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### ON A CLASS OF SYSTEMS OF *n* NEUMANN TWO-POINT BOUNDARY VALUE STURM-LIOUVILLE TYPE EQUATIONS

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ABSTRACT. Employing a three critical points theorem, we prove the existence of multiple solutions for a class of Neumann twopoint boundary value Sturm-Liouville type equations. Using a local minimum theorem for differentiable functionals the existence of at least one non-trivial solution is also ensured.

# 1. Introduction

Consider the following Neumann two-point boundary value Sturm-Liouville type system

$$\begin{cases} -(p_i(x)u'_i(x))' + r_i(x)u'_i(x) + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, ..., u_n) \\ x \in (0, 1), \\ u'_i(0) = u'_i(1) = 0 \end{cases}$$

for  $1 \leq i \leq n$ , where  $n \geq 1$  is an integer,  $p_i \in C^1([0,1)]$ ,  $r_i, q_i \in C^0([0,1)]$ with  $p_i$  and  $q_i$  positive functions for  $1 \leq i \leq n, \lambda$  is a positive parameter,  $F: [0,1] \times \mathbb{R}^n \to \mathbb{R}$  is a function such that  $F(., t_1, ..., t_n)$  is measurable in

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[0,1] for all  $(t_1, ..., t_n) \in \mathbb{R}^n$ , F(x, ..., ..) is  $C^1$  in  $\mathbb{R}^n$  for every  $x \in [0,1]$ and for every  $\varrho > 0$ ,

$$\sup_{|(t_1,...,t_n)| \le \varrho} \sum_{i=1}^n |F_{t_i}(x,t_1,...,t_n)| \in L^1([0,1]),$$

and  $F_{u_i}$  denotes the partial derivative of F with respect to  $u_i$  for  $1 \le i \le n$ .

Based on a three critical point theorem (Theorem 2.1), we establish the existence of at least three weak solutions for the system (1.1) under suitable assumptions on nonlinear term F. Employing a local minimum theorem for differentiable functionals (Theorem 2.2), under appropriate hypotheses on F, we also ensure the existence of at least one non-trivial solution.

Problems of Sturm-Liouville type have been widely investigated. We refer the reader to the papers [3, 5, 6, 18] and the references therein.

For a thorough account on the subject, we also refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In the present paper, our motivation comes from the recent paper [5].

#### 2. Preliminaries and basic notations

Our main tools to prove the results are critical point theorems. First we recall an immediate consequence of [4, Theorem 3.3](see also [1, Theorem 5.2]).

Let X be a nonempty set and  $\Phi, \Psi : X \longrightarrow \mathbb{R}$  be two functions. for all  $r, r_1, r_2 > \inf_X \Phi$ ,  $r_2 > r_1, r_3 > 0$ , we define

$$\begin{split} \varphi(r) &:= \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(u)}{r - \Phi(u)} \\ \beta(r_1, r_2) &:= \inf_{u \in \Phi^{-1}(]-\infty, r_1[)} \sup_{v \in \Phi^{-1}([r_1, r_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\ \gamma(r_2, r_3) &:= \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2 + r_3[)} \Psi(u)}{r_3} \end{split}$$

and

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

**Theorem 2.1.** [4, Theorem 3.3] Let X be a reflexive real Banach space,  $\Phi: X \longrightarrow \mathbb{R}$  be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on  $X^*, \Psi: X \longrightarrow \mathbb{R}$  be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that

1.  $\inf_X \Phi = \Phi(0) = \Psi(0) = 0;$ 

2. for every  $u^*$ ,  $u^{**}$  such that  $\Psi(u^*) \ge 0$  and  $\Psi(u^{**}) \ge 0$ , one has

$$\inf_{s \in [0,1]} \Psi(su^* + (1-s)u^{**}) \ge 0.$$

Assume that there are three positive constants  $r_1, r_2, r_3$  with  $r_1 < r_2$ , such that

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Next, we recall a local minimum theorem for differentiable functionals, Theorem 2.5 of [17] as given in [2, Theorem 5.1] (see also [2, Proposition 2.1] for related results).

For a given non-empty set X, and two functionals  $\Phi, \Psi: X \to \mathbb{R}$ , we define the following functions

$$\delta(r_1, r_2) = \inf_{v \in \Phi^{-1}([r_1, r_2[])} \frac{\sup_{u \in \Phi^{-1}([r_1, r_2[])} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all  $r_1, r_2 \in \mathbb{R}, r_1 < r_2$ .

**Theorem 2.2.** [2, Theorem 5.1] Let X be a reflexive real Banach space,  $\Phi: X \to \mathbb{R}$  be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on  $X^*$  and  $\Psi: X \to \mathbb{R}$  be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put  $I_{\lambda} = \Phi - \lambda \Psi$  and assume that there are  $r_1, r_2 \in \mathbb{R}$ ,  $r_1 < r_2$ , such that

$$\begin{split} \delta(r_1, r_2) < \rho(r_1, r_2). \\ Then, \ for \ each \ \lambda \in ]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\delta(r_1, r_2)}[ \ there \ is \ u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[) \ such \ that \\ I_{\lambda}(u_{0,\lambda}) \le I_{\lambda}(u) \ \forall u \in \Phi^{-1}(]r_1, r_2[) \ and \ I'_{\lambda}(u_{0,\lambda}) = 0. \end{split}$$

Let us introduce some notations which will be used later.

Set  $p_{0i} = \min_{[0,1]} p_i(x)$ ,  $q_{0i} = \min_{[0,1]} q_i(x)$  and  $m_i = \min\{p_{0i}, q_{0i}\}$  for  $1 \le i \le n$ , and set  $\underline{m} = \min\{m_i; 1 \le i \le n\}$ .

Here and in the sequel,

$$X := W^{1,2}([0,1]) \times \dots \times W^{1,2}([0,1]) = (W^{1,2}([0,1]))^n$$

endowed with the norm

$$||(u_1,...,u_n)||_* = \sum_{i=1}^n ||u_i|$$

where  $||u_i|| = \left(\int_0^1 (p_i(x)|u'_i(x)|^2 + \int_0^1 q_i(x)|u_i(x)|^2) dx\right)^{\frac{1}{2}}$ For all  $\gamma > 0$  we denote by  $K(\gamma)$  the set

(2.1) 
$$\left\{ (t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \gamma \right\}$$

#### 3. Existence of three solutions

Consider the following Neumann two-point boundary value Sturm-Liouville type system

(3.1) 
$$\begin{cases} -(p_i(x)u'_i(x))' + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, ..., u_n) & x \in (0, 1), \\ u'_i(0) = u'_i(1) = 0 \end{cases}$$

for  $1 \leq i \leq n$ , where  $n \geq 1$  is an integer,  $p_i \in C^1([0,1)]$ ,  $q_i \in C^0([0,1)]$ with  $p_i$  and  $q_i$  positive functions for  $1 \leq i \leq n$ ,  $\lambda$  is a positive parameter,  $F: [0,1] \times \mathbb{R}^n \to \mathbb{R}$  is a function such that  $F(., t_1, ..., t_n)$  is measurable in [0,1] for all  $(t_1, ..., t_n) \in \mathbb{R}^n$ , F(x, ..., ...) is  $C^1$  in  $\mathbb{R}^n$  for every  $x \in [0,1]$ and for every  $\rho > 0$ ,

$$\sup_{(t_1,...,t_n)|\leq \varrho} \sum_{i=1}^n |F_{t_i}(x,t_1,...,t_n)| \in L^1([0,1]).$$

We say that  $u = (u_1, ..., u_n)$  is a weak solution to the system (3.1) if  $u = (u_1, ..., u_n) \in X$  and

$$\sum_{i=1}^{n} \int_{0}^{1} \left( p_{i}(x)u_{i}'(x)v_{i}'(x) + q_{i}(x)u_{i}(x)v_{i}(x) \right) dx$$
$$-\lambda \sum_{i=1}^{n} \int_{0}^{1} F_{u_{i}}(x, u_{1}(x), \dots, u_{n}(x))v_{i}(x)dx = 0$$

for every  $v = (v_1, \dots, v_n) \in X$ .

For a given positive constant  $\nu$ , put

$$a(\nu) := \frac{\int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu)} F(x, t_1,\dots,t_n) dx}{\nu^2}$$

where  $K(\nu) = \{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \nu\}$  (see (2.1)). We formulate the main result of this section as follows:

**Theorem 3.1.** Let  $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$  satisfies the conditions  $F(x, t_1, ..., t_n) \ge 0$  for all  $(x, t_1, ..., t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$  and F(x, 0, ..., 0) = 0 for every  $x \in [0, 1]$ . Assume that there exist four positive constants  $\nu_1$ ,  $\nu_2$ ,  $\eta$  and  $\tau$  with  $\nu_1 < n\sqrt{\frac{2}{m}||q_n||_1}\tau < \nu_2 < \eta$  such that

(A1)

$$\max\left\{a(\nu_{1}), a(\nu_{2}), \frac{\eta^{2}}{\eta^{2} - \nu_{2}^{2}}a(\eta)\right\}$$

$$< \frac{\underline{m}}{2n^{2}||q_{n}||_{1}} \frac{\int_{0}^{1} F(x, 0, ..., 0, \tau)dx - \int_{0}^{1} \sup_{(t_{1}, \dots, t_{n}) \in K(\nu_{1})} F(x, t_{1}, ..., t_{n})dx}{\tau^{2}}$$

Then, for each

$$\lambda \in \left[ \frac{\frac{1}{2} ||q_n||_1 \tau^2}{\int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu_1)} F(x,t_1,\dots,t_n) dx - \int_0^1 F(x,0,\dots,0,\tau) dx} \right], \frac{\frac{m}{4n^2} \min\left\{\frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2 - \nu_2^2}{\eta^2} \frac{1}{a(\eta)}\right\} \right]$$

the system (3.1) admits at least three weak solutions  $v^j = (v_1^j, \ldots, v_n^j) \in$ X (j = 1, 2, 3) such that

$$\sum_{i=1}^{n} |v_{i}^{j}(x)| \leq \eta \text{ for each } x \in [a,b], \ j = 1,2,3.$$

*Proof.* In order to apply Theorem 2.1 to our problem, we introduce the functionals  $\Phi$ ,  $\Psi: X \to \mathbb{R}$  for each  $u = (u_1, ..., u_n) \in X$ , as follows

$$\Phi(u) = \sum_{i=1}^{n} \frac{||u_i||^2}{2}$$

and

$$\Psi(u) = \int_0^1 F(x, u_1(x), ..., u_n(x)) dx.$$

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It is well known that  $\Phi$  and  $\Psi$  are well defined and continuously differentiable functionals whose derivatives at the point  $u = (u_1, ..., u_n) \in X$ are the functionals  $\Phi'(u), \Psi'(u) \in X^*$ , given by

$$\Phi'(u)(v) = \sum_{i=1}^{n} \int_{0}^{1} \left( p_i(x)u'_i(x)v'_i(x) + q_i(x)u_i(x)v_i(x) \right) dx$$

and

$$\Psi'(u)(v) = \sum_{i=1}^{n} \int_{0}^{1} F_{u_i}(x, u_1(x), ..., u_n(x))v_i(x)dx$$

for every  $v = (v_1, ..., v_n) \in X$ , respectively. Moreover,  $\Phi$  is sequentially weakly lower semicontinuous and  $\Phi'$  admits a continuous inverse on  $X^*$ , as well as,  $\Psi' : X \to X^*$  is a compact operator. Obviously,  $\Phi$  and  $\Psi$ satisfy condition 1. of Theorem 2.1. Moreover, since  $F(x, t_1, ..., t_n) \ge 0$ for all  $(x, t_1, ..., t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$ , for every  $u^*$ ,  $u^{**} \in X$  with  $\Psi(u^*) \ge 0$  and  $\Psi(u^{**}) \ge 0$ , one has

$$\inf_{s \in [0,1]} \Psi(su^* + (1-s)u^{**}) \ge 0.$$

Set  $w(x) = (0, ..., 0, \tau), r_1 = \underline{m}(\frac{\nu_1}{2n})^2, r_2 = \underline{m}(\frac{\nu_2}{2n})^2$  and  $r_3 = \underline{m}\frac{\eta^2 - \nu_2^2}{4n^2}$ . One has  $\Phi(w) = \frac{1}{2}||q_n||_1\tau^2$ . So, bearing the condition  $\nu_1 < n\sqrt{\frac{2}{m}}||q_n||_1\tau^2 < \nu_2 < \eta$  in mind, we get  $r_1 < \Phi(w) < r_2$  and  $r_3 > 0$ . Since for  $1 \le i \le n$ ,

$$|u_i(x)| \le \sqrt{\frac{2}{m_i}} ||u_i|| \quad \forall \ u_i \in W^{1,2}([0,1]),$$

we obtain

(3.2) 
$$\sup_{x \in [0,1]} \sum_{i=1}^{n} |u_i(x)|^2 \le \sum_{i=1}^{n} \frac{2}{m_i} ||u_i||^2 \le \frac{2}{\underline{m}} \sum_{i=1}^{n} ||u_i||^2$$

for each  $u = (u_1, ..., u_n) \in X$ , and so from the definition of  $\Phi$ , we see that

$$\begin{split} \Phi^{-1}(] - \infty, r_1[) &= \{ u = (u_1, ..., u_n) \in X; \Phi(u) < r_1 \} \\ &= \left\{ u \in X; \sum_{i=1}^n ||u_i||^2 < 2r_1 \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)|^2 < \frac{\nu_1^2}{n^2} \text{ for all } x \in [0, 1] \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)| \le \nu_1 \text{ for all } x \in [0, 1] \right\}, \end{split}$$
which follows
$$\begin{aligned} \sup_{(u_1, ..., u_n) \in \Phi^{-1}(] - \infty, r_1[)} \Psi(u) \\ &= \sup_{(u_1, ..., u_n) \in \Phi^{-1}(] - \infty, r_1[)} \int_0^1 F(x, u_1(x), ..., u_n(x)) dx \end{split}$$

$$\leq \int_{0}^{1} \sup_{(t_1,...,t_n) \in K(\nu_1)} F(x,t_1,...,t_n) dx.$$
  
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In a similar way, we get

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$$\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)$$

$$\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)}\int_0^1 F(x,u_1(x),...,u_n(x))dx$$

$$\leq \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_2)} F(x,t_1,...,t_n) dx$$

and

$$\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2+r_3[)}\Psi(u)$$

$$\sup_{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2+r_3[)}\int_0^1 F(x,u_1(x),...,u_n(x))dx$$

$$\leq \int_{-\infty}^{1} \sup F(x, t_1, ..., t_n) dx.$$

$$\int_{0} \sup_{(t_1,\ldots,t_n)\in K(\eta)}$$

So, taking into account that  $0 \in \Phi^{-1}(] - \infty, r_1[)$ , one has

$$\varphi(r_1) \le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_1[]} \Psi(u)}{r_1} \le \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\underline{m}(\frac{\nu_1}{2n})^2}$$
$$\sup_{u \in \Phi^{-1}(]-\infty, r_0[]} \Psi(u) = \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\mu_1)} F(x, t_1, \dots, t_n) dx$$

$$\varphi(r_2) \le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)}{r_2} \le \frac{J_0 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\underline{m}(\frac{\nu_2}{2n})^2}$$

and

$$\gamma(r_1, r_2) \le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2+r_3[)} \Psi(u)}{r_3}$$
$$\le \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\eta)} F(x, t_1, \dots, t_n) dx}{\underline{m} \frac{\eta^2 - \nu_2^2}{4n^2}}$$

On the other hand, for each  $u \in \Phi^{-1}(] - \infty, r_1[)$ , one has

$$\beta(r_1, r_2) \geq \frac{\int_0^1 F(x, w_1(x), \dots, w_n(x)) dx - \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx}{\sum_{i=1}^n \frac{||w_i||^2}{2} - \sum_{i=1}^n \frac{||u_i||^2}{2}}$$
  
$$\geq \frac{\int_0^1 F(x, w_1(x), \dots, w_n(x)) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\frac{1}{2} ||q_n||_1 \tau^2}$$
  
$$= \frac{2}{||q_n||_1} \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\tau^2}.$$

 $||q_n||_1$ Thanks to assumption (A1) we observe that

 $\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$ 

Therefore, taking into account that the weak solutions of the system (3.1) are exactly the solutions of the equation  $\Phi'(u) - \lambda \Psi'(u) = 0$ , and recalling (3.2), Theorem 2.1 follows the conclusion.

**Remark 3.2.** The weak solutions of the system (3.1) where F is a  $C^1$ -function, by using standard methods, belong to  $C^2([0,1])$ . Namely, in this case, the classical and the weak solutions of the system (3.1) coincide.

Now, we point out the following direct consequence of Theorem 3.1.

**Theorem 3.3.** Let  $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$  satisfies the conditions  $F(x,t_1,...,t_n) \geq 0$  for all  $(x,t_1,...,t_n) \in [0,1] \times (\mathbb{R} \cup \{0\})^n$  and F(x,0,...,0) = 0 for every  $x \in [0,1]$ . Assume that there exist three positive constants  $\nu_1$ ,  $\kappa$  and  $\tau$  with  $\nu_1 < n\sqrt{\frac{2}{m}||q_n||_1}\tau < \frac{1}{\sqrt{2}}\kappa$  such that

$$\begin{array}{l} (A2) \ a(\nu_1) < \frac{(\frac{m}{2n^2||q_n||_1})^2}{1 + \frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau) dx}{\tau^2}; \\ (A3) \ a(\kappa) < \frac{1}{2} \frac{\frac{m}{2n^2||q_n||_1}}{1 + \frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau) dx}{\tau^2} \\ Then, \ for \ each \end{array}$$

$$\lambda \in \Big] \frac{1 + \frac{\underline{m}}{2n^2 ||q_n||_1}}{\frac{\underline{m}}{2n^2 ||q_n||_1}} \frac{\frac{1}{2} ||q_n||_1 \tau^2}{\int_0^1 F(x, 0, ..., 0, \tau) dx}, \ \frac{\underline{m}}{4n^2} \min\Big\{\frac{1}{a(\nu_1)}, \frac{1}{2a(\kappa)}\Big\} \Big[$$

the system (3.1) admits at least three weak solutions  $v^j = (v_1^j, \ldots, v_n^j) \in X$  (j = 1, 2, 3) such that

$$\sum_{i=1}^{n} |v_{i}^{j}(x)| \leq \kappa \text{ for each } x \in [a, b], \ j = 1, 2, 3.$$

*Proof.* Since  $\frac{\underline{m}}{2n^2 ||q_n||_1} < 1$ , from (A2) one has

$$\begin{split} a(\nu_1) &= \frac{\int_0^1 \sup_{(t_1,...,t_n) \in K(\nu_1)} F(x,t_1,...,t_n) dx}{\nu_1^2} \\ &< \frac{\int_0^1 \sup_{(t_1,...,t_n) \in K(\nu_1)} F(x,t_1,...,t_n) dx}{\frac{2n^2 ||q_n||_1}{2n^2 ||q_n||_1} \nu_1^2} \\ &< \frac{\frac{m}{2n^2 ||q_n||_1}}{1 + \frac{m}{2n^2 ||q_n||_1}} \frac{\int_0^1 F(x,0,...,0,\tau) dx}{\tau^2}. \end{split}$$

By choosing  $\nu_2 = \frac{1}{\sqrt{2}}\kappa$  and  $\eta = \kappa$ , from (A3) one has

(3.3) 
$$a(\nu_2) = a(\frac{1}{\sqrt{2}}\kappa) \le 2a(\kappa) < \frac{\frac{m}{2n^2||q_n||_1}}{1 + \frac{m}{2n^2||q_n||_1}} \frac{\int_0^1 F(x, 0, ..., 0, \tau) dx}{\tau^2}$$

and

(3.4) 
$$\frac{\eta^2}{\eta^2 - \nu_2^2} a(\eta) = 2a(\kappa) < \frac{\frac{m}{2n^2 ||q_n||_1}}{1 + \frac{m}{2n^2 ||q_n||_1}} \frac{\int_0^1 F(x, 0, ..., 0, \tau) dx}{\tau^2}.$$

Moreover, bearing the condition  $\nu_1 < n\sqrt{\frac{2}{\underline{m}}}||q_n||_1\tau$  in mind, from (A2) we deduce

$$\frac{\underline{m}}{2n^2||q_n||_1} \frac{\int_0^1 F(x,0,...,0,\tau)dx - \int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\tau^2}$$

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$$> \frac{\underline{m}}{2n^2||q_n||_1} \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2} - \frac{\int_0^1 \sup_{(t_1,...,t_n)\in K(\nu_1)} F(x,t_1,...,t_n)dx}{\nu_1^2} \\ > \left(\frac{\underline{m}}{2n^2||q_n||_1} - \frac{\left(\frac{\underline{m}}{2n^2||q_n||_1}\right)^2}{1 + \frac{\underline{m}}{2n^2||q_n||_1}}\right) \frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2} \\ = \frac{\frac{2m^2||q_n||_1}{1 + \frac{\underline{m}}{2n^2||q_n||_1}}}{\frac{\int_0^1 F(x,0,...,0,\tau)dx}{\tau^2}}.$$

Hence, using again Assumptions (A2) and (A3), owing to (3.3) and (3.4), we observe that all assumptions of Theorem 3.1 are satisfied. So, D by applying Theorem 3.1 we have the conclusion.

We here want to point out a simple version of Theorem 3.3 when n = 1.

Let  $p_1 = p$ ,  $q_1 = q$  and  $m_1 = m$ . Let  $f : [0,1] \times \mathbb{R} \to \mathbb{R}$  be an  $L^1$ -Carathéodory function. Let F be the function defined by F(x,t) = $\int_0^t f(x,s)ds$  for each  $(x,t) \in [0,1] \times \mathbb{R}$ . Then, we have the following existence result.

**Theorem 3.4.** Let the function f satisfies the condition  $f(x,t) \ge 0$  for all  $x \in [0,1]$  and for all  $t \ge 0$ . Assume that there exist three positive constants  $\nu_1$ ,  $\kappa$  and  $\tau$  with  $\nu_1 < \sqrt{\frac{2}{m}} ||q||_1 \tau < \frac{1}{\sqrt{2}} \kappa$  such that

$$\begin{split} (A4) \ & \frac{\int_{0}^{1} \sup_{|t| \leq \nu_{1}} F(x,t) dx}{\nu_{1}^{2}} < \frac{(\frac{m}{2||q||_{1}})^{2}}{1 + \frac{m}{2||q||_{1}}} \int_{0}^{1} F(x,\tau) dx}{\tau^{2}}; \\ (A5) \ & \frac{\int_{0}^{1} \sup_{|t| \leq \kappa} F(x,t) dx}{\kappa^{2}} < \frac{1}{2} \frac{\frac{m}{2||q||_{1}}}{1 + \frac{m}{2||q||_{1}}} \frac{\int_{0}^{1} F(x,\tau) dx}{\tau^{2}}. \\ Then, for each \\ & \lambda \in \Big] \frac{1 + \frac{m}{2||q||_{1}}}{\frac{m}{2||q||_{1}}} \frac{\frac{1}{2}||q||_{1}\tau^{2}}{\int_{0}^{1} F(x,\tau) dx} \\ & , \ \frac{m}{4} \min \Big\{ \frac{\nu_{1}^{2}}{\int_{0}^{1} \sup_{|t| \leq \nu_{1}} F(x,t) dx}, \frac{\kappa^{2}}{2 \int_{0}^{1} \sup_{|t| \leq \kappa} F(x,t) dx} \Big\} \Big[ \\ the \ problem \end{split}$$

(3.5) 
$$\begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(x,u) & x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three weak solutions  $v^j \in W^{1,2}([0,1])$  (j = 1,2,3) such that  $\downarrow i \land \downarrow$ 

$$|v^{j}(x)| \leq \kappa \text{ for each } x \in [a, b], \ j = 1, 2, 3.$$

The following result gives the existence of at least three weak solutions in  $W^{1,2}([0,1])$  to the problem (3.5) in the autonomous case.

Let  $f : \mathbb{R} \to \mathbb{R}$  be a continuous function satisfying  $f(t) \ge 0$  for every  $t \ge 0$ , and put  $F(t) = \int_0^t f(\xi) d\xi$  for all  $t \in \mathbb{R}$ . We have the following result as a straightforward consequence of Theorem 3.4.

**Theorem 3.5.** Assume that there exist three positive constants  $\nu_1$ ,  $\kappa$ and  $\tau$  with  $\nu_1 < \sqrt{\frac{2}{m}} ||q||_1 \tau < \frac{1}{\sqrt{2}} \kappa$  such that

$$\begin{array}{l} (A6) \ f(t) \geq 0 \ for \ each \ t \in [-\kappa, \ 0]; \\ (A7) \ \frac{F(\nu_1)}{\nu_1^2} < \frac{(\frac{m}{2||q||_1})^2}{1 + \frac{m}{2||q||_1}} \frac{F(\tau)}{\tau^2}; \\ (A8) \ \frac{F(\kappa)}{\kappa^2} < \frac{1}{2} \frac{\frac{2||q||_1}{1 + \frac{m}{2||q||_1}}}{1 + \frac{m}{2||q||_1}} \frac{F(\tau)}{\tau^2}. \\ Then, \ for \ each \\ & \lambda \in \frac{1}{2} 2||q||_1^2 + ||q||_1 m \ \tau^2 \quad m \ \min \left\{ \begin{array}{c} \nu_1^2 & 1 \ \kappa^2 \end{array} \right\} \left[ \end{array} \right.$$

 $\lambda \in \left[ \frac{1}{2m} \frac{1}{F(\tau)}, \frac{1}{4} \min\left\{ \frac{1}{F(\nu_1)}, \frac{1}{2} \frac{1}{F(\kappa)} \right\} \right]$ lem

the problem

(3.6) 
$$\begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(u) & x \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions  $v^j \in W^{1,2}([0,1])$  (j = 1, 2, 3)such that

 $|v^{j}(x)| \leq \kappa \text{ for each } x \in [a, b], \ j = 1, 2, 3.$ 

As an example of the results, the following consequence, ensures the existence of at least two non-trivial classical solutions to the problem (3.6).

$$\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to +\infty} \frac{f(t)}{t} = 0.$$

**Theorem 3.6.** Let  $f: \mathbb{R} \to ]0, +\infty[$  be a continuous function such that  $\lim_{t \to 0^+} \frac{f(t)}{t} = \lim_{t \to +\infty} \frac{f(t)}{t} = 0.$ Then, for every  $\lambda > \inf \left\{ \frac{2||q||_1^2 + ||q||_1 m}{2m} \frac{\tau^2}{F(\tau)} : \tau > 0, F(\tau) > 0 \right\}$ , the problem (3.6) admits at least two non-trivial elements  $t \in [0, \infty)$ problem (3.6) admits at least two non-trivial classical solutions.

*Proof.* Fix  $\lambda > \inf\left\{\frac{2||q||_1^2 + ||q||_1 m}{2m} \frac{\tau^2}{F(\tau)} : \tau > 0, F(\tau) > 0\right\}$ , and let  $\tau$  be a positive constant such that  $F(\tau) > 0$  and  $\lambda > \frac{2||q||_1^2 + ||q||_1 m}{2m} \frac{\tau^2}{F(\tau)}$ . Since  $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$ , there is a positive constant  $\nu_1$  with  $\nu_1 < \sqrt{\frac{2}{m}} ||q||_1 \tau$ 

such that  $\frac{F(\nu_1)}{\nu_1^2} < \frac{m}{4\lambda}$ , and since  $\lim_{t\to+\infty} \frac{f(t)}{t} = 0$ , there is a positive constant  $\kappa$  with  $\sqrt{\frac{2}{m}||q||_1}\tau < \frac{1}{\sqrt{2}}\kappa$  such that  $\frac{F(\kappa)}{\kappa^2} < \frac{m}{8\lambda}$ . Therefore, Theorem 3.5 follows the conclusion.

We end this section by giving the following result which provides the existence of at least three weak solutions for the system (1.1).

Put  $m'_i = \min\left\{\min_{[0,1]} e^{-R_i} p_i, \min_{[0,1]} e^{-R_i} q_i\right\}$  where  $R_i$  is a primitive of  $\frac{r_i}{p_i}$  for  $1 \le i \le n$ , and put  $\underline{m}' = \min\{m'_i; 1 \le i \le n\}$ . Then, we have the following result.

**Theorem 3.7.** Let  $F: [0,1] \times \mathbb{R}^n \to \mathbb{R}$  satisfy the conditions  $F(x,t_1,...,t_n) \geq 0$  for all  $(x,t_1,...,t_n) \in [0,1] \times (\mathbb{R} \cup \{0\})^n$  and F(x,0,...,0) = 0 for every  $x \in [0,1]$ . Assume that there exist four positive constants  $\nu_1$ ,  $\nu_2$ ,  $\eta$  and  $\tau$  with  $\nu_1 < n\sqrt{\frac{2}{m'}}||e^{-R_n}q_n||_1 \tau < \nu_2 < \eta$  where  $||e^{-R_n}q_n||_1 = \int_0^1 e^{-R_n(x)}q_n(x)dx$  such that

(A9)

$$\max\left\{a(\nu_{1}), a(\nu_{2}), \frac{\eta^{2}}{\eta^{2} - \nu_{2}^{2}}a(\eta)\right\}$$

$$<\underline{m}'\frac{\int_{0}^{1}F(x,0,...,0,\tau)dx-\int_{0}^{1}\sup_{(t_{1},...,t_{n})\in K(\nu_{1})}F(x,t_{1},...,t_{n})dx}{2n^{2}||e^{-R_{n}}q_{n}||_{1}\tau^{2}}$$

Then, for each

$$\lambda \in \left[\frac{\frac{1}{2}||e^{-R_n}q_n||_1\tau^2}{\max\{||e^{-R_i}||_1; \ 1 \le i \le n\}} \times \frac{1}{\left(\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x,t_1,\dots,t_n)dx - \int_0^1 F(x,0,\dots,0,\tau)dx\right)}\right]$$

$$(t_1, \dots, t_n) = \sum_{i=1}^{n} \frac{1}{4n^2 \min\{||e^{-R_i}||_1; \ 1 \le i \le n\}} \min\left\{\frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2 - \nu_2^2}{\eta^2} \frac{1}{a(\eta)}\right\} = \sum_{i=1}^{n} \frac{1}{a(\nu_1)} \sum_{$$

the system (1.1) admits at least three weak solutions  $v^j = (v_1^j, \ldots, v_n^j) \in X$  (j = 1, 2, 3) such that

$$\sum_{i=1}^{n} |v_i^j(x)| \le \eta \text{ for each } x \in [a, b], \ j = 1, 2, 3.$$

*Proof.* Taking into account that the solutions of the system

$$\begin{cases} -(e^{-R_i}p_i(x)u'_i(x))' + e^{-R_i}q_i(x)u_i(x) = \lambda e^{-R_i}F_{u_i}(x, u_1, ..., u_n) \\ x \in (0, 1), \\ u'_i(0) = u'_i(1) = 0 \end{cases}$$

and the solutions of the system (1.1) coincide, Theorem 3.1 follows the conclusion. 

#### 4. Existence of a non-trivial solution

In this section, by the use of Theorem 2.2, we prove that under appropriate assumptions on F, the system (1.1) admits at least one non-trivial weak solution.

For given a nonnegative constant  $\nu$  and a positive constant  $\tau$  with  $\underline{m}(\frac{\nu}{n})^2 \neq 2\tau^2 ||q_n||_1$ , put

$$b_{\tau}(\nu) := \frac{\int_{0}^{1} \sup_{(t_1,\dots,t_n) \in K(\nu)} F(x,t_1,\dots,t_n) dx - \int_{0}^{1} F(x,0,\dots,0,\tau) dx}{\frac{\underline{m}}{2} (\frac{\nu}{n})^2 - \tau^2 ||q_n||_1}$$

where  $K(\nu) = \{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \nu\}$  (see (2.1)) and  $||q_n||_1 = \int_0^1 q_n(x) dx$ . We formulate the main result of this section as follows:

**Theorem 4.1.** Assume that there exist a non-negative constant  $\nu_1$  and two positive constants  $\nu_2$  and  $\tau$  with  $\nu_1 < n\sqrt{\frac{2}{\underline{m}}||q_n||_1}\tau < \nu_2$  such that

(B1) 
$$b_{\tau}(\nu_2) < b_{\tau}(\nu_1)$$
.

Then, for each  $\lambda \in \left] \frac{1}{2} \frac{1}{b_{\tau}(\nu_1)}, \frac{1}{2} \frac{1}{b_{\tau}(\nu_2)} \right[$  the system (3.1) admits at least one non-trivial weak solution  $u_0 = (u_{01}, \dots, u_{0n}) \in X$  such that  $\frac{m}{2} (\frac{\nu_1}{n})^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m}{2} (\frac{\nu_2}{n})^2$ .

*Proof.* In order to apply Theorem 2.2 to our problem, we take  $\Phi$  and  $\Psi$  as in the proof of Theorem 3.1. Set  $w(x) = (0, ..., 0, \tau)$ ,  $r_1 = \underline{m}(\frac{\nu_1}{2n})^2$ and  $r_2 = \underline{m}(\frac{\nu_2}{2n})^2$ . One has  $\Phi(w) = \frac{1}{2}||q_n||_1\tau^2$ . So, bearing the condition  $\nu_1 < n \sqrt{\frac{2}{m} ||q_n||_1} \tau^2 < \nu_2$  in mind, we get

$$r_1 < \Phi(w) < r_2.$$

From the definition of  $\Phi$  and recalling (3.2), we see that

$$\begin{split} \Phi^{-1}(] - \infty, r_2[) &= \{ u = (u_1, ..., u_n) \in X; \Phi(u) < r_2 \} \\ &= \left\{ u \in X; \sum_{i=1}^n ||u_i||^2 < 2r_2 \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)|^2 < \frac{\nu_2^2}{n^2} \text{ for all } x \in [0, 1] \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)| \le \nu_2 \text{ for all } x \in [0, 1] \right\}, \end{split}$$
hich follows
$$\begin{aligned} \sup_{(u_1, ..., u_n) \in \Phi^{-1}(] - \infty, r_2[)} \Psi(u) \end{aligned}$$

which follows

$$\sup_{\substack{(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)\\(u_1,...,u_n)\in\Phi^{-1}(]-\infty,r_2[)}} \frac{\Psi(u)}{\int_0^1 F(x,u_1(x),...,u_n(x))dx}$$
  
$$\leq \int_0^1 \sup_{\substack{(t_1,...,t_n)\in K(\nu_2)\\(t_1,...,t_n)\in K(\nu_2)}} F(x,t_1,...,t_n)dx.$$

So, one has

$$\delta(r_1, r_2) \leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ \leq \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_2)} F(x, t_1, \dots, t_n) dx - \Psi(w)}{r_2 - \Phi(w)} \\ \leq 2b_{\tau}(\nu_2).$$

On the other hand, by similar reasoning as before, one has

$$\rho(r_1, r_2) \geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(w) - r_1} \\
\geq \frac{\Psi(w) - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\Phi(w) - r_1} \\
\geq 2b_{\tau}(\nu_1).$$

Hence, from Assumption (B1), one has  $\delta(r_1, r_2) < \rho(r_1, r_2)$ . Therefore, applying Theorem 2.2, taking into account that the weak solution of the system (3.1) are exactly the solution of the equation  $\Phi'(u) - \lambda \Psi'(u) = 0$ , we have the conclusion. 

Now we point out the following consequence of Theorem 4.1.

**Theorem 4.2.** Assume that there exist two positive constants  $\nu$  and  $\tau$  with  $n\sqrt{\frac{2}{m}||q_n||_1}\tau < \nu$  such that

$$\begin{array}{l} (B2) \ \frac{\int_{0}^{1} \sup_{(t_{1},...,t_{n})\in K(\nu)} F(x,t_{1},...,t_{n})dx}{\nu^{2}} < \frac{m}{2n^{2}||q_{n}||_{1}} \frac{\int_{0}^{1} F(x,0,...,0,\tau)dx}{\tau^{2}}; \\ (B3) \ F(x,0,...,0) = 0 \ for \ every \ x \in [0,1]. \end{array}$$

Then, for each

$$\lambda \in \left] \frac{\frac{1}{2} ||q_n||_1 \tau^2}{\int_0^1 F(x, 0, ..., 0, \tau) dx}, \ \frac{\underline{m}(\frac{\nu}{2n})^2}{\int_0^1 \sup_{(t_1, ..., t_n) \in K(\nu)} F(x, t_1, ..., t_n) dx} \right[$$

the system (3.1) admits at least one non-trivial weak solution  $u_0 = (u_{01}, \ldots, u_{0n}) \in X$  such that  $\sum_{i=1}^{n} ||u_i||_{\infty} < \nu$ .

*Proof.* By the use of Theorem 4.1 and taking  $\nu_1 = 0$ ,  $\nu_2 = \nu$  we get the conclusion. Indeed, owing to our assumptions, one has

$$b_{\tau}(\nu_{2}) < \frac{\left(1 - \frac{2n^{2}||q_{n}||_{1}\tau^{2}}{\underline{m}\nu^{2}}\right)\int_{0}^{1}\sup_{(t_{1},...,t_{n})\in K(\nu)}F(x,t_{1},...,t_{n})dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^{2} - \tau^{2}||q_{n}||_{1}}$$

$$= \frac{\left(1 - \frac{||q_{n}||_{1}\tau^{2}}{\underline{m}(\frac{\nu}{n})^{2}}\right)\int_{0}^{1}\sup_{(t_{1},...,t_{n})\in K(\nu)}F(x,t_{1},...,t_{n})dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^{2} - \tau^{2}||q_{n}||_{1}}$$

$$= \frac{\int_{0}^{1}\sup_{(t_{1},...,t_{n})\in K(\nu)}F(x,t_{1},...,t_{n})dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^{2}}.$$

On the other hand, taking Assumption (B3) into account, one has

$$\frac{\int_0^1 F(x,0,...,0,\tau)dx}{||q_n||_1\tau^2} = b_\tau(\nu_1).$$

Moreover, taking (3.2) into account,  $\sum_{i=1}^{n} ||u_i||_{\infty} < \nu$  whenever  $\Phi(u) < r_2$ . Now, owing to assumption (B2), it is sufficient to invoke Theorem 4.1 for concluding the proof.

Here we want to point out a simple version of Theorem 4.1 when n = 1.

Let  $p_1 = p$ ,  $q_1 = q$  and  $m_1 = m$ . Let  $f : [0,1] \times \mathbb{R} \to \mathbb{R}$  be an  $L^1$ -Carathéodory function. Let F be the function defined by F(x,t) =

 $\int_0^t f(x,s)ds$  for each  $(x,t) \in [0,1] \times \mathbb{R}$ . For given nonnegative constant  $\nu$  and a positive constant  $\tau$  with  $m\nu^2 \neq 2\tau^2 ||q||_1$ , put

$$c_{\tau}(\nu) := \frac{\int_0^1 \sup_{|t| \le \nu} F(x, t) dx - \int_0^1 F(x, \tau) dx}{\frac{m}{2}\nu^2 - \tau^2 ||q||_1}$$

We have the following result.

**Theorem 4.3.** Assume that there exist a non-negative constant  $\nu_1$  and two positive constants  $\nu_2$  and  $\tau$  with  $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \nu_2$  such that

$$(B_4) c_{\tau}(\nu_2) < c_{\tau}(\nu_1).$$

Then, for each  $\lambda \in \left] \frac{1}{2} \frac{1}{c_{\tau}(\nu_1)}, \frac{1}{2} \frac{1}{c_{\tau}(\nu_2)} \right[$  the problem (3.5) admits at least one non-trivial weak solution  $u_0 \in W^{1,2}([0,1])$  such that

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m}{2}\nu_2^2.$$

The last result gives the existence of at least one non-trivial weak solution in  $W^{1,2}([0,1])$  to the problem (3.5) in the autonomous case.

Let  $f : \mathbb{R} \to \mathbb{R}$  be a continuous function, and put  $F(t) = \int_0^t f(\xi) d\xi$  for all  $t \in \mathbb{R}$ . We have the following result as a direct consequence of Theorem 4.3.

**Theorem 4.4.** Assume that there exist a non-negative constant  $\nu_1$  and two positive constants  $\nu_2$  and  $\tau$  with  $\nu_1 < \sqrt{\frac{2}{m}||q||_1}\tau < \nu_2$  such that

 $\begin{array}{l} (B5) \ f(t) \geq 0 \ for \ each \ t \in [-\nu_2, \ \max\{\nu_2, \tau\}]; \\ (B6) \ \frac{F(\nu_2) - F(\tau)}{\frac{m}{2} (\frac{\nu}{n})^2 - \tau^2 ||q||_1} < \frac{F(\nu_1) - F(\tau)}{\frac{m}{2} (\frac{\nu}{n})^2 - \tau^2 ||q||_1}. \end{array}$ 

Then, for each  $\lambda \in \left] \frac{1}{2} \frac{\frac{m}{2} (\frac{\nu}{n})^2 - \tau^2 ||q||_1}{F(\nu_1) - F(\tau)}, \ \frac{1}{2} \frac{\frac{m}{2} (\frac{\nu}{n})^2 - \tau^2 ||q||_1}{F(\nu_2) - F(\tau)} \right[$  the problem (3.6) admits at least one non-trivial classical solution  $u_0 \in W^{1,2}([0,1])$  such that

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m}{2}\nu_2^2.$$

As an example, we point out the following special case of the main result.

**Theorem 4.5.** Let  $g : \mathbb{R} \to \mathbb{R}$  be a nonnegative function such that  $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$ .

Then, for each 
$$\lambda \in \left]0, \ \frac{m}{4} \sup_{\nu>0} \frac{\nu^2}{\int_0^{\nu} g(\xi) d\xi} \right[$$
, the problem  
 $\left\{ \begin{array}{l} -(p(x)u'(x))' + q(x)u'(x) = \lambda g(u) \quad x \in (0,1), \\ u'(0) = u'(1) = 0 \end{array} \right.$ 

admits at least one non-trivial classical solution in  $W^{1,2}([0,1])$ .

*Proof.* For fixed  $\lambda$  as in the conclusion, there exists a positive constant  $\nu$  such that

$$\lambda < \frac{m}{4} \frac{\nu^2}{\int_0^\nu g(\xi) d\xi}$$

Moreover, the condition  $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$  implies  $\lim_{t\to 0^+} \frac{\int_0^t g(\xi)d\xi}{t^2} = +\infty$ . Therefore, a positive constant  $\tau$  satisfying  $\tau < \frac{\nu}{\sqrt{2}||q||_1}$  can be

chosen such that

$$\frac{1}{\lambda}(\frac{||q||_1}{2}) < \frac{\int_0^{\tau} g(\xi) d\xi}{\tau^2}$$

Hence, the conclusion follows from Theorem 4.4 with  $\nu_1 = 0$ ,  $\nu_2 = \nu$  and f(t) = g(t) for every  $t \in \mathbb{R}$ .

**Remark 4.6.** For fixed  $\rho$  put  $\lambda_{\rho} := \frac{m}{4} \sup_{\nu \in ]0,\rho[} \frac{\nu^2}{\int_0^{\nu} g(\xi)d\xi}$ . The result of Theorem 4.5 for every  $\lambda \in ]0, \lambda_{\rho}[$  holds with  $|u_0(x)| < \rho$  for all  $x \in [0, 1]$  where  $u_0$  is the ensured non-trivial classical solution in  $W^{1,2}([0,1])$  (see [7, Remark 4.3]).

We present the following example to illustrate the result.

Example 1. Consider the problem

(4.1) 
$$\begin{cases} -(e^x u')' + e^x u = \lambda (1 + e^{-u^+} (u^+)^2 (3 - u^+)) & x \in (0, 1), \\ u'(0) = u'(1) = 0. \end{cases}$$

where  $u^{+} = \max\{u, 0\}$ . Let

$$g(t) = 1 + e^{-t^+} (t^+)^2 (3 - t^+)$$

for all  $t \in \mathbb{R}$  where  $t^+ = \max\{t, 0\}$ . It is clear that  $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$ . Hence, by applying Theorem 4.5, since m = 1, for every  $\lambda \in ]0, \frac{e}{4(1+e)}[$ the problem (4.1) has at least one non-trivial classical solution  $u_0 \in W^{1,2}([0,1])$  such that  $||u_0||_{\infty} < 1$ . Finally, we give the following existence property of the system (1.1).

Put  $m'_i = \min\left\{\min_{[0,1]} e^{-R_i} p_i, \min_{[0,1]} e^{-R_i} q_i\right\}$  where  $R_i$  is a primitive of  $\frac{r_i}{p_i}$  for  $1 \le i \le n$ , and put  $\underline{m}' = \min\{m'_i; 1 \le i \le n\}$ . For given a nonnegative constant  $\nu$  and a positive constant  $\tau$  with

 $\underline{m}'(\frac{\nu}{n})^2 \neq 2\tau^2 ||q_n||_1$ , put

$$d_{\tau}(\nu) := \frac{\int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu)} F(x,t_1,\dots,t_n) dx - \int_0^1 F(x,0,\dots,0,\tau) dx}{\frac{m'}{2} (\frac{\nu}{\pi})^2 - \tau^2 ||e^{-R_n}q_n||_1}$$

where  $K(\nu) = \{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \nu\}$  (see (2.1)) and  $||e^{-R_n}q_n||_1 = \int_0^1 e^{-R_n(x)}q_n(x)dx$ . Then, we have the following result.

**Theorem 4.7.** Assume that there exist a non-negative constant  $\nu_1$  and two positive constants  $\nu_2$  and  $\tau$  with  $\nu_1 < n\sqrt{\frac{2}{m'}}||e^{-R_n}q_n||_1\tau < \nu_2$  such that

(*B*7) 
$$d_{\tau}(\nu_2) < d_{\tau}(\nu_1)$$
.

Then, for each

 $\lambda \in \left] \frac{1}{2\max\{||e^{-R_i}||_1; \ 1 \le i \le n\}} \frac{1}{d_{\tau}(\nu_1)}, \ \frac{1}{2\min\{||e^{-R_i}||_1; \ 1 \le i \le n\}} \frac{1}{d_{\tau}(\nu_2)} \right[ the system (1.1) admits at least one non-trivial weak solution <math>u_0 = (u_{01}, \dots, u_{0n}) \in U_{01}$ X such that  $\frac{m'}{2}(\frac{\nu_1}{n})^2 < \sum_{i=1}^n ||u_{0i}||^2 < \frac{m'}{2}(\frac{\nu_2}{n})^2$ .

Proof. By the same reasoning as in the proof of Theorem 3.7, Theorem 4.1 follows the conclusion. 

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