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SOME COMBINATORIAL ASPECTS OF FINITE HAMILTONIAN GROUPS

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Abstract. In this paper we provide explicit formulas for the number of elements/subgroups/cyclic subgroups of a given order and for the total number of subgroups/cyclic subgroups in a finite Hamiltonian group. The coverings with three proper subgroups and the principal series of such a group are also counted. Finally, we give a complete description of the lattice of characteristic subgroups of a finite Hamiltonian group.

1. Introduction

One of the most important family of finite groups is the finite Hamiltonian groups, that is finite non-Abelian groups all of whose subgroups are normal. The structure of such a group H is well-known: it can be written as the direct product of the quaternion group $Q_8 = \langle x, y \mid x^4 = y^4 = 1, yxy^{-1} = x^{-1}\rangle$, an elementary Abelian 2-group and a finite Abelian group A of odd order, i.e.,

$$H \cong Q_8 \times \mathbb{Z}_2^n \times A. \tag{1.1}$$

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Since $Q_8 \times \mathbb{Z}_2^n$ and A have relatively prime orders, this leads to a similar decomposition of the subgroup lattice $L(H)$ of H, namely

$$L(H) \cong L(Q_8 \times \mathbb{Z}_2^n) \times L(A). \tag{1.2}$$

According to the above direct decompositions, many combinatorial problems related to Abelian groups can naturally be extended to Hamiltonian groups. Their study is the main goal of our paper.

The paper is organized as follows. In Section 2 we study the subgroup lattice $L(H)$ of a finite Hamiltonian group H. By using (1.2), we determine the number of subgroups/cyclic subgroups of a given order and the total number of subgroups/cyclic subgroups of H. We also count the coverings with three proper subgroups and the principal series of H. Section 3 deals with the characteristic subgroups of finite Hamiltonian groups. We give a complete description of the automorphism group $\text{Aut}(H)$ and of the characteristic subgroup lattice $\text{Char}(H)$ of H. In particular, their cardinality will be explicitly found. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will not be repeated here. Basic definitions and results on group theory can be found in [8] and [14]. The similar combinatorial aspects for finite Abelian groups have been investigated in [4, 5, 6], as well as in our previous papers [2], [15] and [17, 18]. For subgroup lattice notions we refer the reader to [13] and [16].

2. The lattice of subgroups of a finite Hamiltonian group

First of all, we determine explicitly the number of elements of a given order in a finite Hamiltonian group. It can be computed similarly as it was done in corollary 4.5 of [17].

Theorem 2.1. *Let $H = Q_8 \times \mathbb{Z}_2^n \times A$ be a finite Hamiltonian group. Then, for every divisor d of $\mid A \mid$, H possesses:*

a) *$e_d(A)$ elements of order d,*
b) *$(2^{n+1} - 1)\, e_d(A)$ elements of order $2d$,*
c) *$3 \cdot 2^{n+1} e_d(A)$ elements of order $4d$,*

where $e_d(A)$ denotes the number of elements of order d in A.

Proof. Fix a divisor d of $\mid A \mid$. Since d is odd, the elements of order d of $H = Q_8 \times \mathbb{Z}_2^n \times A$ are of the form $(1, \hat{0}, a)$ (where 1 denotes the identity of Q_8) with $o(a) = d$ in A. This proves a). An element $(x, y, a) \in H$

is of order $2d$ if and only if $o(x,y)=2$ in $Q_8\times\mathbb{Z}_2^n$ and $o(a)=d$ in A. We can directly see that the number of elements of order 2 in $Q_8\times\mathbb{Z}_2^n$ is $2^{n+1}-1$, proving b). We remark that $o(x,y,a)=4d$ if and only if $o(x)=4$ in Q_8, $y\in\mathbb{Z}_2^n$ is arbitrary and $o(a)=d$ in A. Clearly, c) follows immediately because Q_8 possesses six elements of order 4. □

Remark that the above theorem allows us to compute the number of cyclic subgroups of a given order d in H. This is closely connected with the number $cs_d(A)$ of cyclic subgroups of order d in A, computed in Corollary 4.5 of [17].

Corollary 2.2. *Under the notation of Theorem* 2.1*, for every divisor d of* $\mid A\mid$*, H possesses:*

a) $cs_d(A)$ *cyclic subgroups of order d,*
b) $(2^{n+1}-1)\,cs_d(A)$ *cyclic subgroups of order* $2d$*,*
c) $3\cdot 2^n cs_d(A)$ *cyclic subgroups of order* $4d$*.*

Obviously, the total number of cyclic subgroups of H can be obtained from Corollary 2.2.

Theorem 2.3. *The total number of cyclic subgroups of the finite Hamiltonian group* $H=Q_8\times\mathbb{Z}_2^n\times A$ *is given by the equality*

$$cs(H)=5\cdot 2^n\,cs(A),$$

where $cs(A)$ *denotes the total number of cyclic subgroups of A.*

Next, we will focus on arbitrary subgroups of a finite Hamiltonian group H. The main ingredient that will be used is the direct decomposition (1.2), which gives a powerful connection between $L(H)$ and $L(A)$. The subgroups of a finite Abelian group have been exhaustively studied by Birkhoff [4]. Since then, many methods to compute their number have been developed (for example, see [5] or [6]). One of them is based on an arithmetical description of the subgroup lattice of a finite Abelian group and produces explicit results in several particular cases (see [17]). We recall here only the precise formulas for the number of subgroups of order 2^k, $k=1,2,...,n$, in the elementary Abelian 2-group $\mathbb{Z}_2^n$

$$a_{n,2}(k)=\sum_{1\le i_1<i_2<...<i_k\le n}2^{\,i_1+i_2+...+i_k-\frac{k(k+1)}{2}},$$

respectively for the total number of subgroups of $\mathbb{Z}_2^n$

$$a_{n,2}=\sum_{k=0}^{n}a_{n,2}(k)=1+\sum_{k=1}^{n}\sum_{1\le i_1<i_2<...<i_k\le n}2^{\,i_1+i_2+...+i_k-\frac{k(k+1)}{2}}.$$

Also, we notice that the subgroups of the direct product $Q_8 \times \mathbb{Z}_2^n$ can be easily counted by using (4.19) of [14], I.

Lemma 2.4. *For every $k = 0, 1, ..., n+3$, the number of subgroups of order 2^k of $Q_8 \times \mathbb{Z}_2^n$ is*

$$b_{n,2}(k) = 2^k a_{n,2}(k) + 2^{2k-2} a_{n,2}(k-1) + 3 \cdot 2^{k-2} a_{n,2}(k-2) + a_{n,2}(k-3),$$

where, by convention, we have $a_{n,2}(r) = 0$ if $r < 0$ or $r > n$. In particular, the total number of subgroups of $Q_8 \times \mathbb{Z}_2^n$ is

$$b_{n,2} = 2^{n+2} + 1 + 8 \sum_{i=0}^{n-2} (2^{n-i} - 2^{2i+1} + 2^i) a_{i,2} + 2^{n+2} a_{n-1,2} + a_{n,2}.$$

Proof. From (4.19) of [14], I, we know that a subgroup K of $Q_8 \times \mathbb{Z}_2^n$ is uniquely determined by two subgroups $K_1 \subseteq K_1'$ of Q_8, two subgroups $K_2 \subseteq K_2'$ of $\mathbb{Z}_2^n$ and a group isomorphism $\varphi : K_1'/K_1 \longrightarrow K_2'/K_2$ (more exactly, $K = \{(x_1, x_2) \in K_1' \times K_2' \mid \varphi(x_1 K_1) = x_2 + K_2\}$). Moreover, we have

$$\mid K \mid = \mid K_1 \mid\mid K_2' \mid = \mid K_1' \mid\mid K_2 \mid .$$

By asking that $\mid K \mid = 2^k$, $k \in \{0, 1, ..., n+3\}$, we distinguish the following four cases.

Case 1. $\mid K_1 \mid = 1$.
In this case $\mid K_2' \mid = 2^k$ and so K_2' can be chosen in $a_{n,2}(k)$ ways. If $\mid K_1' \mid = 1$, then both K_1' and φ are trivial, and $K_2 = K_2'$ can be chosen in $a_{n,2}(k)$ ways. These determine $a_{n,2}(k)$ distinct subgroups K. If $\mid K_1' \mid = 2$, then K_1' can be chosen in a unique way, φ is the identity map and K_2 can be chosen in $2^k - 1$ ways because its order is 2^{k-1}. These determine $(2^k - 1) a_{n,2}(k)$ distinct subgroups K. If $\mid K_1' \mid = 4$ or $\mid K_1' \mid = 8$, then there is no way to choose K_2 and K_2'. Hence in this case one obtains

$$a_{n,2}(k) + (2^k - 1) a_{n,2}(k) = 2^k a_{n,2}(k)$$

distinct subgroups of $Q_8 \times \mathbb{Z}_2^n$.

Case 2. $\mid K_1 \mid = 2$.
In this case K_1 can uniquely be chosen and K_2' can be chosen in $a_{n,2}(k-1)$ ways because its order is 2^{k-1}. If $\mid K_1' \mid = 2$, then $K_1' = K_1$, φ is trivial and $K_2 = K_2'$ can be chosen in $a_{n,2}(k-1)$ ways. These determine $a_{n,2}(k-1)$ distinct subgroups K. If $\mid K_1' \mid = 4$, then K_1' can be one of the (three) cyclic subgroups of order 4 in Q_8, φ is the identity map and K_2 can be chosen in $2^{k-1} - 1$ ways because its order is 2^{k-2}. These determine

$3(2^{k-1}-1)a_{n,2}(k-1)$ distinct subgroups K. If $\mid K_1' \mid = 8$, then $K_1' = Q_8$, φ can be chosen in 6 ways ($\mathbb{Z}_2 \times \mathbb{Z}_2$ has six automorphisms) and K_2 can be chosen in $a_{k-1,2}(k-3) = \frac{1}{3}\,(2^{2k-3} - 3 \cdot 2^{k-2} + 1)$ ways (it is a subgroup of order 2^{k-3} of $K_2' \cong \mathbb{Z}_2^{k-1}$). These determine $(2^{2k-2} - 3 \cdot 2^{k-1} + 2)a_{n,2}(k-1)$ distinct subgroups K. Hence in this case one obtains

$$a_{n,2}(k-1) + 3(2^{k-1}-1)a_{n,2}(k-1) + (2^{2k-2} - 3 \cdot 2^{k-1} + 2)a_{n,2}(k-1) =$$

$$= 2^{2k-2}a_{n,2}(k-1)$$

distinct subgroups of $Q_8 \times \mathbb{Z}_2^n$.

Case 3. $\mid K_1 \mid = 4$.
In this case K_1 can be chosen in 3 ways and K_2' can be chosen in $a_{n,2}(k-2)$ ways because its order is 2^{k-2}. If $\mid K_1' \mid = 4$, then it can uniquely be chosen, φ is trivial and $K_2 = K_2'$ can be chosen in $a_{n,2}(k-2)$ ways. These determine $3a_{n,2}(k-2)$ distinct subgroups K. If $\mid K_1' \mid = 8$, then $K_1' = Q_8$, φ can uniquely be chosen ($\mathbb{Z}_2$ has a unique automorphism) and K_2 can be chosen in $2^{k-2}-1$ because its order is 2^{k-3}. These determine $3(2^{k-2}-1)a_{n,2}(k-2)$ distinct subgroups K. Hence in this case one obtains

$$3a_{n,2}(k-2) + 3(2^{k-2}-1)a_{n,2}(k-2) = 3 \cdot 2^{k-2}a_{n,2}(k-2)$$

distinct subgroups of $Q_8 \times \mathbb{Z}_2^n$.

Case 4. $\mid K_1 \mid = 8$.
In this case $K_1 = K_1'$ can uniquely be chosen, $K_2 = K_2'$ can be chosen in $a_{n,2}(k-3)$ ways because their order is 2^{k-3} and φ is trivial. Hence they determine

$$a_{n,2}(k-3)$$

distinct subgroups of $Q_8 \times \mathbb{Z}_2^n$.

Now, by summing all above quantities, we infer that

$$b_{n,2}(k) = 2^k a_{n,2}(k) + 2^{2k-2}a_{n,2}(k-1) + 3 \cdot 2^{k-2}a_{n,2}(k-2) + a_{n,2}(k-3).$$

In the following this equality will be used to compute the total number $b_{n,2}$ of subgroups of $Q_8 \times \mathbb{Z}_2^n$. We have

$$b_{n,2} = \sum_{k=0}^{n+3} b_{n,2}(k) = a_{n,2} + 4x_n + y_n,$$

where

$$x_n = \sum_{k=0}^{n} 2^k a_{n,2}(k) \text{ and } y_n = \sum_{k=0}^{n} 2^{2k} a_{n,2}(k).$$

The connections between the numbers $a_{n,2}(k)$ and $a_{n,2}$ are well-known:

- $a_{n,2}(k) = a_{n-1,2}(k) + 2^{n-k} a_{n-1,2}(k-1)$, for all $k = \overline{1, n-1}$,
- $a_{n,2} = 2a_{n-1,2} + (2^{n-1} - 1)a_{n-2,2}$.

These lead to some recurrence relations satisfied by the chains $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$:

$$x_n = x_{n-1} + 2^n a_{n-1,2} \text{ and } y_n = y_{n-1} + 2^{n+1} x_{n-1},$$

respectively. It follows that

$$x_n = 1 + \sum_{i=0}^{n-1} 2^{i+1} a_{i,2} \text{ and } y_n = 2^{n+2} - 3 + \sum_{i=0}^{n-2} (2^{n+3-i} - 2^{2i+4}) a_{i,2},$$

which implies

$$b_{n,2} = 2^{n+2} + 1 + 8 \sum_{i=0}^{n-2} (2^{n-i} - 2^{2i+1} + 2^i) a_{i,2} + 2^{n+2} a_{n-1,2} + a_{n,2},$$

completing the proof. □

Clearly, Lemma 2.4 leads to the following theorem.

Theorem 2.5. *The total number of subgroups of the finite Hamiltonian group $H = Q_8 \times \mathbb{Z}_2^n \times A$ is given by the equality*

$$s(H) = b_{n,2}\, s(A),$$

where $s(A)$ denotes the total number of subgroups of A.

Another interesting problem involving the subgroup structure of a (finite) group is to study whether it can be written as the union of $r \geq 3$ proper subgroups. Such a decomposition is called a *subgroup covering* and there are many papers on the coverings of finite groups with different numbers/types of subgroups. We recall here a famous Scorza's result (see [19]) which states that a finite group has a covering with three proper subgroups if and only if it possesses a quotient isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Also, we recall the problem of counting the coverings of a finite group with three proper subgroups, formulated in [18]. We are now able to solve it for finite Hamiltonian groups.

Theorem 2.6. *The number of coverings with three proper subgroups of the finite Hamiltonian group $H = Q_8 \times \mathbb{Z}_2^n \times A$ is given by the equality*

$$c_3(H) = \frac{2^{2n+3} - 3 \cdot 2^{n+1} + 1}{3}.$$

Proof. As we have seen in [18], the number $c_3(H)$ of coverings with three proper subgroups of H coincides with the number of quotients of H that are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, by (1.1) we infer that $c_3(H)$ is also equal to the number of quotients of $Q_8 \times \mathbb{Z}_2^n$ that are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let K be a subgroup of order 2^{n+1} of $Q_8 \times \mathbb{Z}_2^n$ and assume that the quotient $Q_8 \times \mathbb{Z}_2^n/K$ contains an element xK of order 4. Then $o(x) = 4$ in $Q_8 \times \mathbb{Z}_2^n$ and $\langle x \rangle \cap K = 1$. On the other hand, $Q_8 \times \mathbb{Z}_2^n/K$ is Abelian because its order is 4. This implies that $\langle x^2 \rangle = (Q_8 \times \mathbb{Z}_2^n)' \subseteq K$, a contradiction. It follows that the quotient $Q_8 \times \mathbb{Z}_2^n/K$ is isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}_2$, for any subgroup K of order 2^{n+1} in $Q_8 \times \mathbb{Z}_2^n$. Hence

$$c_3(H) = b_{n,2}(n+1) = 2^{2n} a_{n,2}(n) + 3 \cdot 2^{n-1} a_{n,2}(n-1) + a_{n,2}(n-2) =$$
$$= 2^{2n} + 3 \cdot 2^{n-1}(2^n - 1) + \frac{2^{2n-1} - 3 \cdot 2^{n-1} + 1}{3} = \frac{2^{2n+3} - 3 \cdot 2^{n+1} + 1}{3},$$

as desired. □

Remark 2.7.
1. *Under the above notation, we have $c_3(H) = 1$ if and only if $n = 1$, that gives another proof of Corollary F of* [18].
2. *The partitions of a group constitute a particular type of subgroup coverings (remind that a subgroup covering $(H_i)_{i=\overline{1,m}}$ of a group H is called a partition of H if $H_i \cap H_j = 1$, for all $i \neq j$ – see Section 3.5 of* [13]*; moreover, if $H_i \neq H$ for all $i = \overline{1,m}$, then the partition $(H_i)_{i=\overline{1,m}}$ is called non-trivial). By inspecting the subgroups of a finite Hamiltonian group described above, we infer that it has no nontrivial partitions (notice that an alternative way to get this conclusion is given by Theorem 3.5.10 of* [13]*).*

Since we know the subgroup structure of finite Hamiltonian groups, the number of principal series of these groups can also be counted. This follows the most general topic of counting maximal chains of subgroups in finite nilpotent groups, studied in [2] and [15].

Let $H = Q_8 \times \mathbb{Z}_2^n \times A$ be a finite Hamiltonian group, $p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ be the decomposition of $\mid A \mid$ as a product of (odd) prime factors and put $\alpha = \sum_{i=1}^{r} \alpha_i$. According to Theorem 2.1 of [15], one obtains that the numbers $ps(H)$ and $ps(A)$ of all principal series of H and A, respectively, are connected by the equality

$$ps(H) = \binom{n+3+\alpha}{n+3, \alpha} ps(Q_8 \times \mathbb{Z}_2^n) ps(A).$$

Because the principal series of a finite Abelian group have been counted in our previous papers [2] and [15], we infer that the problem of determining $ps(H)$ is reduced to the computation of $ps(Q_8 \times \mathbb{Z}_2^n)$.

Lemma 2.8. *The number of principal series of subgroups of $Q_8 \times \mathbb{Z}_2^n$ is*

$$ps(Q_8 \times \mathbb{Z}_2^n) = (2^{2n+4} - 3 \cdot 2^{n+2} - 3n \cdot 2^{n+1} - 1) \prod_{i=1}^{n} (2^i - 1).$$

Proof. In order to count the number $ps(Q_8 \times \mathbb{Z}_2^n)$ of all principal series of $Q_8 \times \mathbb{Z}_2^n$, we remark that every such series contains a unique maximal subgroup of $Q_8 \times \mathbb{Z}_2^n$. Consequently

$$ps(Q_8 \times \mathbb{Z}_2^n) = \sum_{M \leq Q_8 \times \mathbb{Z}_2^n,\, |M| = 2^{n+2}} ps(M).$$

On the other hand, by the proof of Lemma 2.4 we infer that $Q_8 \times \mathbb{Z}_2^n$ possesses $b_{n,2}(n+2) = 2^{n+2} - 1$ maximal subgroups, namely $4(2^n - 1)$ isomorphic to $Q_8 \times \mathbb{Z}_2^{n-1}$ and 3 isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2^n$. One obtains

$$ps(Q_8 \times \mathbb{Z}_2^n) = 4(2^n - 1) ps(Q_8 \times \mathbb{Z}_2^{n-1}) + 3 ps(\mathbb{Z}_4 \times \mathbb{Z}_2^n).$$

By using the recurrence relation established in Lemma D of [2], we easily get

$$ps(\mathbb{Z}_4 \times \mathbb{Z}_2^n) = (n \cdot 2^{n+1} + 1) \prod_{i=1}^{n} (2^i - 1),$$

which implies that the chain $z_n = ps(Q_8 \times \mathbb{Z}_2^n)$ satisfies the following recurrence relation

$$z_n = 4(2^n - 1) z_{n-1} + 3(n \cdot 2^{n+1} + 1) \prod_{i=1}^{n} (2^i - 1), \text{ for all } n \geq 1.$$

Since $z_0 = 3$, it results

$$z_n = (2^{2n+4} - 3 \cdot 2^{n+2} - 3n \cdot 2^{n+1} - 1) \prod_{i=1}^{n} (2^i - 1),$$

which completes the proof. □

Now, an explicit formula for $ps(H)$ is obtained.

Theorem 2.9. *The total number of principal series of subgroups of the finite Hamiltonian group $H = Q_8 \times \mathbb{Z}_2^n \times A$ is given by the equality*

$$ps(H)=\left(\begin{array}{c} n+3+\alpha \\ n+3,\alpha \end{array}\right)(2^{2n+4}-3\cdot 2^{n+2}-3n\cdot 2^{n+1}-1)\prod_{i=1}^{n}(2^i-1)ps(A).$$

3. The lattice of characteristic subgroups of a finite Hamiltonian group

Let G be a group, $L(G)$ be the subgroup lattice of G and $N(G)$ be the normal subgroup lattice of G. Then the characteristic subgroups of G form a remarkable sublattice of $L(G)$, usually denoted by $\mathrm{Char}(G)$, which can be seen as the set of fixed points of $L(G)$ under the natural action of $\mathrm{Aut}(G)$. We also know that $\mathrm{Char}(G)$ is contained in $N(G)$ and therefore it is a modular lattice. In contrast with $L(G)$ and $N(G)$ that are known for many classes of groups (for example, see [13]), $\mathrm{Char}(G)$ has been exhaustively described only for few classes of groups G. One of them is constituted by the finite Abelian groups, and important contributions have had Miller [11, 12], Baer [1], Birkhoff [4], or the more recent paper by Kerby and Rode [10] (see also [9]). In this section we will extend this study to finite Hamiltonian groups, in view of the form of subgroups of a direct product.

Let $H \cong Q_8 \times \mathbb{Z}_2^n \times A$ be a finite Hamiltonian group. One needs first to know the structure of automorphisms of H. According to Lemma 2.1 of [7], we infer that a direct decomposition of type (1.1) also holds for $\mathrm{Aut}(H)$, namely

$$\mathrm{Aut}(H) \cong \mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n) \times \mathrm{Aut}(A). \tag{3.1}$$

Since the automorphisms of finite Abelian groups are precisely determined and counted in [7], we must focus only on describing the automorphisms of $Q_8 \times \mathbb{Z}_2^n$. By Theorem 3.2 of [3] and by using the elementary isomorphism $Z(Q_8) = \langle x^2 \rangle \cong \mathbb{Z}_2$, one obtains that $\mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n)$ is isomorphic to the multiplicative group of all matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{with} \quad \begin{array}{cc} \alpha \in \mathrm{Aut}(Q_8) & \beta \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \\ \gamma \in \mathrm{Hom}(Q_8, \mathbb{Z}_2^n) & \delta \in \mathrm{Aut}(\mathbb{Z}_2^n) \end{array}.$$

More exactly, if $f \in \mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n)$ is determined by the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $f(x_1, x_2) = (\alpha(x_1)\beta(x_2), \gamma(x_1) + \delta(x_2))$, for all $(x_1, x_2) \in Q_8 \times \mathbb{Z}_2^n$. In particular, we have

$$|\mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n)| = |\mathrm{Aut}(Q_8)||\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)||\mathrm{Hom}(Q_8, \mathbb{Z}_2^n)||\mathrm{Aut}(\mathbb{Z}_2^n)|\ .$$

Now, we can easily see that

- $|\mathrm{Aut}(Q_8)| = 24$,
- $|\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)| = 2^n$,
- $|\mathrm{Hom}(Q_8, \mathbb{Z}_2^n)| = 2^{2n}$,
- $|\mathrm{Aut}(\mathbb{Z}_2^n)| = 2^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}(2^i - 1)$.

Hence we have proved the following result.

Theorem 3.1. *The number of automorphisms of the finite Hamiltonian group $H \cong Q_8 \times \mathbb{Z}_2^n \times A$ is given by the equality*

$$|\mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n)| = 3 \cdot 2^{\frac{(n+2)(n+3)}{2}} \prod_{i=1}^{n}(2^i - 1)\ |\mathrm{Aut}(A)|,$$

where $|\mathrm{Aut}(A)|$ can be computed by Theorem 4.1 of [7].

Since the Sylow subgroups of a finite nilpotent group are characteristic, the lattice $\mathrm{Char}(H)$ of characteristic subgroups of $H \cong Q_8 \times \mathbb{Z}_2^n \times A$ is also decomposable:

$$\mathrm{Char}(H) \cong \mathrm{Char}(Q_8 \times \mathbb{Z}_2^n) \times \mathrm{Char}(A). \tag{3.2}$$

In our case we can give an explicit description of the lattice $\mathrm{Char}(A)$. The condition $|A| \equiv 1 \pmod 2$ implies that all characteristic subgroups of A are regular (see [1] and [11, 12]). It follows that they form a sublattice of a direct product of chains, that is a distributive lattice. Moreover, we note that the number $|\,\mathrm{Char}(A)|$ can precisely be determined by Corollary 1.7 of [10].

So, in order to study the characteristic subgroups of H we must look again only on the characteristic subgroups of its 2-component. The following result shows that there are few possibilities for a subgroup K of $Q_8 \times \mathbb{Z}_2^n$ to be characteristic.

Lemma 3.2. *Let G be a characteristic subgroup of $Q_8 \times \mathbb{Z}_2^n$ and assume that it is determined by K_1, K_1', K_2, K_2' and φ, as in the proof of Lemma 2.4. Then $K_1, K_1' \in \mathrm{Char}(Q_8)$ and $K_2, K_2' \in \mathrm{Char}(\mathbb{Z}_2^n)$.*

Proof. First of all, by using the form of automorphisms of $Q_8 \times \mathbb{Z}_2^n$ described above, we infer that for every $f_1 \in \mathrm{Aut}(Q_8)$ there is $f \in \mathrm{Aut}(Q_8 \times \mathbb{Z}_2^n)$ such that $f \mid Q_8 = f_1$ and $f \mid \mathbb{Z}_2^n = 1_{\mathbb{Z}_2^n}$ (namely, $f(x_1, x_2) = (f_1(x_1), x_2)$, for all $(x_1, x_2) \in Q_8 \times \mathbb{Z}_2^n$). It is clear that for $x_1 \in K_1 = K \cap Q_8$ we have $f_1(x_1) = \pi_1(f(x_1, 0)) \in K_1$, because K is characteristic in $Q_8 \times \mathbb{Z}_2^n$. This proves that $K_1 \in \mathrm{Char}(Q_8)$. If $x_1 \in K_1' = \pi_1(K)$, then $(x_1, x_2) \in K$ for some $x_2 \in \mathbb{Z}_2^n$. One obtains $f_1(x_1) = \pi_1(f(x_1, x_2)) \in K_1'$ and so K_2' is characteristic, too.

The second conclusion follows similarly. □

Remark 3.3.

1. *Under the above notation, for a characteristic subgroup K of $Q_8 \times \mathbb{Z}_2^n$ we must have $K_1, K_1' \in \{1, \langle x^2 \rangle, Q_8\}$ and $K_2, K_2' \in \{0, \mathbb{Z}_2^n\}$, respectively.*
2. *Lemma 3.1 can naturally be generalized to a finite direct product of arbitrary groups. We also remark that its converse fails (for example, the subgroup $K = Q_8 \times 0 \in L(Q_8 \times \mathbb{Z}_2^n)$ has $K_1 = K_1' = Q_8 \in \mathrm{Char}(Q_8)$ and $K_2 = K_2' = 0 \in \mathrm{Char}(\mathbb{Z}_2^n)$, but it is not characteristic in $Q_8 \times \mathbb{Z}_2^n$).*

Next, we observe that if $K \in \mathrm{Char}(Q_8 \times \mathbb{Z}_2^n)$ and $n \geq 3$, then $K_1 = K_1'$, $K_2 = K_2'$ and φ is trivial. In this way

$$\mathrm{Char}(Q_8 \times \mathbb{Z}_2^n) \subseteq \mathrm{Char}(Q_8) \times \mathrm{Char}(\mathbb{Z}_2^n).$$

Again, a simple exercise involving the form of automorphisms of $Q_8 \times \mathbb{Z}_2^n$ shows that $\langle x^2 \rangle \times\ 0 \in \mathrm{Char}(Q_8 \times \mathbb{Z}_2^n)$, while the subgroups $Q_8 \times 0$ and $1 \times \mathbb{Z}_2^n$ are not characteristic. Since the other subgroups the direct product $\mathrm{Char}(Q_8) \times \mathrm{Char}(\mathbb{Z}_2^n)$ obviously belong to $\mathrm{Char}(Q_8 \times \mathbb{Z}_2^n)$ (mention that $\langle x^2 \rangle \times\ \mathbb{Z}_2^n = \Phi(Q_8 \times \mathbb{Z}_2^n)$), one obtains that for $n \geq 3$ the lattice $\mathrm{Char}(Q_8 \times \mathbb{Z}_2^n)$ is reduced to the following chain:

$$1 \times 0 \subset \langle x^2 \rangle \times\ 0 \subset \langle x^2 \rangle \times\ \mathbb{Z}_2^n \subset Q_8 \times \mathbb{Z}_2^n.$$

If K is a subgroup of $Q_8 \times \mathbb{Z}_2^n$ and $n = 1$ or $n = 2$, then the quotients K_1'/K_1 and K_2'/K_2 are not necessarily trivial. In these two cases we have the additional subgroups $\langle x^2, 1 \rangle$ corresponding to the (unique) isomorphism $K_1'/K_1 \cong K_2'/K_2 \cong \mathbb{Z}_2$ and six subgroups $\cong Q_8$ corresponding to the isomorphisms $K_1'/K_1 \cong K_2'/K_2 \cong \mathbb{Z}_2^2$, respectively. By a direct inspection, we infer that none of these subgroups are characteristic.

We are now able to give a complete description of the lattice of characteristic subgroups of H.

Theorem 3.4. *The lattice $\mathrm{Char}(H)$ of characteristic subgroups of the finite Hamiltonian group $H \cong Q_8 \times \mathbb{Z}_2^n \times A$ is distributive. More precisely,* *it possesses a direct decomposition of type*

$$\mathrm{Char}(H) \cong \mathrm{Char}(Q_8 \times \mathbb{Z}_2^n) \times \mathrm{Char}(A),$$

where $\mathrm{Char}(Q_8 \times \mathbb{Z}_2^n)$ *is a chain of length* 3 *and* $\mathrm{Char}(A)$ *is a sublattice of a direct product of chains.*

In particular, Theorem 3.4 allows us to determine explicitly the cardinality of $\mathrm{Char}(H)$.

Theorem 3.5. *The total number of characteristic subgroups of the finite Hamiltonian group* $H = Q_8 \times \mathbb{Z}_2^n \times A$ *is given by the equality*

$$|\ \mathrm{Char}(H)| = 4\ |\ \mathrm{Char}(A)|,$$

where $|\ \mathrm{Char}(A)|$ *is computed in Corollary* 1.7 *of* [10].

Finally, by fixing the finite Hamiltonian group $H \cong Q_8 \times \mathbb{Z}_2^n \times A$, we remark that there are many finite groups G whose lattices of characteristic subgroups are isomorphic to $\mathrm{Char}(H)$ (for example, $G = \mathbb{Z}_8^n \times A$, where $n \in \mathbb{N}^*$, or $G = D_8 \times A$, where D_8 is the dihedral group of order 8). Hence the finite Hamiltonian groups are not determined by their lattices of characteristic subgroups.

4. Conclusions and further research

The study of some combinatorial aspects of subgroup lattices is a significant research direction of group theory. It is clear that all problems studied in the current paper for finite Hamiltonian groups can be extended to more large classes of (finite) groups. These will surely constitute the subject of some further research.

Several open problems with respect to this topic are the following.

Problem 1. Improve Theorem 2.6, by finding the number of coverings with r proper subgroups of H, where $r \geq 3$ is arbitrary.

Problem 2. Improve Theorem 2.8, by finding the total number of series of subgroups of an arbitrary length in H.

Problem 3. A classical result due to Miller, Baer and Birkhoff (see Corollary 1.7 of [10]) gives an explicit formula for $|\mathrm{Char}(G)|$, when G is an Abelian p-group with $p \neq 2$. Thus, the number of characteristic subgroups of any Abelian group of odd order is known. Moreover, [10] gives an algorithm to determine $|\,\mathrm{Char}(G)\,|$ when G is an Abelian 2-group, but not a precise expression of this number. Find such a precise expression and therefore determine an explicit formula for the number of characteristic subgroups of an *arbitrary* finite Abelian group.

Problem 4. Let H and K be two finite groups with no common direct factor and set $G = H \times K$. By using the form of automorphisms of G (described in Theorem 3.2 of [3]) and the form of subgroups of G (described in (4.19) of [14], I), determine the lattice $\mathrm{Char}(G)$. As we already observed, in general the lattice isomorphism $\mathrm{Char}(G) \cong \mathrm{Char}(H) \times \mathrm{Char}(K)$ fails. When is it true (of course, except the elementary case when H and K are of relatively prime orders)?

Problem 5. In Section 3 of [10] finite Abelian p-groups with isomorphic lattices of characteristic subgroups have been investigated. What can be said about two *arbitrary* finite groups whose lattices of characteristic subgroups are isomorphic?

Problem 6. As we have seen above, there exist finite groups whose lattices of characteristic subgroups are distributive (for example, Abelian groups of odd order, Hamiltonian groups, DLN-groups, ... and so on). We remark that also exist finite groups whose lattices of characteristic subgroups are not distributive (for example, a large class of Abelian 2-groups - see [10], or the quasi-dihedral groups S_{2^n} with $n \geq 4$). Which are *all* finite groups G such that $\mathrm{Char}(G)$ is a distributive lattice? More particularly, which are the finite groups G such that $\mathrm{Char}(G)$ is a chain (of a prescribed length)?

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References

[1] R. Baer, Types of elements and characteristic subgroups of Abelian groups, *Proc. London Math. Soc.* **39** (1934), no. 1, 481–514.

[2] L. Bentea and M. Tărnăuceanu, On the total number of principal series of a finite Abelian group, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **18** (2010), no. 2, 41–52.

[3] J. N. S. Bidwell, M. J. Curran and D. J. McCaughan, Automorphisms of direct products of finite groups, *Arch. Math.* **86** (2006), no. 6, 481–489.
[4] G. Birkhoff, Subgroups of Abelian groups, *Proc. London Math. Soc.* **38** (1934), no. 1, 385–401.
[5] M. L. Butler, Subgroup lattices and symmetric functions, *Mem. Amer. Math. Soc.* **112** (1994), no. 539, 160 pages.
[6] Gr. G. Călugăreanu, The total number of subgroups of a finite Abelian group, *Sci. Math. Jpn.* **60** (2004), no. 1, 157–167.
[7] C. J. Hillar and D. L. Rhea, Automorphisms of finite Abelian groups, *Amer. Math. Monthly* **114** (2007), no. 10, 917–923.
[8] B. Huppert, Endliche Gruppen, I, Springer- Verlag, Berlin-New York, 1967.
[9] B. L. Kerby, Rational Schur Rings Over Abelian Groups, Master's Thesis, Brigham Young University, 2008.
[10] B. L. Kerby and E. Rode, Characteristic subgroups of finite Abelian groups, *Comm. Algebra* **39** (2011), no. 4, 1315–1343.
[11] G. A. Miller, Determination of all the characteristic subgroups of any Abelian group, *Amer. J. Math.* **27** (1905), no. 1, 15–24.
[12] G. A. Miller, Characteristic subgroups of an Abelian prime power group, *Amer. J. Math.* **42** (1920), 278–286.
[13] R. Schmidt, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics, 14, Walter de Gruyter & Co., Berlin, 1994.
[14] M. Suzuki, Group Theory, I, II, Springer- Verlag, Berlin-New York, 1982.
[15] M. Ştefănescu and M. Tărnăuceanu, Counting maximal chains of subgroups of finite nilpotent groups, *Carpathian J. Math.* **25** (2009), no. 1, 119–127.
[16] M. Tărnăuceanu, Groups Determined by Posets of Subgroups, Matrix Rom, Bucharest, 2006.
[17] M. Tărnăuceanu, An arithmetic method of counting the subgroups of a finite Abelian group, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **101** (2010), no. 4, 373–386.
[18] M. Tărnăuceanu, A note on subgroup coverings of finite groups, *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **49** (2011), no. 2, 129–135.
[19] G. Zappa, The papers of Gaetano Scorza on group theory, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei. (9) Mat. Appl.* **2** (1991), no. 2, 95–101.

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