim that for each j, r $(1 \leq j \leq m, 0 \leq r \leq s)$ we have $(a_j^\circ, b_j^\circ) \cap (\beta_r^\circ, \alpha_{r+1}^\circ) = \varnothing$. If not, assume y belongs to this set, then only two cases may occur:

First case; we have $F(y) < d_r^\circ < c_{r+1}^\circ$ for $1 \leq r \leq s-1$, $F(y) < c_1^\circ$ for $r = 0$ and $F(y) < d_s^\circ$ for $r = s$. The case $r = 0$ has no sense because $y \in (a_j^\circ, b_j^\circ)$ and the images of all (a_j°, b_j°) are greater than c_1°. When $r = s$ since $F(y) < d_s^\circ \leq F(\alpha_{s+1}^\circ) = F(x_i)$, the intermediate value theorem implies that there exists a point $z \in (y, x_i]$ such that $F(z) = d_s^\circ$. But according to the definition of β_s° we must have $z \leq \beta_s^\circ$ which contradicts with the position of y. Finally when $1 \leq r \leq s-1$, since $F(y) < d_r^\circ < F(\alpha_{r+1}^\circ) = c_{r+1}^\circ$, the intermediate value theorem implies that there exists a point $z' \in (y, \alpha_{r+1}^\circ)$ such that $F(z') = d_r^\circ$ but according to the definition of β_r° we must have $z' \leq \beta_r^\circ$ which is a contradiction.

On the other hand in the second case we may have $d_r^\circ < c_{r+1}^\circ < F(y)$ for $1 \leq r \leq s-1$, $c_1^\circ < F(y)$ for $r = 0$ and $d_s^\circ < F(y)$ for $r = s$. The case $r = s$ has no sense because the images of all (a_j°, b_j°) s are less than d_s°. When $r = 0$ since $F(\beta_0^\circ) = F(x_{i-1}) \leq c_1^\circ < F(y)$, the intermediate value theorem implies that there exists a point $t \in [x_{i-1}, y)$ such that $F(t) = c_1^\circ$. But according to the definition of α_1° we must have $\alpha_1^\circ \leq t$ which is in contradiction with the position of y.

Finally when $1 \leq r \leq s-1$, since $F(\beta_r^\circ) = d_r^\circ < c_{r+1}^\circ < F(y)$, the intermediate value theorem implies that there exists a point $t' \in (\beta_r^\circ, y)$ such that $F(t') = c_{r+1}^\circ$ but according to the definition of α_{r+1}° we must have $\alpha_{r+1}^\circ \leq t'$ which is a contradiction. Thus, our claim is proved, that is, non of the points a_j° or b_j° lie inside the intervals $(\beta_r^\circ, \alpha_{r+1}^\circ)$ or in other words, all points a_j° and b_j° lie only inside the intervals $[\alpha_r^\circ, \beta_r^\circ]$.

The above fact admits the definition of partition S_2 as

$$\begin{aligned} S_2: \ x_{i-1} \ &= \ \beta_0^\circ \leq \alpha_1^\circ \leq a_1^\circ < b_1^\circ < \cdots < a_{j_1}^\circ < b_{j_1}^\circ \leq \beta_1^\circ \\ &< \ \alpha_2^\circ \leq a_{j_1+1}^\circ < b_{j_1+1}^\circ < \cdots < a_{j_2}^\circ < b_{j_2}^\circ \leq \beta_2^\circ \\ &< \ \alpha_3^\circ < \cdots < \alpha_s^\circ \leq \cdots < a_m^\circ < b_m^\circ \leq \beta_s^\circ \leq \alpha_{s+1}^\circ = x_i, \end{aligned} \tag{4.10}$$

which is clearly a refinement of partition S_1 defined in (4.9). Thus, according to our hypothesis we see that $|F(S_2)| - |F(S_1)| < \epsilon$ which by a simple but careful observation results in the following relation

$$\sum_{j=1}^{m} |F(b_j^\circ) - F(a_j^\circ)| < \epsilon + \sum_{r=1}^{s} |F(\beta_r^\circ) - F(\alpha_r^\circ)|.$$

Recalling the definitions of α_r° and β_r° and since $J_{k_r}^\circ = (c_r^\circ, d_r^\circ)$, the above relation converts to

$$\sum_{j=1}^{m} |F(b_j^\circ) - F(a_j^\circ)| < \epsilon + \sum_{r=1}^{s} \lambda(J_{k_r}^\circ),$$

and due to relation (4.1) one obtains

$$\sum_{j=1}^{m} |F(b_j^\circ) - F(a_j^\circ)| < 2\,\epsilon.$$

Since the above relation is true for the end points of any finite number (here m) of I_{kl}° it is also valid for all of them, that is

$$\sum_{k,l} |F(b_{kl}^\circ) - F(a_{kl}^\circ)| < 2\,\epsilon. \tag{4.11}$$

Now by gathering the relations (4.6), (4.7) and (4.11) it is found that

$$\sum_{k,l=1}^{\infty} |F(b_{kl}) - F(a_{kl})| < 4\,\epsilon. \tag{4.12}$$

Inequalities (4.2), (4.4) and (4.12) yield

$$\lambda(p(N_i)) \le \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) < 5\,\epsilon. \tag{4.13}$$

This establishes the zero measure of $p(N_i)$ when $F(x_{i-1}) \neq F(x_i)$.

It only remains to show for the non-decreasing function $n = p - F$, that $\lambda(n(N_i)) = 0$. In an exactly similar way of obtaining relation (4.2) one easily finds that

$$\lambda(n(N_i)) \le \sum_{k,l=1}^{\infty} \lambda(n(I_{kl})),$$

where still $I_{kl} = (a_{kl}, b_{kl})$ and thus $n(I_{kl}) \subset [n(a_{kl}), n(b_{kl})]$. Then we note that for any two points $x, y \in [x_{i-1}, x_i]$ since $n = p - F$ we have

$$|n(y) - n(x)| \le |p(y) - p(x)| + |F(y) - F(x)|.$$

Substituting a_{kl}, b_{kl} s to resp. x, y the latter relation yields

$$\lambda(n(N_i)) \leq \sum_{k,l=1}^{\infty} \lambda(n(I_{kl})) \leq \sum_{k,l=1}^{\infty} \lambda(p(I_{kl})) + \sum_{k,l=1}^{\infty} |F(b_{kl}) - F(a_{kl})|.$$

The upper bounds for the first and second terms on the right hand side of the above relation due to (4.12) and (4.13) proves the zero measure of $n(N_i)$.

5. Conclusion

As one little step towards understanding the regularity of hydrodynamical quantities, it was attempted to see a more direct and clear dependency of continuity and integrability through the Lebesgue integral while there is a hope to generalize the method to find the situation for other types of integration. Indeed, there probably exists an alternative kind of absolute continuity in connection with other types of integration rather than the Lebesgue one.

Even further, since the used method here essentially employed the general measure-type informations, it looks to have sense to include the issue of measurability of fluid functions under the mechanism of singularity. In other words, the problem of blow up usually deals with singularities and therefore infinite integrals while it is not yet known if this dynamics can change even the measurability of solutions or not.

It was seen here that the absolute continuity can be extracted directly as a consequence of measure-type properties of functions. There was nowhere used the idea of differentiability which is the result of the Vitali covering lemma. Instead, the Radon–Nikodym theorem was the main tool which relies solely on the excellent consistency between the Lebesgue integral and a measure space.

In addition, Lemma 2.2 was proven showing a slightly more general result than needed for the proof of the Banach-Zarecki theorem. Although the classical version of this theorem was proven here but it is not surprising if one can generalize this proof to more general spaces and even higher dimensions.

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References

[1] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, Real Analysis, Prentice-Hall, New Jersey, 2007.

[2] N. L. Carothers, Real Analysis, Cambridge Univ. Press, Cambridge, 2000.

[3] J. Duda, Absolutely continuous functions with values in a metric space, *Real Anal. Exchange* **32** (2006), no. 2, 569–582.

[4] J. Duda and L. Zajíček, The Banach-Zarecki theorem for functions with values in metric spaces, *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3631–3633.

[5] O. Dzagnidze, Some new results on the continuity and differentiability of functions of several real variables, *Proc. A. Razmadze Math. Inst.* **134** (2004) 1–144.

[6] H. Federer, Geometric Measure Theory, Grundlehren der Mathematischen Wissenschaften, 153, Springer-Verlag, New York, 1969.

[7] R. Jahanipour and B. Zohouri-Zanganeh, Real Analysis: An introduction to measure and integration theory (in persian), Sharif Univ. Press, Tehran, 2009.

[8] N. N. Luzin, The integral and trigonometric series (Russian), *Moskva, Mat. Sborn.* **30** (1916) 48–212.

[9] J. Malý, Absolutely continuous functions of several variables, *J. Math. Anal. Appl.* **231** (1999), no. 2, 492–508.

[10] I. P. Natanson, Theory of functions of a real variable, (Translated from Russian), Frederick Ungar Publishing Co., New York, 1961.

[11] H. L. Royden and P. M. Fitzpatrick, Real Analysis, Prentice–Hall, New Jersey, 2010.

[12] J. Šremr, A Note on Absolutely Continuous Functions of Two Variables in the Sense of Carathéodory, Preprint, Institute of Math., Prague, 2008.

[13] J. Yeh, Lectures on Real Analysis, World Scientific Publishing Co., Inc., River Edge, 2000.

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