Bulletin of the Iranian Mathematical Society Vol. 39 *No.* 5 (*2013*), *pp 855-864.*

ANALYTIC SOLUTIONS FOR THE STEPHEN'S INVERSE PROBLEM WITH LOCAL BOUNDARY CONDITIONS INCLUDING ELLIPTIC AND HYPERBOLIC EQUATIONS

M. JAHANSHAHI* AND M. SAJJADMANESH

Communicated by Mohammad Asadzadeh

ABSTRACT. In this paper, two inverse problems of Stephen kind with local (Dirichlet) boundary conditions are investigated. In the first problem only a part of the boundary is unknown and in the second problem, the whole of the boundary is unknown. For both of the problems, first, analytic expressions for unknown boundary are presented, then by using these analytic expressions for unknown boundaries and boundary conditions of the main problem, analytic solution of the main inverse problem is derived.

1. Introduction

Lavrentiev and Stephen types inverse problems for elliptic equations are appeared in a number of fields, such as boundary value problems in engineering and physics [9] and [4]. Also, the inverse Cauchy problem for elliptic equation arises in many applications such as vibration of structure, nondestructive testing technique, electro-cardiology, electromagnetic scattering and so on.

MSC(2010): Primary: 35R30; Secondary: 31A25, 35E05

Keywords: Local boundary conditions, inverse problem, fundamental solution, Dirac's delta function.

Received: 25 December 2011, Accepted: 1 July 2012.

*Corresponding author

Liu [8] considered a Cauchy problem in a rectangular domain and reduced it into a first-kind Fredholm integral equation and then transformed it into a second -kind Fredholm integral equation by Lavrentiev type regularization.
A. Eden and V.K. Kalantarov [3] have considered the global in time behavior of solutions of an inverse problem in a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ such that in addition to the solution of the equation, the right-hand side of the equation is also unknown.
In [11], authors have studied the Cauchy-Riemann equation on the region $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\, x_1 \in \mathbb{R},\ x_2 \in (0,1)\}$ with two nonlocal boundary conditions on $\partial\Omega$ such that in addition to the solution of the equation, the right-hand side of the second boundary condition is also unknown:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = 0 \qquad x = (x_1, x_2),\ x_1 \in \mathbb{R},\ x_2 \in (0,1),$$
$$\alpha_j(x_1)\, u(x_1, 0) + \beta_j(x_1)\, u(x_1, 1) = \phi_j(x_1) \qquad j = 1, 2,\ x_1 \in \mathbb{R},$$

where $i = \sqrt{-1}$, u and ϕ_2 are unknown functions and ϕ_1, α_j and $\beta_j;\ j = 1, 2$ are given continuous functions. We have obtained a second kind Fredholm integral equation for $u(x_1, x_2)$ and $\phi_2(x_1)$.
In the present paper we attempt to solve a boundary value problem of the Stephen's kind for the elliptic Cauchy-Riemann equation on a bounded domain $\Omega \subset \mathbb{R}^2$, with local boundary conditions on the smooth boundary $\partial\Omega$.

In the first section, we assume that only a part of boundary $\partial\Omega$ is unknown but in the second section, the whole of boundary $\partial\Omega$ will be unknown. To this approach, consider the boundary value problem in the following form

$$\frac{\partial u(x)}{\partial x_2} + i\,\frac{\partial u(x)}{\partial x_1} = 0, \quad x \in \Omega, \tag{1.1}$$
$$u(x_1, 0) = \phi(x_1) \tag{1.2}$$
$$u(x_1, \gamma(x_1)) = \psi(x_1) \tag{1.3}$$

where $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 \in (0,1),\ x_2 \in (0, \gamma(x_1))\}$ is a bounded convex region in $\mathbb{R}^2$, ϕ and ψ are known real continuous functions on $[0,1]$ that can be extended to complex plane. Beside the solution $u(x_1, x_2)$, the upper boundary γ is unknown.

2. **Analytic solution for calculating the unknown boundary**

The following lemma provides a way to calculate the unknown boundary γ.

Lemma 2.1. *For the boundary values of the unknown function $u(x_1, x_2)$ in the problem (1.1)-(1.3) we have*

$$\begin{aligned} u(\xi_1, 0) &= -2\int_0^1 u(x_1, 0)\, U(x_1 - \xi_1, 0) dx_1 \\ &\quad +2\int_0^1 u(x_1, \gamma(x_1))\, U(x_1 - \xi_1, \gamma(x_1))(-1 + i\,\gamma'(x_1)) dx_1 \end{aligned} \tag{2.1}$$

$$\begin{aligned} u(\xi_1, \gamma(\xi_1)) = &-2\int_0^1 u(x_1, 0)\, U(x_1 - \xi_1, -\gamma(\xi_1))\, dx_1 \\ &+2\int_0^1 u(x_1, \gamma(x_1))\, U(x_1 - \xi_1, \gamma(x_1) - \gamma(\xi_1))(-1 + i\,\gamma'(x_1))\, dx_1 \end{aligned} \tag{2.2}$$

where $U(x - \xi)$ is the fundamental (generalized) solution of the Cauchy-Riemann equation (1.1) which is given by (see [12]*)*

$$U(x - \xi) = \frac{1}{2\pi(x_2 - \xi_2 + i(x_1 - \xi_1))}. \tag{2.3}$$

Proof. Multiplying both sides of (1.1) by the fundamental solution (2.3) and integrating on Ω, then applying the Ostrogradski-Gauss's formula [2] and using Dirac's delta function properties (similar to [11], [1]-[7]) we obtain

$$\begin{aligned} 0 &= \int_\Omega \Big(\frac{\partial u(x)}{\partial x_2} + i\,\frac{\partial u(x)}{\partial x_1}\Big)\, U(x - \xi)\, dx \\ &= \int_\Omega \frac{\partial u(x)}{\partial x_2}\, U(x - \xi)\, dx + i\int_\Omega \frac{\partial u(x)}{\partial x_1}\, U(x - \xi)\, dx \\ &= \int_\Gamma u(x)\, U(x - \xi) \cos(n, x_2)\, dx - \int_\Omega u(x)\, \frac{\partial U(x - \xi)}{\partial x_2}\, dx \\ &\quad + i\int_\Gamma u(x)\, U(x - \xi) \cos(n, x_1)\, dx - i\int_\Omega u(x) \frac{\partial U(x - \xi)}{\partial x_1}\, dx. \end{aligned}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is the boundary of region Ω where $\Gamma_1 = \{\, (x_1, 0) \in \mathbb{R}^2;\; x_1 \in [0, 1]\,\}$, $\Gamma_2 = \{\, (x_1, \gamma(x_1)) \in \mathbb{R}^2;\; x_1 \in [0, 1]\,\}$ and (n, x_j), $\;j =$

$1,2$ denotes the angle between outward unit normal vector to the boundary Γ and coordinates axes $x_j,\ j=1,2$.
Now, using the property of Dirac's delta function we obtain

$$\begin{aligned}\int_\Gamma u(x)U(x-\xi)\,[\cos(n,x_2)+i\,\cos(n,x_1)]\,dx &= \int_\Omega u(x)\delta(x-\xi)dx\\ &= \begin{cases} u(\xi), & \xi\in\Omega,\\ 1/2\,u(\xi), & \xi\in\Gamma,\\ 0, & \xi\notin\bar{\Omega}.\end{cases}\end{aligned}\tag{2.4}$$

Let (n_i,x_j) and $(\tau_i,x_j),\ j=1,2$ be the angles between outward unit normal and unit tangent vectors on the boundaries $\Gamma_i(i=1,2)$ respectively. Then

$$\begin{aligned}\cos(n_1,x_1)=0 \quad &, \quad \cos(n_1,x_2)=-1\\ \cos(n_2,x_1)=\sin(\tau_2,x_1) \quad &, \quad \cos(n_2,x_2)=-\cos(\tau_2,x_1)\\ \cos(\tau_2,x_1)=\frac{dx_1}{dx} \quad &, \quad \tan(\tau_2,x_1)=\gamma'(x_1)\end{aligned}\tag{2.5}$$

From the second case on the right-hand side of (2.4) we obtain

$$u(\xi)=2\int_\Gamma u(x)U(x-\xi)\,[\cos(n,x_2)+i\cos(n,x_1)]\,dx\,;\quad \xi\in\Gamma$$

Therefore for $\xi\in\Gamma_1$ and $\xi\in\Gamma_2$, and by using (2.5) we obtain

$$\begin{aligned}u(\xi_1,0) &= 2\int_{\Gamma_1} u(x_1,0)U(x_1-\xi_1,0)[\cos(n_1,x_2)+i\cos(n_1,x_1)]\,dx\\ &\quad +2\int_{\Gamma_2} u(x_1,\gamma(x_1))U(x_1-\xi_1,\gamma(x_1))(\cos(n_2,x_2)\\ &\qquad +i\cos(n_2,x_1))\,dx\\ &= -2\int_0^1 u(x_1,0)\,U(x_1-\xi_1,0)dx_1\\ &\quad +2\int_0^1 u(x_1,\gamma(x_1))\,U(x_1-\xi_1,\gamma(x_1))(-1+i\gamma'(x_1))dx_1\\ u(\xi_1,\gamma(\xi_1)) &= 2\int_{\Gamma_1} u(x_1,0)U(x_1-\xi_1,-\gamma(\xi_1))[\cos(n_1,x_2)\\ &\qquad +i\cos(n_1,x_1)]\,dx\\ &\quad +2\int_{\Gamma_2} u(x_1,\gamma(x_1))U(x_1-\xi_1,\gamma(x_1)-\gamma(\xi_1))[\cos(n_2,x_2)\\ &\qquad +i\cos(n_2,x_1)]\,dx\end{aligned}$$

$$
\begin{aligned}
= \quad & 2\int_0^1 u(x_1,0)\,U(x_1-\xi_1,-\gamma(\xi_1))dx_1 \\
& +2\int_0^1 u(x_1,\gamma(x_1))\,U(x_1-\xi_1,\gamma(x_1)-\gamma(\xi_1))(-1+i\gamma'(x_1))dx_1
\end{aligned}
$$

that implies relations (2.1)-(2.2) of lemma. □

Now, we state and prove the main theorem.

Theorem 2.2. *Let the inverse problem be given by equation (1.1) and boundary conditions (1.2)-(1.3), where ϕ and the invertible function ψ are known continuous functions on $[0,1]$. Then, the unknown boundary γ is given by*

$$
\gamma(\xi_1) = i\xi_1 - i\phi^{-1}(\psi(\xi_1)). \tag{2.6}
$$

Proof. By substituting the boundary conditions (1.2)-(1.3) and fundamental solution (2.3) in (2.1)-(2.2) we get

$$
\begin{aligned}
\phi(\xi_1) \quad = \quad & -1/(\pi\, i)\int_0^1 \frac{\phi(x_1)}{x_1-\xi_1}\,dx_1 \\
& +1/\pi\int_0^1 \frac{\psi(x_1)}{\gamma(x_1)+i(x_1-\xi_1)}\,(-1+i\gamma'(x_1))dx_1 \\
\psi(\xi_1) \quad = \quad & -1/\pi\int_0^1 \frac{\phi(x_1)}{-\gamma(\xi_1)+i(x_1-\xi_1)}\,dx_1 \\
& +1/\pi\int_0^1 \frac{\psi(x_1)}{\gamma(x_1)-\gamma(\xi_1)+i(x_1-\xi_1)}\,(-1+i\gamma'(x_1))dx_1
\end{aligned}
$$

Comparing the above two relations we obtain

$$
\phi(\xi_1 - i\gamma(\xi_1)) = \psi(\xi_1).
$$

Therefore by using implicit and inverse functions theorems [10] we have

$$
\gamma(\xi_1) = i\xi_1 - i\phi^{-1}(\psi(\xi_1)).
$$

□

Remark 2.3. *By utilizing (2.6) and (2.5) in first case on the right-hand side of (2.4), we obtain the solution for the main problem (1.1)-(1.3)* *represented by*

$$\begin{aligned} u(\xi) &= -\int_0^1 u(x_1,0)\,U(x_1-\xi_1,-\xi_2)dx_1 \\ &\quad +\int_0^1 u(x_1,\gamma(x_1))\,U(x_1-\xi_1,\gamma(x_1)-\xi_2)(-1+i\,\gamma'(x_1))dx_1 \\ &= 1/(2\pi)\int_0^1 \frac{\phi(x_1)}{\xi_2-i(x_1-\xi_1)}\,dx_1 \\ &\quad +1/(2\pi)\int_0^1\frac{\psi(x_1)}{-\xi_2+i(2x_1-\xi_1-\phi^{-1}(\psi(x_1)))}\,(-2+\frac{1}{\phi'(x_1)})\,dx_1. \end{aligned}$$

3. The Stephen's problem with two unknown boundaries

Consider the inverse boundary value problem of the Stephen's kind

$$\frac{\partial u_1(x)}{\partial x_2}+i\,\frac{\partial u_1(x)}{\partial x_1} = 0, \qquad x\in\Omega_1, \tag{3.1}$$

$$\frac{\partial u_2(x)}{\partial x_1}+\frac{\partial u_2(x)}{\partial x_2} = 0, \qquad x\in\Omega_2, \tag{3.2}$$

$$u_1(x_1,\gamma_1(x_1)) = \psi_1(x_1) \tag{3.3}$$

$$u_1(x_1,0)=u_2(x_1,0) = \psi_0(x_1) \tag{3.4}$$

$$u_2(x_1,\gamma_2(x_1)) = \psi_2(x_1) \tag{3.5}$$

where $\Omega_1 = \{(x_1,x_2)\in\mathbb{R}^2;\ x_1\in(0,1),\ x_2\in(0,\gamma_1(x_1))\}$, $\Omega_2 = \{(x_1,x_2)\in\mathbb{R}^2;\ x_1\in(0,1),\ x_2\in(\gamma_2(x_1),0)\}$ are bounded convex regions in $\mathbb{R}^2$. Also γ_1 and γ_2 are unknown boundaries but the real continuous functions ψ_0,ψ_1 and ψ_2 are known on $[0,1]$.

3.1. **Calculating the Unknown Boundaries γ_1 and γ_2.** By the previous section, the unknown boundary γ_1 is given by

$$\gamma_1(\xi_1)=i\xi_1-i\psi_0^{-1}(\psi_1(\xi_1)). \tag{3.6}$$

Here we try to present an analytic expression for the second part of boundary Γ, which is denoted by $\Gamma_2 : x_2=\gamma_2(x_1)$.

Lemma 3.1. *The expression for the boundary values of unknown function $u_2(x_1, x_2)$ of problem (3.2) and (3.4) – (3.5) satisfies the following relations*

$$
\begin{aligned}
u_2(\xi_1, 0) &= 2\int_0^1 u_2(x_1, 0)\, U_2(x_1 - \xi_1, 0)dx_1 \\
&\quad +2\int_0^1 u_2(x_1, \gamma_2(x_1))\, U_2(x_1 - \xi_1, \gamma_2(x_1))(\gamma_2'(x_1) - 1)dx_1 \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
u_2(\xi_1, \gamma_2(\xi_1)) &= 2\int_0^1 u_2(x_1, 0)\, U_2(x_1 - \xi_1, -\gamma_2(\xi_1))dx_1 \\
&\quad +2\int_0^1 u_2(x_1, \gamma_2(x_1))\, U_2(x_1 - \xi_1, \gamma_2(x_1) \qquad (3.8) \\
&\quad -\gamma_2(\xi_1))(\gamma_2'(x_1) - 1)dx_1
\end{aligned}
$$

where $U_2(x - \xi)$ is the fundamental solution for the equation(3.2) given by

$$U_2(x - \xi) = \theta(x_2 - \xi_2)\delta(x_1 - \xi_1 - (x_2 - \xi_2)) \qquad (3.9)$$

in which θ and δ are the symmetric Heaviside and Dirac's delta functions respectively [12].

Proof. Multiplying both sides of equation (3.2) by the fundamental solution (3.10), integrating over region Ω_2 and applying the Ostrogradskii-Gauss's formula (similar to previous section) we get

$$
\begin{aligned}
0 &= \int_{\Omega_2} \left(\frac{\partial u_2(x)}{\partial x_1} + \frac{\partial u_2(x)}{\partial x_2}\right) U_2(x - \xi)\, dx \\
&= \int_{\Omega_2} \frac{\partial u_2(x)}{\partial x_1} U_2(x - \xi)\, dx + \int_{\Omega_2} \frac{\partial u_2(x)}{\partial x_2} U_2(x - \xi)\, dx \\
&= \int_\Gamma u_2(x)\, U_2(x - \xi) \cos(n, x_2)\, dx - \int_{\Omega_2} u_2(x) \frac{\partial U_2(x - \xi)}{\partial x_2}\, dx \\
&\quad + \int_\Gamma u_2(x)\, U_2(x - \xi) \cos(n, x_1)\, dx - \int_{\Omega_2} u_2(x) \frac{\partial U_2(x - \xi)}{\partial x_1}\, dx.
\end{aligned}
$$

where Γ is the boundary of region Ω_2 as $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \{(x_1, 0) \in \mathbb{R}^2;\ x_1 \in [0, 1]\}, \Gamma_2 = \{(x_1, \gamma_2(x_1)) \in \mathbb{R}^2;\ x_1 \in [0, 1]\}$. Now, using the properties of the fundamental solution and Dirac's delta function we obtain

$$\int_\Gamma u_2(x)U_2(x-\xi)\left[\cos(n,x_1)+\cos(n,x_2)\right]dx = \int_{\Omega_2} u_2(x)\delta(x-\xi)dx$$

$$(3.10) \qquad = \begin{cases} u_2(\xi), & \xi\in\Omega_2, \\ 1/2\,u_2(\xi), & \xi\in\Gamma, \\ 0, & \xi\notin\bar\Omega_2. \end{cases}$$

Let $(n_i,x_j),\ (\tau_i,x_j)\ j=1,2$ be the angle between outward unit normal and tangent vectors on the boundary $\Gamma_i\,(i=1,2)$ respectively with coordinates axes. Then we obtain

$$\cos(n_1,x_1)=0 \quad , \quad \cos(n_1,x_2)=1$$
$$\cos(n_2,x_1)=\sin(\tau_2,x_1) \quad , \quad \cos(n_2,x_2)=-\cos(\tau_2,x_1)$$
$$(3.11) \qquad \cos(\tau_2,x_1)=\frac{dx_1}{dx} \quad , \quad \tan(\tau_2,x_1)=\gamma_2'(x_1)$$

From the second case on the right-hand side of (3.11) we have

$$u_2(\xi)=2\int_\Gamma u_2(x)U_2(x-\xi)[\cos(n,x_1)+\cos(n,x_2)]\,dx, \quad \xi\in\Gamma$$

which leads to (3.7)-(3.9) of lemma 3.1 by considering $\xi\in\Gamma_1$ and $\xi\in\Gamma_2$, respectively, and using (3.12). □

Theorem 3.2. *Let the Stephen inverse problem be given by equation (3.2) with boundary conditions (3.4)-(3.5) and known continuous functions ψ_0,ψ_1 and ψ_2 on $[0,1]$. Further assume that the function ψ_2 is invertible. Then, the unknown boundary γ_2 is given by*

$$(3.12) \qquad \gamma_2(\sigma(\xi_1))=\sigma(\xi_1)-\xi_1$$

where $\sigma(\xi_1)=\psi_2^{-1}o\,\psi_0(\xi_1)$.

Proof. Using the fundamental solution (3.10) and the boundary conditions (3.4)-(3.5) in (3.7)-(3.9) we have

$$\psi_0(\xi_1) = 2\int_0^1 \psi_0(x_1)\theta(0)\delta(x_1-\xi_1)dx_1$$
$$+2\int_0^1 \psi_2(x_1)\theta(\gamma_2(x_1))\delta(x_1-\xi_1-\gamma_2(x_1))(\gamma_2'(x_1)-1)dx_1$$
$$(3.13) \qquad = -\int_0^1 \psi_2(x_1)\delta(x_1-\xi_1-\gamma_2(x_1))\,(\gamma_2'(x_1)-1)dx_1.$$

$$\begin{aligned}\psi_2(\xi_1) &= 2\int_0^1 \psi_0(x_1)\theta(-\gamma_2(\xi_1))\delta(x_1-\xi_1+\gamma_2(\xi_1))dx_1 \\ &\quad +2\int_0^1 \psi_2(x_1)\theta(\gamma_2(x_1)-\gamma_2(\xi_1))\delta(x_1-\xi_1 \\ &\quad -(\gamma_2(x_1)-\gamma_2(\xi_1))(\gamma_2'(x_1)-1)dx_1 \\ &= \int_0^1 \psi_0(x_1)\delta(x_1-\xi_1+\gamma_2(\xi_1))dx_1 \\ &\quad +2\int_0^1 \psi_2(x_1)\theta(\gamma_2(x_1) \\ &\quad -\gamma_2(\xi_1))\delta(x_1-\xi_1-(\gamma_2(x_1)-\gamma_2(\xi_1)))(\gamma_2'(x_1)-1)dx_1.\end{aligned}$$

(3.14)

Let $1-\gamma_2'(x_1)>0$ which is the relative derivation of $x_1-\xi_1-\gamma_2(x_1)$ with respect to x_1. Note that $x_1-\xi_1-\gamma_2(x_1)$ is the argument of Dirac delta function appeared in (3.14). Therefore, $x_1-\xi_1-\gamma_2(x_1)$ is a strictly increasing function on $[0,1]$. On the other hand we have

$$x_1-\xi_1-\gamma_2(x_1)|_{x_1=0}=-\xi_1<0 \quad , \quad x_1-\xi_1-\gamma_2(x_1)|_{x_1=1}=1-\xi_1>0$$

Therefore this function has only one zero in $[0,1]$ which is denoted by $x_1=\sigma(\xi_1)$.
From (3.14)-(3.15), by using the property of the Dirac's delta function, we obtain two necessary conditions

$$\begin{aligned}\psi_0(\xi_1) &= \psi_2(\sigma(\xi_1)) \\ \psi_2(\xi_1) &= u_1(\xi_1-\gamma_2(\xi_1),0).\end{aligned}$$

One can prove that the above two relations are equivalent and so we obtain the unknown boundary γ_2 as (3.13). □

Acknowledgments

The authors are thankful to Prof. N. Aliev for his comments and suggestions to use the necessary conditions in well-posedness of Stephen inverse problems with local boundary conditions and also to Dr. M. K. Mirnia for his suggestions and helps to edit the text of paper.

References

[1] N. Aliev, M. Jahanshahi, Sufficient conditions for reduction of the BVP including a mixed PDE with non-local boundary conditions to Fredholm integral equations, *Internat. J. Math. Ed. Sci. Tech.* **28** (1997), no. 3, 419–425.

[2] R. Courant and D. Gilbert, Methods of Mathematical Physics, Interscience publisher, Moscow-Leningrad, 1951.

[3] A. Eden and V. K. Kalantarov, On global behavior of solutions to an inverse problem for nonlinear parabolic equations, *J. Math. Anal. Appl.* **307** (2005), no. 1, 120–133.

[4] V. B. Glasko, Inverse Problems of Mathematical Physics, Moskov. Gos. Univ., Moscow, 1984.

[5] M. Jahanshahi, S. M. Hosseini and N. A1iev, An analytic method for investigation and solving two-dimensional steady state Navier-Stokes equations, *Southeast Asian Bull. Math.* **33** (2009), no. 6, 1075–1089.

[6] M. Jahanshahi and M. Fatehi, Analytic solution for the Cauchy-Riemann equation with nonlocal boundary conditions in the first quarter, *Int. J. Pure Appl. Math.* **46** (2005), no. 2, 135–145.

[7] M. Jahanshahi, Reduction of Newmann and Poincare B.V.Ps to Dirichlet B.V.P, Proceedings of 4th ISSAC Toronto, Canada, 2003.

[8] CS. Liu, An analytical method for the inverse Cauchy problem of Laplace equation in a rectangular domain, *J. Mech.* In press.

[9] A. G. Ramm, Inverse Problems, Springer, New York, 2005.

[10] W. Rudin *Principles of Mathematical Analysis* McGraw-Hill Book Co., Inc., New York-Toronto-London, 1953.

[11] M. Sajjadmanesh, M. Jahanshahi and N. Aliev, Tikhonov-Lavrentev type inverse problem including Cauchy-Riemann equation, *Azerb. J. Math.* **3** (2013), no. 1, 104–110.

[12] V. S. Vladimirov, Equatuations of Mathematical Physics, Mir Publishers, Moscow, 1981.

M. Jahanshahi
Department of Mathematics, Azarbaijan university of Tarbiat Moallem, P.O. Box 53714-161, Tabriz, Iran
Email: jahanshahi@azaruniv.edu

M. Sajjadmanesh
Department of Mathematics, Azarbaijan university of Tarbiat Moallem, P.O. Box 53714-161, Tabriz, Iran
Email: s.sajjadmanesh@azaruniv.edu