

ON A CLASS OF SYSTEMS OF n NEUMANN
TWO-POINT BOUNDARY VALUE STURM-LIOUVILLE
TYPE EQUATIONS

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ABSTRACT. Employing a three critical points theorem, we prove the existence of multiple solutions for a class of Neumann two-point boundary value Sturm-Liouville type equations. Using a local minimum theorem for differentiable functionals the existence of at least one non-trivial solution is also ensured.

1. Introduction

Consider the following Neumann two-point boundary value Sturm-Liouville type system

$$(1.1) \quad \begin{cases} -(p_i(x)u_i'(x))' + r_i(x)u_i'(x) + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, \dots, u_n) \\ x \in (0, 1), \\ u_i'(0) = u_i'(1) = 0 \end{cases}$$

for $1 \leq i \leq n$, where $n \geq 1$ is an integer, $p_i \in C^1([0, 1])$, $r_i, q_i \in C^0([0, 1])$ with p_i and q_i positive functions for $1 \leq i \leq n$, λ is a positive parameter, $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $F(., t_1, \dots, t_n)$ is measurable in

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$[0, 1]$ for all $(t_1, \dots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \dots, \cdot)$ is C^1 in \mathbb{R}^n for every $x \in [0, 1]$ and for every $\varrho > 0$,

$$\sup_{|(t_1, \dots, t_n)| \leq \varrho} \sum_{i=1}^n |F_{t_i}(x, t_1, \dots, t_n)| \in L^1([0, 1]),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \leq i \leq n$.

Based on a three critical point theorem (Theorem 2.1), we establish the existence of at least three weak solutions for the system (1.1) under suitable assumptions on nonlinear term F . Employing a local minimum theorem for differentiable functionals (Theorem 2.2), under appropriate hypotheses on F , we also ensure the existence of at least one non-trivial solution.

Problems of Sturm-Liouville type have been widely investigated. We refer the reader to the papers [3, 5, 6, 18] and the references therein.

For a thorough account on the subject, we also refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In the present paper, our motivation comes from the recent paper [5].

2. Preliminaries and basic notations

Our main tools to prove the results are critical point theorems.

First we recall an immediate consequence of [4, Theorem 3.3] (see also [1, Theorem 5.2]).

Let X be a nonempty set and $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two functions. for all $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, $r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r])} \frac{\sup_{u \in \Phi^{-1}(-\infty, r])} \Psi(u) - \Psi(u)}{r - \Phi(u)}$$

$$\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(-\infty, r_1])} \sup_{v \in \Phi^{-1}([r_1, r_2])} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2+r_3])} \Psi(u)}{r_3}$$

and

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 2.1. [4, Theorem 3.3] *Let X be a reflexive real Banach space, $\Phi : X \rightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. for every u^* , u^{**} such that $\Psi(u^*) \geq 0$ and $\Psi(u^{**}) \geq 0$, one has

$$\inf_{s \in [0,1]} \Psi(su^* + (1-s)u^{**}) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Then, for each $\lambda \in \left] \frac{1}{\beta(r_1, r_2)}, \frac{1}{\alpha(r_1, r_2, r_3)} \right[$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points u, v, w such that $u \in \Phi^{-1}(]-\infty, r_1[)$, $v \in \Phi^{-1}(]r_1, r_2[)$ and $w \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Next, we recall a local minimum theorem for differentiable functionals, Theorem 2.5 of [17] as given in [2, Theorem 5.1] (see also [2, Proposition 2.1] for related results).

For a given non-empty set X , and two functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$, we define the following functions

$$\delta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.2. [2, Theorem 5.1] *Let X be a reflexive real Banach space, $\Phi : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\delta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left] \frac{1}{\rho(r_1, r_2)}, \frac{1}{\delta(r_1, r_2)} \right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u) \forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let us introduce some notations which will be used later.

Set $p_{0i} = \min_{[0,1]} p_i(x)$, $q_{0i} = \min_{[0,1]} q_i(x)$ and $m_i = \min\{p_{0i}, q_{0i}\}$ for $1 \leq i \leq n$, and set $\underline{m} = \min\{m_i; 1 \leq i \leq n\}$.

Here and in the sequel,

$$X := W^{1,2}([0, 1]) \times \dots \times W^{1,2}([0, 1]) = (W^{1,2}([0, 1]))^n$$

endowed with the norm

$$\|(u_1, \dots, u_n)\|_* = \sum_{i=1}^n \|u_i\|$$

where $\|u_i\| = \left(\int_0^1 (p_i(x)|u_i'(x)|^2 + q_i(x)|u_i(x)|^2) dx \right)^{\frac{1}{2}}$.

For all $\gamma > 0$ we denote by $K(\gamma)$ the set

$$(2.1) \quad \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \gamma \right\}.$$

3. Existence of three solutions

Consider the following Neumann two-point boundary value Sturm-Liouville type system

$$(3.1) \quad \begin{cases} -(p_i(x)u_i'(x))' + q_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, \dots, u_n) & x \in (0, 1), \\ u_i'(0) = u_i'(1) = 0 \end{cases}$$

for $1 \leq i \leq n$, where $n \geq 1$ is an integer, $p_i \in C^1([0, 1])$, $q_i \in C^0([0, 1])$ with p_i and q_i positive functions for $1 \leq i \leq n$, λ is a positive parameter, $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $F(\cdot, t_1, \dots, t_n)$ is measurable in $[0, 1]$ for all $(t_1, \dots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \dots, \cdot)$ is C^1 in \mathbb{R}^n for every $x \in [0, 1]$ and for every $\varrho > 0$,

$$\sup_{|(t_1, \dots, t_n)| \leq \varrho} \sum_{i=1}^n |F_{t_i}(x, t_1, \dots, t_n)| \in L^1([0, 1]).$$

We say that $u = (u_1, \dots, u_n)$ is a weak solution to the system (3.1) if $u = (u_1, \dots, u_n) \in X$ and

$$\begin{aligned} & \sum_{i=1}^n \int_0^1 (p_i(x)u_i'(x)v_i'(x) + q_i(x)u_i(x)v_i(x)) dx \\ & - \lambda \sum_{i=1}^n \int_0^1 F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x) dx = 0 \end{aligned}$$

for every $v = (v_1, \dots, v_n) \in X$.

For a given positive constant ν , put

$$a(\nu) := \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx}{\nu^2}$$

where $K(\nu) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \nu\}$ (see (2.1)).

We formulate the main result of this section as follows:

Theorem 3.1. *Let $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the conditions $F(x, t_1, \dots, t_n) \geq 0$ for all $(x, t_1, \dots, t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x, 0, \dots, 0) = 0$ for every $x \in [0, 1]$. Assume that there exist four positive constants ν_1, ν_2, η and τ with $\nu_1 < n\sqrt{\frac{2}{m}}\|q_n\|_1\tau < \nu_2 < \eta$ such that*

(A1)

$$\max \left\{ a(\nu_1), a(\nu_2), \frac{\eta^2}{\eta^2 - \nu_2^2} a(\eta) \right\} < \frac{\frac{m}{2n^2\|q_n\|_1} \int_0^1 F(x, 0, \dots, 0, \tau) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\tau^2}.$$

Then, for each

$$\lambda \in \left] \frac{\frac{1}{2}\|q_n\|_1\tau^2}{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx - \int_0^1 F(x, 0, \dots, 0, \tau) dx}, \frac{m}{4n^2} \min \left\{ \frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2 - \nu_2^2}{\eta^2} \frac{1}{a(\eta)} \right\} \right[$$

the system (3.1) admits at least three weak solutions $v^j = (v_1^j, \dots, v_n^j) \in X$ ($j = 1, 2, 3$) such that

$$\sum_{i=1}^n |v_i^j(x)| \leq \eta \text{ for each } x \in [a, b], j = 1, 2, 3.$$

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|^2}{2}$$

and

$$\Psi(u) = \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \sum_{i=1}^n \int_0^1 (p_i(x)u'_i(x)v'_i(x) + q_i(x)u_i(x)v_i(x)) dx$$

and

$$\Psi'(u)(v) = \sum_{i=1}^n \int_0^1 F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x) dx$$

for every $v = (v_1, \dots, v_n) \in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous and Φ' admits a continuous inverse on X^* , as well as, $\Psi' : X \rightarrow X^*$ is a compact operator. Obviously, Φ and Ψ satisfy condition 1. of Theorem 2.1. Moreover, since $F(x, t_1, \dots, t_n) \geq 0$ for all $(x, t_1, \dots, t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$, for every $u^*, u^{**} \in X$ with $\Psi(u^*) \geq 0$ and $\Psi(u^{**}) \geq 0$, one has

$$\inf_{s \in [0,1]} \Psi(su^* + (1-s)u^{**}) \geq 0.$$

Set $w(x) = (0, \dots, 0, \tau)$, $r_1 = \underline{m}(\frac{\nu_1}{2n})^2$, $r_2 = \underline{m}(\frac{\nu_2}{2n})^2$ and $r_3 = \underline{m}\frac{\eta^2 - \nu_2^2}{4n^2}$. One has $\Phi(w) = \frac{1}{2}\|q_n\|_1\tau^2$. So, bearing the condition $\nu_1 < n\sqrt{\frac{2}{\underline{m}}\|q_n\|_1}\tau^2 < \nu_2 < \eta$ in mind, we get $r_1 < \Phi(w) < r_2$ and $r_3 > 0$. Since for $1 \leq i \leq n$,

$$|u_i(x)| \leq \sqrt{\frac{2}{m_i}}\|u_i\| \quad \forall u_i \in W^{1,2}([0, 1]),$$

we obtain

$$(3.2) \quad \sup_{x \in [0,1]} \sum_{i=1}^n |u_i(x)|^2 \leq \sum_{i=1}^n \frac{2}{m_i} \|u_i\|^2 \leq \frac{2}{\underline{m}} \sum_{i=1}^n \|u_i\|^2$$

for each $u = (u_1, \dots, u_n) \in X$, and so from the definition of Φ , we see that

$$\begin{aligned} \Phi^{-1}(] - \infty, r_1]) &= \{u = (u_1, \dots, u_n) \in X; \Phi(u) < r_1\} \\ &= \left\{ u \in X; \sum_{i=1}^n \|u_i\|^2 < 2r_1 \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)|^2 < \frac{\nu_1^2}{n^2} \text{ for all } x \in [0, 1] \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)| \leq \nu_1 \text{ for all } x \in [0, 1] \right\}, \end{aligned}$$

which follows

$$\begin{aligned} &\sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_1])} \Psi(u) \\ &= \sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_1])} \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx \\ &\leq \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx. \end{aligned}$$

In a similar way, we get

$$\begin{aligned} &\sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_2])} \Psi(u) \\ &= \sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_2])} \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx \\ &\leq \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_2)} F(x, t_1, \dots, t_n) dx \end{aligned}$$

and

$$\begin{aligned} &\sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_2 + r_3])} \Psi(u) \\ &= \sup_{(u_1, \dots, u_n) \in \Phi^{-1}(] - \infty, r_2 + r_3])} \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx \\ &\leq \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\eta)} F(x, t_1, \dots, t_n) dx. \end{aligned}$$

So, taking into account that $0 \in \Phi^{-1}(]-\infty, r_1[)$, one has

$$\varphi(r_1) \leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{r_1} \leq \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\underline{m}(\frac{\nu_1}{2n})^2},$$

$$\varphi(r_2) \leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u)}{r_2} \leq \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\underline{m}(\frac{\nu_2}{2n})^2}$$

and

$$\begin{aligned} \gamma(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2+r_3[)} \Psi(u)}{r_3} \\ &\leq \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\eta)} F(x, t_1, \dots, t_n) dx}{\underline{m} \frac{n^2 - \nu_2^2}{4n^2}} \end{aligned}$$

On the other hand, for each $u \in \Phi^{-1}(]-\infty, r_1[)$, one has

$$\begin{aligned} \beta(r_1, r_2) &\geq \frac{\int_0^1 F(x, w_1(x), \dots, w_n(x)) dx - \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx}{\sum_{i=1}^n \frac{\|w_i\|^2}{2} - \sum_{i=1}^n \frac{\|u_i\|^2}{2}} \\ &\geq \frac{\int_0^1 F(x, w_1(x), \dots, w_n(x)) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\frac{1}{2} \|q_n\|_1 \tau^2} \\ &= \frac{2 \int_0^1 F(x, 0, \dots, 0, \tau) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\|q_n\|_1 \tau^2}. \end{aligned}$$

Thanks to assumption (A1) we observe that

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Therefore, taking into account that the weak solutions of the system (3.1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$, and recalling (3.2), Theorem 2.1 follows the conclusion. \square

Remark 3.2. *The weak solutions of the system (3.1) where F is a C^1 -function, by using standard methods, belong to $C^2([0, 1])$. Namely, in this case, the classical and the weak solutions of the system (3.1) coincide.*

Now, we point out the following direct consequence of Theorem 3.1.

Theorem 3.3. *Let $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the conditions $F(x, t_1, \dots, t_n) \geq 0$ for all $(x, t_1, \dots, t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x, 0, \dots, 0) = 0$ for every $x \in [0, 1]$. Assume that there exist three positive constants ν_1, κ and τ with $\nu_1 < n \sqrt{\frac{2}{\underline{m}}} \|q_n\|_1 \tau < \frac{1}{\sqrt{2}} \kappa$ such that*

$$(A2) \quad a(\nu_1) < \frac{\left(\frac{m}{2n^2\|q_n\|_1}\right)^2 \int_0^1 F(x,0,\dots,0,\tau)dx}{1 + \frac{m}{2n^2\|q_n\|_1} \tau^2},$$

$$(A3) \quad a(\kappa) < \frac{1}{2} \frac{\frac{m}{2n^2\|q_n\|_1} \int_0^1 F(x,0,\dots,0,\tau)dx}{1 + \frac{m}{2n^2\|q_n\|_1}}$$

Then, for each

$$\lambda \in \left] \frac{1 + \frac{m}{2n^2\|q_n\|_1}}{2n^2\|q_n\|_1} \frac{\frac{1}{2}\|q_n\|_1\tau^2}{\int_0^1 F(x,0,\dots,0,\tau)dx}, \frac{m}{4n^2} \min \left\{ \frac{1}{a(\nu_1)}, \frac{1}{2a(\kappa)} \right\} \right[$$

the system (3.1) admits at least three weak solutions $v^j = (v_1^j, \dots, v_n^j) \in X$ ($j = 1, 2, 3$) such that

$$\sum_{i=1}^n |v_i^j(x)| \leq \kappa \quad \text{for each } x \in [a, b], \quad j = 1, 2, 3.$$

Proof. Since $\frac{m}{2n^2\|q_n\|_1} < 1$, from (A2) one has

$$\begin{aligned} a(\nu_1) &= \frac{\int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\nu_1^2} \\ &< \frac{\int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\frac{m}{2n^2\|q_n\|_1} \nu_1^2} \\ &< \frac{\frac{m}{2n^2\|q_n\|_1} \int_0^1 F(x, 0, \dots, 0, \tau) dx}{1 + \frac{m}{2n^2\|q_n\|_1} \tau^2}. \end{aligned}$$

By choosing $\nu_2 = \frac{1}{\sqrt{2}}\kappa$ and $\eta = \kappa$, from (A3) one has

$$(3.3) \quad a(\nu_2) = a\left(\frac{1}{\sqrt{2}}\kappa\right) \leq 2a(\kappa) < \frac{\frac{m}{2n^2\|q_n\|_1} \int_0^1 F(x, 0, \dots, 0, \tau) dx}{1 + \frac{m}{2n^2\|q_n\|_1} \tau^2}$$

and

$$(3.4) \quad \frac{\eta^2}{\eta^2 - \nu_2^2} a(\eta) = 2a(\kappa) < \frac{\frac{m}{2n^2\|q_n\|_1} \int_0^1 F(x, 0, \dots, 0, \tau) dx}{1 + \frac{m}{2n^2\|q_n\|_1} \tau^2}.$$

Moreover, bearing the condition $\nu_1 < n\sqrt{\frac{2}{m}\|q_n\|_1}\tau$ in mind, from (A2) we deduce

$$\frac{m}{2n^2\|q_n\|_1} \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx - \int_0^1 \sup_{(t_1,\dots,t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\tau^2}$$

$$\begin{aligned}
&> \frac{m}{2n^2 \|q_n\|_1} \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx}{\tau^2} - \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\nu_1^2} \\
&> \left(\frac{m}{2n^2 \|q_n\|_1} - \frac{\left(\frac{m}{2n^2 \|q_n\|_1}\right)^2}{1 + \frac{m}{2n^2 \|q_n\|_1}} \right) \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx}{\tau^2} \\
&= \frac{\frac{m}{2n^2 \|q_n\|_1}}{1 + \frac{m}{2n^2 \|q_n\|_1}} \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx}{\tau^2}.
\end{aligned}$$

Hence, using again Assumptions (A2) and (A3), owing to (3.3) and (3.4), we observe that all assumptions of Theorem 3.1 are satisfied. So, by applying Theorem 3.1 we have the conclusion. \square

We here want to point out a simple version of Theorem 3.3 when $n = 1$.

Let $p_1 = p$, $q_1 = q$ and $m_1 = m$. Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be an L^1 -Carathéodory function. Let F be the function defined by $F(x, t) = \int_0^t f(x, s) ds$ for each $(x, t) \in [0, 1] \times \mathbb{R}$. Then, we have the following existence result.

Theorem 3.4. *Let the function f satisfies the condition $f(x, t) \geq 0$ for all $x \in [0, 1]$ and for all $t \geq 0$. Assume that there exist three positive constants ν_1 , κ and τ with $\nu_1 < \sqrt{\frac{2}{m} \|q\|_1} \tau < \frac{1}{\sqrt{2}} \kappa$ such that*

$$(A_4) \quad \frac{\int_0^1 \sup_{|t| \leq \nu_1} F(x, t) dx}{\nu_1^2} < \frac{\left(\frac{m}{2\|q\|_1}\right)^2}{1 + \frac{m}{2\|q\|_1}} \frac{\int_0^1 F(x, \tau) dx}{\tau^2};$$

$$(A_5) \quad \frac{\int_0^1 \sup_{|t| \leq \kappa} F(x, t) dx}{\kappa^2} < \frac{1}{2} \frac{\frac{m}{2\|q\|_1}}{1 + \frac{m}{2\|q\|_1}} \frac{\int_0^1 F(x, \tau) dx}{\tau^2}.$$

Then, for each

$$\lambda \in \left] \frac{1 + \frac{m}{2\|q\|_1}}{\frac{m}{2\|q\|_1}} \frac{\frac{1}{2} \|q\|_1 \tau^2}{\int_0^1 F(x, \tau) dx}, \frac{m}{4} \min \left\{ \frac{\nu_1^2}{\int_0^1 \sup_{|t| \leq \nu_1} F(x, t) dx}, \frac{\kappa^2}{2 \int_0^1 \sup_{|t| \leq \kappa} F(x, t) dx} \right\} \right[$$

the problem

$$(3.5) \quad \begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(x, u) & x \in (0, 1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three weak solutions $v^j \in W^{1,2}([0, 1])$ ($j = 1, 2, 3$) such that

$$|v^j(x)| \leq \kappa \quad \text{for each } x \in [a, b], \quad j = 1, 2, 3.$$

The following result gives the existence of at least three weak solutions in $W^{1,2}([0, 1])$ to the problem (3.5) in the autonomous case.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $f(t) \geq 0$ for every $t \geq 0$, and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following result as a straightforward consequence of Theorem 3.4.

Theorem 3.5. *Assume that there exist three positive constants ν_1, κ and τ with $\nu_1 < \sqrt{\frac{2}{m}}\|q\|_1\tau < \frac{1}{\sqrt{2}}\kappa$ such that*

(A6) $f(t) \geq 0$ for each $t \in [-\kappa, 0]$;

(A7) $\frac{F(\nu_1)}{\nu_1^2} < \frac{(\frac{m}{2\|q\|_1})^2 F(\tau)}{1 + \frac{m}{2\|q\|_1} \tau^2}$;

(A8) $\frac{F(\kappa)}{\kappa^2} < \frac{1}{2} \frac{(\frac{m}{2\|q\|_1})^2 F(\tau)}{1 + \frac{m}{2\|q\|_1} \tau^2}$.

Then, for each

$$\lambda \in \left] \frac{2\|q\|_1^2 + \|q\|_1 m}{2m} \frac{\tau^2}{F(\tau)}, \frac{m}{4} \min \left\{ \frac{\nu_1^2}{F(\nu_1)}, \frac{1}{2} \frac{\kappa^2}{F(\kappa)} \right\} \right[$$

the problem

$$(3.6) \quad \begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda f(u) & x \in (0, 1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions $v^j \in W^{1,2}([0, 1])$ ($j = 1, 2, 3$) such that

$$|v^j(x)| \leq \kappa \text{ for each } x \in [a, b], j = 1, 2, 3.$$

As an example of the results, the following consequence, ensures the existence of at least two non-trivial classical solutions to the problem (3.6).

Theorem 3.6. *Let $f : \mathbb{R} \rightarrow]0, +\infty[$ be a continuous function such that*

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t} = \lim_{t \rightarrow +\infty} \frac{f(t)}{t} = 0.$$

Then, for every $\lambda > \inf \left\{ \frac{2\|q\|_1^2 + \|q\|_1 m}{2m} \frac{\tau^2}{F(\tau)} : \tau > 0, F(\tau) > 0 \right\}$, the problem (3.6) admits at least two non-trivial classical solutions.

Proof. Fix $\lambda > \inf \left\{ \frac{2\|q\|_1^2 + \|q\|_1 m}{2m} \frac{\tau^2}{F(\tau)} : \tau > 0, F(\tau) > 0 \right\}$, and let τ be a positive constant such that $F(\tau) > 0$ and $\lambda > \frac{2\|q\|_1^2 + \|q\|_1 m}{2m} \frac{\tau^2}{F(\tau)}$. Since $\lim_{t \rightarrow 0^+} \frac{f(t)}{t} = 0$, there is a positive constant ν_1 with $\nu_1 < \sqrt{\frac{2}{m}}\|q\|_1\tau$

such that $\frac{F(\nu_1)}{\nu_1^2} < \frac{m}{4\lambda}$, and since $\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = 0$, there is a positive constant κ with $\sqrt{\frac{2}{m} \|q\|_1 \tau} < \frac{1}{\sqrt{2}} \kappa$ such that $\frac{F(\kappa)}{\kappa^2} < \frac{m}{8\lambda}$. Therefore, Theorem 3.5 follows the conclusion. \square

We end this section by giving the following result which provides the existence of at least three weak solutions for the system (1.1).

Put $m'_i = \min \left\{ \min_{[0,1]} e^{-R_i} p_i, \min_{[0,1]} e^{-R_i} q_i \right\}$ where R_i is a primitive of $\frac{r_i}{p_i}$ for $1 \leq i \leq n$, and put $\underline{m}' = \min \{m'_i; 1 \leq i \leq n\}$. Then, we have the following result.

Theorem 3.7. *Let $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy the conditions $F(x, t_1, \dots, t_n) \geq 0$ for all $(x, t_1, \dots, t_n) \in [0, 1] \times (\mathbb{R} \cup \{0\})^n$ and $F(x, 0, \dots, 0) = 0$ for every $x \in [0, 1]$. Assume that there exist four positive constants ν_1, ν_2, η and τ with $\nu_1 < n \sqrt{\frac{2}{m'} \|e^{-R_n} q_n\|_1 \tau} < \nu_2 < \eta$ where $\|e^{-R_n} q_n\|_1 = \int_0^1 e^{-R_n(x)} q_n(x) dx$ such that*

(A9)

$$\max \left\{ a(\nu_1), a(\nu_2), \frac{\eta^2}{\eta^2 - \nu_2^2} a(\eta) \right\} < \frac{\underline{m}' \int_0^1 F(x, 0, \dots, 0, \tau) dx - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{2n^2 \|e^{-R_n} q_n\|_1 \tau^2}.$$

Then, for each

$$\lambda \in \left] \frac{\frac{1}{2} \|e^{-R_n} q_n\|_1 \tau^2}{\max \{ \|e^{-R_i}\|_1; 1 \leq i \leq n \}} \right. \\ \times \frac{1}{\left(\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx - \int_0^1 F(x, 0, \dots, 0, \tau) dx \right)} \\ \left. , \frac{m}{4n^2 \min \{ \|e^{-R_i}\|_1; 1 \leq i \leq n \}} \min \left\{ \frac{1}{a(\nu_1)}, \frac{1}{a(\nu_2)}, \frac{\eta^2 - \nu_2^2}{\eta^2} \frac{1}{a(\eta)} \right\} \right[$$

the system (1.1) admits at least three weak solutions $v^j = (v_1^j, \dots, v_n^j) \in X$ ($j = 1, 2, 3$) such that

$$\sum_{i=1}^n |v_i^j(x)| \leq \eta \quad \text{for each } x \in [a, b], \quad j = 1, 2, 3.$$

Proof. Taking into account that the solutions of the system

$$\begin{cases} -(e^{-R_i} p_i(x) u_i'(x))' + e^{-R_i} q_i(x) u_i(x) = \lambda e^{-R_i} F_{u_i}(x, u_1, \dots, u_n) \\ x \in (0, 1), \\ u_i'(0) = u_i'(1) = 0 \end{cases}$$

and the solutions of the system (1.1) coincide, Theorem 3.1 follows the conclusion. \square

4. Existence of a non-trivial solution

In this section, by the use of Theorem 2.2, we prove that under appropriate assumptions on F , the system (1.1) admits at least one non-trivial weak solution.

For given a nonnegative constant ν and a positive constant τ with $\underline{m}(\frac{\nu}{n})^2 \neq 2\tau^2 \|q_n\|_1$, put

$$b_\tau(\nu) := \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx - \int_0^1 F(x, 0, \dots, 0, \tau) dx}{\frac{\underline{m}}{2}(\frac{\nu}{n})^2 - \tau^2 \|q_n\|_1}$$

where $K(\nu) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \nu\}$ (see (2.1)) and $\|q_n\|_1 = \int_0^1 q_n(x) dx$.

We formulate the main result of this section as follows:

Theorem 4.1. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < n\sqrt{\frac{2}{\underline{m}} \|q_n\|_1} \tau < \nu_2$ such that*

$$(B1) \quad b_\tau(\nu_2) < b_\tau(\nu_1).$$

Then, for each $\lambda \in]\frac{1}{2} \frac{1}{b_\tau(\nu_1)}, \frac{1}{2} \frac{1}{b_\tau(\nu_2)}[$ the system (3.1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\frac{\underline{m}}{2}(\frac{\nu_1}{n})^2 < \sum_{i=1}^n \|u_{0i}\|^2 < \frac{\underline{m}}{2}(\frac{\nu_2}{n})^2$.

Proof. In order to apply Theorem 2.2 to our problem, we take Φ and Ψ as in the proof of Theorem 3.1. Set $w(x) = (0, \dots, 0, \tau)$, $r_1 = \underline{m}(\frac{\nu_1}{2n})^2$ and $r_2 = \underline{m}(\frac{\nu_2}{2n})^2$. One has $\Phi(w) = \frac{1}{2} \|q_n\|_1 \tau^2$. So, bearing the condition $\nu_1 < n\sqrt{\frac{2}{\underline{m}} \|q_n\|_1} \tau < \nu_2$ in mind, we get

$$r_1 < \Phi(w) < r_2.$$

From the definition of Φ and recalling (3.2), we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_2]) &= \{u = (u_1, \dots, u_n) \in X; \Phi(u) < r_2\} \\ &= \left\{ u \in X; \sum_{i=1}^n \|u_i\|^2 < 2r_2 \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)|^2 < \frac{\nu_2^2}{n^2} \text{ for all } x \in [0, 1] \right\} \\ &\subseteq \left\{ u \in X; \sum_{i=1}^n |u_i(x)| \leq \nu_2 \text{ for all } x \in [0, 1] \right\},\end{aligned}$$

which follows

$$\begin{aligned}& \sup_{(u_1, \dots, u_n) \in \Phi^{-1}(]-\infty, r_2])} \Psi(u) \\ &= \sup_{(u_1, \dots, u_n) \in \Phi^{-1}(]-\infty, r_2])} \int_0^1 F(x, u_1(x), \dots, u_n(x)) dx \\ &\leq \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_2)} F(x, t_1, \dots, t_n) dx.\end{aligned}$$

So, one has

$$\begin{aligned}\delta(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2])} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\leq \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_2)} F(x, t_1, \dots, t_n) dx - \Psi(w)}{r_2 - \Phi(w)} \\ &\leq 2b_\tau(\nu_2).\end{aligned}$$

On the other hand, by similar reasoning as before, one has

$$\begin{aligned}\rho(r_1, r_2) &\geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(w) - r_1} \\ &\geq \frac{\Psi(w) - \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu_1)} F(x, t_1, \dots, t_n) dx}{\Phi(w) - r_1} \\ &\geq 2b_\tau(\nu_1).\end{aligned}$$

Hence, from Assumption (B1), one has $\delta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, applying Theorem 2.2, taking into account that the weak solution of the system (3.1) are exactly the solution of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we have the conclusion. \square

Now we point out the following consequence of Theorem 4.1.

Theorem 4.2. *Assume that there exist two positive constants ν and τ with $n\sqrt{\frac{2}{m}}\|q_n\|_1\tau < \nu$ such that*

- (B2) $\frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx}{\nu^2} < \frac{m}{2n^2\|q_n\|_1} \frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx}{\tau^2};$
- (B3) $F(x, 0, \dots, 0) = 0$ for every $x \in [0, 1]$.

Then, for each

$$\lambda \in \left[\frac{\frac{1}{2}\|q_n\|_1\tau^2}{\int_0^1 F(x, 0, \dots, 0, \tau) dx}, \frac{m\left(\frac{\nu}{2n}\right)^2}{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx} \right]$$

the system (3.1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\sum_{i=1}^n \|u_i\|_\infty < \nu$.

Proof. By the use of Theorem 4.1 and taking $\nu_1 = 0, \nu_2 = \nu$ we get the conclusion. Indeed, owing to our assumptions, one has

$$\begin{aligned} b_\tau(\nu_2) &< \frac{\left(1 - \frac{2n^2\|q_n\|_1\tau^2}{m\nu^2}\right) \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx}{\frac{m}{2}\left(\frac{\nu}{n}\right)^2 - \tau^2\|q_n\|_1} \\ &= \frac{\left(1 - \frac{\|q_n\|_1\tau^2}{\frac{m}{2}\left(\frac{\nu}{n}\right)^2}\right) \int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx}{\frac{m}{2}\left(\frac{\nu}{n}\right)^2 - \tau^2\|q_n\|_1} \\ &= \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx}{\frac{m}{2}\left(\frac{\nu}{n}\right)^2}. \end{aligned}$$

On the other hand, taking Assumption (B3) into account, one has

$$\frac{\int_0^1 F(x, 0, \dots, 0, \tau) dx}{\|q_n\|_1\tau^2} = b_\tau(\nu_1).$$

Moreover, taking (3.2) into account, $\sum_{i=1}^n \|u_i\|_\infty < \nu$ whenever $\Phi(u) < r_2$. Now, owing to assumption (B2), it is sufficient to invoke Theorem 4.1 for concluding the proof. \square

Here we want to point out a simple version of Theorem 4.1 when $n = 1$.

Let $p_1 = p, q_1 = q$ and $m_1 = m$. Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be an L^1 -Carathéodory function. Let F be the function defined by $F(x, t) =$

$\int_0^t f(x, s)ds$ for each $(x, t) \in [0, 1] \times \mathbb{R}$. For given nonnegative constant ν and a positive constant τ with $m\nu^2 \neq 2\tau^2\|q\|_1$, put

$$c_\tau(\nu) := \frac{\int_0^1 \sup_{|t| \leq \nu} F(x, t)dx - \int_0^1 F(x, \tau)dx}{\frac{m}{2}\nu^2 - \tau^2\|q\|_1}.$$

We have the following result.

Theorem 4.3. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < \sqrt{\frac{2}{m}\|q\|_1}\tau < \nu_2$ such that*

(B4) $c_\tau(\nu_2) < c_\tau(\nu_1)$.

Then, for each $\lambda \in \left] \frac{1}{2} \frac{1}{c_\tau(\nu_1)}, \frac{1}{2} \frac{1}{c_\tau(\nu_2)} \right[$ the problem (3.5) admits at least one non-trivial weak solution $u_0 \in W^{1,2}([0, 1])$ such that

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n \|u_{0i}\|^2 < \frac{m}{2}\nu_2^2.$$

The last result gives the existence of at least one non-trivial weak solution in $W^{1,2}([0, 1])$ to the problem (3.5) in the autonomous case.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following result as a direct consequence of Theorem 4.3.

Theorem 4.4. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < \sqrt{\frac{2}{m}\|q\|_1}\tau < \nu_2$ such that*

(B5) $f(t) \geq 0$ for each $t \in [-\nu_2, \max\{\nu_2, \tau\}]$;

(B6) $\frac{F(\nu_2) - F(\tau)}{\frac{m}{2}(\frac{\nu_2}{n})^2 - \tau^2\|q\|_1} < \frac{F(\nu_1) - F(\tau)}{\frac{m}{2}(\frac{\nu_1}{n})^2 - \tau^2\|q\|_1}$.

Then, for each $\lambda \in \left] \frac{1}{2} \frac{\frac{m}{2}(\frac{\nu_1}{n})^2 - \tau^2\|q\|_1}{F(\nu_1) - F(\tau)}, \frac{1}{2} \frac{\frac{m}{2}(\frac{\nu_2}{n})^2 - \tau^2\|q\|_1}{F(\nu_2) - F(\tau)} \right[$ the problem (3.6) admits at least one non-trivial classical solution $u_0 \in W^{1,2}([0, 1])$ such that

$$\frac{m}{2}\nu_1^2 < \sum_{i=1}^n \|u_{0i}\|^2 < \frac{m}{2}\nu_2^2.$$

As an example, we point out the following special case of the main result.

Theorem 4.5. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function such that $\lim_{t \rightarrow 0^+} \frac{g(t)}{t} = +\infty$.

Then, for each $\lambda \in]0, \frac{m}{4} \sup_{\nu > 0} \frac{\nu^2}{\int_0^\nu g(\xi) d\xi} [$, the problem

$$\begin{cases} -(p(x)u'(x))' + q(x)u'(x) = \lambda g(u) & x \in (0, 1), \\ u'(0) = u'(1) = 0 \end{cases}$$

admits at least one non-trivial classical solution in $W^{1,2}([0, 1])$.

Proof. For fixed λ as in the conclusion, there exists a positive constant ν such that

$$\lambda < \frac{m}{4} \frac{\nu^2}{\int_0^\nu g(\xi) d\xi}.$$

Moreover, the condition $\lim_{t \rightarrow 0^+} \frac{g(t)}{t} = +\infty$ implies $\lim_{t \rightarrow 0^+} \frac{\int_0^t g(\xi) d\xi}{t^2} = +\infty$. Therefore, a positive constant τ satisfying $\tau < \frac{\nu}{\sqrt{\frac{2}{m} \|q\|_1}}$ can be

chosen such that

$$\frac{1}{\lambda} \left(\frac{\|q\|_1}{2} \right) < \frac{\int_0^\tau g(\xi) d\xi}{\tau^2}.$$

Hence, the conclusion follows from Theorem 4.4 with $\nu_1 = 0$, $\nu_2 = \nu$ and $f(t) = g(t)$ for every $t \in \mathbb{R}$. \square

Remark 4.6. For fixed ρ put $\lambda_\rho := \frac{m}{4} \sup_{\nu \in]0, \rho[} \frac{\nu^2}{\int_0^\nu g(\xi) d\xi}$. The result of Theorem 4.5 for every $\lambda \in]0, \lambda_\rho[$ holds with $|u_0(x)| < \rho$ for all $x \in [0, 1]$ where u_0 is the ensured non-trivial classical solution in $W^{1,2}([0, 1])$ (see [7, Remark 4.3]).

We present the following example to illustrate the result.

Example 1. Consider the problem

$$(4.1) \quad \begin{cases} -(e^x u')' + e^x u = \lambda(1 + e^{-u^+} (u^+)^2 (3 - u^+)) & x \in (0, 1), \\ u'(0) = u'(1) = 0. \end{cases}$$

where $u^+ = \max\{u, 0\}$. Let

$$g(t) = 1 + e^{-t^+} (t^+)^2 (3 - t^+)$$

for all $t \in \mathbb{R}$ where $t^+ = \max\{t, 0\}$. It is clear that $\lim_{t \rightarrow 0^+} \frac{g(t)}{t} = +\infty$. Hence, by applying Theorem 4.5, since $m = 1$, for every $\lambda \in]0, \frac{e}{4(1+e)} [$ the problem (4.1) has at least one non-trivial classical solution $u_0 \in W^{1,2}([0, 1])$ such that $\|u_0\|_\infty < 1$.

Finally, we give the following existence property of the system (1.1).

Put $m'_i = \min \left\{ \min_{[0,1]} e^{-R_i} p_i, \min_{[0,1]} e^{-R_i} q_i \right\}$ where R_i is a primitive of $\frac{r_i}{p_i}$ for $1 \leq i \leq n$, and put $\underline{m}' = \min\{m'_i; 1 \leq i \leq n\}$.

For given a nonnegative constant ν and a positive constant τ with $\underline{m}'(\frac{\nu}{n})^2 \neq 2\tau^2 \|q_n\|_1$, put

$$d_\tau(\nu) := \frac{\int_0^1 \sup_{(t_1, \dots, t_n) \in K(\nu)} F(x, t_1, \dots, t_n) dx - \int_0^1 F(x, 0, \dots, 0, \tau) dx}{\frac{\underline{m}'}{2} \left(\frac{\nu}{n}\right)^2 - \tau^2 \|e^{-R_n} q_n\|_1}$$

where $K(\nu) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \nu\}$ (see (2.1)) and $\|e^{-R_n} q_n\|_1 = \int_0^1 e^{-R_n(x)} q_n(x) dx$.

Then, we have the following result.

Theorem 4.7. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < n \sqrt{\frac{2}{\underline{m}'}} \|e^{-R_n} q_n\|_1 \tau < \nu_2$ such that*

$$(B7) \quad d_\tau(\nu_2) < d_\tau(\nu_1).$$

Then, for each

$\lambda \in \left] \frac{1}{2 \max\{\|e^{-R_i}\|_1; 1 \leq i \leq n\}} \frac{1}{d_\tau(\nu_1)}, \frac{1}{2 \min\{\|e^{-R_i}\|_1; 1 \leq i \leq n\}} \frac{1}{d_\tau(\nu_2)} \right[$ the system (1.1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\frac{\underline{m}'}{2} \left(\frac{\nu_1}{n}\right)^2 < \sum_{i=1}^n \|u_{0i}\|^2 < \frac{\underline{m}'}{2} \left(\frac{\nu_2}{n}\right)^2$.

Proof. By the same reasoning as in the proof of Theorem 3.7, Theorem 4.1 follows the conclusion. \square

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