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LINEAR PRESERVERS OF G-ROW AND G-COLUMN MAJORIZATION ON $\mathbf{M}_{n,m}$

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Abstract. Let A and B be $n \times m$ matrices. The matrix B is said to be g-row majorized (respectively g-column majorized) by A, denoted by $B \prec_g^{row} A$ (respectively $B \prec_g^{column} A$), if every row (respectively column) of B, is g-majorized by the corresponding row (respectively column) of A. In this paper all kinds of g-majorization are studied on $\mathbf{M}_{n,m}$, and the possible structure of their linear preservers will be found. Also all linear operators $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserving (or strongly preserving) g-row or g-column majorization will be characterized.

1. Introduction

An $n \times n$ matrix R (not necessarily nonnegative) is called g-row stochastic if $Re = e$, where $e = (1, 1, \ldots, 1)^t$. A matrix D is called g-doubly stochastic if both D and D^t are g-row stochastic matrices. The collection of all $n \times n$ g-row stochastic matrices, and $n \times n$ g-doubly stochastic matrices are denoted by $\mathbf{GR}_n$ and $\mathbf{GD}_n$ respectively. Throughout the paper, $\mathbf{M}_{n,m}$ is the set of all $n \times m$ matrices with entries in $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), and $\mathbf{M}_n := \mathbf{M}_{n,n}$. The set of all $n \times 1$ column vectors is denoted by $\mathbb{F}^n$, and the set of all $1 \times n$ row vectors is denoted by $\mathbb{F}_n$. The symbol $\mathbb{N}_k$ is used for the set $\{1, \ldots, k\}$. The symbol e_i is the row (or

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column) vector with 1 as i^{th} component and 0 elsewhere. The summation of all components of a vector x in $\mathbb{F}^n$ or $\mathbb{F}_n$ is denoted by tr(x). The symbol $[x_1/x_2/\dots/x_n]$ (resp. $[x_1 \mid x_2 \mid \dots \mid x_m]$) is used for the $n \times m$ matrix whose rows (resp. columns) are $x_1, x_2, \dots, x_n \in \mathbb{F}_m$ (resp. $x_1, x_2, \dots, x_m \in \mathbb{F}^n$). For a matrix $X = [x_{ij}] \in \mathbf{M}_{n,m}$, its average (column) vector $\overline{X} = [\overline{x_1}/\dots/\overline{x_n}] \in \mathbb{F}^n$ is defined by the components $\overline{x_i} = m^{-1}(x_{i1} + x_{i2} + \cdots + x_{im})$, for $i \in \mathbb{N}_n$. The letter $\mathbf{J}$ stands for the (rank-1) square matrix all of whose entries are 1.

For $A, B \in \mathbf{M}_{n,m}$, it is said that A is lgs-majorized (resp. rgs-majorized) by B and denoted by $A \prec_{lgs} B$ (resp. $A \prec_{rgs} B$) if there exists an $n \times n$ (resp. $m \times m$) g-doubly stochastic matrix D such that $A = DB$ (resp. $A = BD$), see [4, 6].

Let $A, B \in \mathbf{M}_{n,m}$. The matrix A is said to be lgw-majorized (resp. rgw-majorized) by B and denoted by $\prec_{lgw}$ (resp. $\prec_{rgw}$) if there exists an $n \times n$ (resp. $m \times m$) g-row stochastic matrix R such that $A = RB$ (resp. $A = BR$), for more details see [2, 5].

Let $\prec$ be a relation on $\mathbf{M}_{n,m}$. A linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is said to be a linear preserver (resp. strong linear preserver) of $\prec$, if $X \prec Y$ implies $TX \prec TY$ (resp. $X \prec Y$ if and only if $TX \prec TY$).

The linear preservers and strong linear preservers of lgs-majorization are characterized in [6] as follows:

Proposition 1.1. [6, Theorem 3.3] *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator that preserves lgs-majorization. Then one of the following statements holds:*

(i) There exist $A_1, A_2, \dots, A_m \in \boldsymbol{M}_{n,m}$ such that $TX = \sum_{j=1}^{m} \operatorname{tr}(x_j)A_j$, where $X = [x_1 \mid \dots \mid x_m]$;

(ii) There exist $S \in \boldsymbol{M}_m$, $a_1, \dots, a_m \in \mathbb{F}^m$ and invertible matrices $B_1, B_2, \dots, B_m \in \boldsymbol{GD}_n$, such that $TX = [B_1 X a_1 \mid \dots \mid B_m X a_m] + \boldsymbol{J}XS$.

Proposition 1.2. [6, Theorem 3.7] *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T strongly preserves $\prec_{lgs}$ if and only if $TX = AXR + \boldsymbol{J}XS$, for some $R, S \in \boldsymbol{M}_m$ and invertible matrix $A \in \boldsymbol{GD}_n$ such that $R(R + nS)$ is invertible.*

In [2, 5], the authors proved that a linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ strongly preserves lgw-majorization (resp. rgw-majorization) if and only if $TX = AXM$ (resp. $TX = MXA$), for some invertible matrices $M \in \mathbf{M}_m$ (resp. $M \in \mathbf{M}_n$) and $A \in \mathbf{GR}_n$ (resp. $A \in \mathbf{GR}_m$).

In the present paper, we find the possible structure of linear operators that preserve lgw, rgw or rgs-majorization. Also, all linear preservers and strong linear preservers of g-row and g-column majorization will be characterized. To see some kinds of majorization and their linear preservers we refer the readers to [1], [3] and [7]-[11].

2. Lgs-column (rgs-row) majorization on $\mathbf{M}_{n,m}$

In this section we characterize all linear operators on $\mathbf{M}_{n,m}$ that preserve or strongly preserve lgs-column (rgs-row) majorization.

Definition 2.1. Let $A, B \in \mathbf{M}_{n,m}$. It is said that B is lgs-column (resp. rgs-row) majorized by A, written as $B \prec_{lgs}^{column} A$ (resp. $B \prec_{rgs}^{row} A$), if every column (resp. row) of B is lgs- (resp. rgs-) majorized by the corresponding column (resp. row) of A.

We use the following statements to prove the main result of this section.

Proposition 2.2. [6, Theorem 2.4] *Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T preserves gs-majorization if and only if one of the following statements holds:*

(a) $Tx = \mathrm{tr}(x)a$, for some $a \in \mathbb{F}^n$;

(b) $Tx = \alpha Dx + \beta\ \boldsymbol{J}x$, for some $\alpha, \beta \in \mathbb{F}$ and invertible matrix $D \in \boldsymbol{GD}_n$.

Proposition 2.3. [6, Lemma 3.1] *Let $A \in \boldsymbol{GD}_n$ be invertible. Then the following conditions are equivalent:*

(a) $A = \alpha\boldsymbol{I} + \beta\boldsymbol{J}$, for some $\alpha, \beta \in \mathbb{F}$;

(b) $(Dx + ADy) \prec_{gs} (x + Ay)$, for all $D \in \boldsymbol{GD}_n$ and for all $x, y \in \mathbb{F}^n$.

Proposition 2.4. [6, Lemma 3.2] *Let $T_1, T_2 : \mathbb{F}^n \to \mathbb{F}^n$ satisfy $T_1(x) = \alpha Ax + \beta\boldsymbol{J}x$ and $T_2(x) = \mathrm{tr}(x)a$, for some $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$, invertible matrix $A \in \boldsymbol{GD}_n$ and $a \in (\mathbb{F}^n \setminus \mathrm{Span}\{e\})$. Then there exists a g-doubly stochastic matrix D and a vector $x \in \mathbb{F}^n$ such that $T_1(Dx) + T_2(Dx) \prec_{gs} T_1(x) + T_2(x)$.*

Lemma 2.5. *Let $a \in \mathbb{F}^m$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [Xa \mid \dots \mid Xa]$, preserves lgs-column majorization if and only if $a \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$.*

Proof. If $a \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$, it is easy to show that T preserves $\prec_{lgs}^{column}$. Conversely, let T preserve $\prec_{lgs}^{column}$. Assume that $a = (a_1, \dots, a_m)^t \notin \cup_{i=1}^m \mathrm{Span}\{e_i\}$. Then there exist distinct $i, j \in \mathbb{N}_m$ such that $a_i, a_j \neq 0$.

Without loss of generality assume that $a_1, a_2 \neq 0$. Put

$$X := \begin{pmatrix} -a_2 & -a_1 \\ a_2 & a_1 \end{pmatrix} \oplus 0,\ Y := \begin{pmatrix} a_2 & -a_1 \\ -a_2 & a_1 \end{pmatrix} \oplus 0 \in \mathbf{M}_{n,m}.$$

It is clear that $X \prec_{lgs}^{column} Y$, so $Xa \prec_{lgs} Ya$. But $Ya = 0$ and $Xa \neq 0$, which is a contradiction. □

For every $i, j \in \mathbb{N}_m$, consider the embedding $E^j : \mathbb{F}^m \to \mathbf{M}_{n,m}$ by $E^j(x) = xe_j$ and projection $E_i : \mathbf{M}_{n,m} \to \mathbb{F}^n$ by $E_i(A) = Ae_i$. It is easy to show that for every linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$,

$$TX = \left[\sum_{j=1}^{m} T_1^j x_j \middle| \dots \middle| \sum_{j=1}^{m} T_m^j x_j \right],$$

where $T_i^j = E_i \circ T \circ E^j$ and $X = [x_1 \mid \dots \mid x_m]$. If T preserves $\prec_{lgs}^{column}$, it is clear that $T_i^j : \mathbb{F}^n \to \mathbb{F}^n$ preserves $\prec_{lgs}$.
Now, we state the main theorem of this section.

Theorem 2.6. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves lgs-column majorization if and only if there exist $A_1, \dots, A_m \in \boldsymbol{M}_{n,m}$, $b_1, \dots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$, invertible matrices $B_1, \dots, B_m \in \boldsymbol{GD}_n$, and $S \in \boldsymbol{M}_m$ such that for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$ and for all $X = [x_1 \mid \dots \mid x_m] \in \boldsymbol{M}_{n,m}$,*

$$TX = \sum_{j=1}^{m} \mathrm{tr}(x_j) A_j + [B_1 X b_1 \mid \dots \mid B_m X b_m] + \boldsymbol{J} X S. \tag{2.1}$$

Proof. First, assume that the condition (2.1) holds. Suppose $X = [x_1 \mid \dots \mid x_m], Y = [y_1 \mid \dots \mid y_m] \in \mathbf{M}_{n,m}$ and $X \prec_{lgs}^{column} Y$. Since for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$, it is easy to see that $TXe_i \prec_{lgs} TYe_i$ and hence $TX \prec_{lgs}^{column} TY$. Conversely, assume that T preserves $\prec_{lgs}^{column}$. For every $i, j \in \mathbb{N}_m$, $T_i^j : \mathbb{F}^n \to \mathbb{F}^n$ preserves $\prec_{lgs}$. Then, each T_i^j is of the form (a) or (b) in Proposition 2.2. Let

$$\mathbf{I} = \{k \in \mathbb{N}_m : \exists l \in \mathbb{N}_m \text{ such that } T_k^l \text{ is of the form } (b)\ with\ \alpha_k^l \neq 0\}.$$

For every $k \in \mathbf{I}$ there exists $l \in \mathbb{N}_m$ such that $T_k^l x = \alpha_k^l B_k x + \beta_k^l \mathbf{J} x$ for some invertible matrix $B_k \in \mathbf{GD}_n$ and $\alpha_k^l \neq 0, \beta_k^l \in \mathbb{F}$.

We show that if $k \in \mathbf{I}$, then T_k^j is of form (b) with same invertible matrix $B_k \in \mathbf{GD}_n$, for every $j \in \mathbb{N}_m$.

Suppose $k \in \mathbf{I}$, then there exist $l \in \mathbb{N}_n$, $\alpha_k^l \neq 0, \beta_k^l \in \mathbb{F}$, invertible matrix $B_k \in \mathbf{GD}_n$ such that $T_k^l x = \alpha_k^l B_k x + \beta_k^l \mathbf{J}x$. For every $x, y \in \mathbb{F}^n$ define $X = xe_j + ye_l \in \mathbf{M}_{n,m}$. It is clear that $DX \prec_{lgs}^{column} X$, and hence $TDX \prec_{lgs}^{column} TX$, for all $D \in \mathbf{GD}_n$. This implies that $T_k^j Dx + T_k^l Dy \prec_{lgs} T_k^j x + T_k^l y$. Then by Propositions 2.3 and 2.4, there exist $\alpha_k^j, \beta_k^j \in \mathbb{F}$ such that $T_k^j x = \alpha_k^j B_k x + \beta_k^j \mathbf{J}x$. For $k \in \mathbf{I}$, set $b_k := (\alpha_k^1, \ldots, \alpha_k^m)^t$, $s_k := (\beta_k^1, \ldots, \beta_k^m)^t \in \mathbb{F}^m$ and for $k \in (\mathbb{N}_m \setminus \mathbf{I})$ set $b_k = s_k := 0 \in \mathbb{F}^m$. Define $S := [s_1 \mid \ldots \mid s_m] \in \mathbf{M}_m$.

If $k \notin \mathbf{I}$, then T_k^j is of form (a) for every $j \in \mathbb{N}_m$ and hence $T_k^j x = (\mathrm{tr} x) a_k^j$, for some $a_k^j \in \mathbb{F}^n$. For $k \in \mathbf{I}$, put $a_k^j = 0$ and define $A_j := [a_1^j \mid \ldots \mid a_m^j] \in \mathbf{M}_{n,m}$.

It is clear that for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$ and by a straightforward calculation one may show that for any $X = [x_1 \mid \ldots \mid x_m] \in \mathbf{M}_{n,m}$,

$$TX = \sum_{j=1}^{m} \mathrm{tr}(x_j) A_j + [B_1 X b_1 \mid \ldots \mid B_m X b_m] + \mathbf{J}XS.$$

If $b_j \notin \cup_{i=1}^m \mathrm{Span}\{e_i\}$ for some $j \in \mathbb{N}_m$, then Lemma 3.7 implies that T is not a linear preserver of $\prec_{lgs}^{column}$ which is a contradiction. Therefore $b_1, \ldots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$, as desired. □

The structure of strong linear preservers of lgs-column majorization is characterized as follows:

Theorem 2.7. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T strongly preserves lgs-column majorization if and only if there exist invertible matrices $B_1, \ldots, B_m \in \boldsymbol{GD}_n$, $S \in \boldsymbol{M}_m$ and, $b_1, \ldots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$ such that $D(D + nS)$ is invertible and*

$$TX = [B_1 X b_1 \mid \ldots \mid B_m X b_m] + \boldsymbol{J}XS, \tag{2.2}$$

where $D = [b_1 \mid \ldots \mid b_m]$.

Proof. The fact that the condition (2.2) is sufficient for T to be a strong linear preserver of $\prec_{lgs}^{column}$ is easy to prove. So, we prove the necessity of the conditions. Assume that T is a strong linear preserver of $\prec_{lgs}^{column}$. It can be easily seen that T is invertible. By Theorem 2.6, there exist $A_1, \ldots, A_m \in \mathbf{M}_{n,m}$, $b_1, \ldots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$, $S \in \mathbf{M}_m$, and invertible matrices $B_1, \ldots, B_m \in \mathbf{GD}_n$ such that for all $X = [x_1 \mid \ldots \mid x_m] \in \mathbf{M}_{n,m}$, $TX = \sum_{j=1}^n \mathrm{tr}(x_j) A_j + [B_1 X b_1 \mid \ldots \mid B_m X b_m] + \mathbf{J}XS$ and for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1 e_i = \cdots = A_m e_i = 0$. We show that for every $j \in \mathbb{N}_m$, $A_j = 0$. Assume that there exists $j \in \mathbb{N}_m$, such that $A_j \neq 0$. Without loss of generality suppose that $A_j e_1 \neq 0$, then $b_1 = 0$. Set $V := \text{Span}\{b_2, \ldots, b_m\}$, so $\dim V \leqslant m-1$. It follows that there exists $0 \neq s \in V^\perp$. Set $X := [s^t / -s^t / 0 / \ldots / 0] \in \mathbf{M}_{n,m}$. Then X is nonzero and $TX = 0$, which is a contradiction. Therefore $A_j = 0$, for every $j \in \mathbb{N}_m$.

Now, we prove (by contradiction) that D is invertible. Indeed, assume that D is not invertible. Choose a nonzero $s \in (\text{Span}\{b_1, \ldots, b_m\})^\perp$ and put $X := [s^t / -s^t / 0 / \ldots / 0] \in \mathbf{M}_{n,m}$. Then X is nonzero and $TX = 0$, which is a contradiction. Therefore D is invertible.

Finally, we show that $D + nS$ is invertible. Assume, by contradiction, that $D + nS$ is not invertible. Choose a nonzero $x \in \mathbb{F}_m$ such that $(D + nS)x = 0$ and put $X := [x / \ldots / x] \in \mathbf{M}_{n,m}$. Then X is nonzero and

$$TX = [B_1 X b_1 \mid \ldots \mid B_m X b_m] + \mathbf{J} X S = X(D + nS) = 0,$$

which is a contradiction. Therefore $D + nS$ is invertible and the proof is complete. □

Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Define $\tau : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ by $\tau X = (TX^t)^t$. It is easy to see that T is a (strong) linear preserver of $\prec_{rgs}^{row}$ if and only if τ is a (strong) linear preserver of $\prec_{lgs}^{column}$. Combining this fact and previous theorems, we have the following corollaries:

Corollary 2.8. *Let $T : \mathbb{F}_n \to \mathbb{F}_n$ be a linear operator. Then T preserves rgs-majorization if and only if one of the following statements holds:*

(*a*) $Tx = \text{tr}(x)a$, *for some* $a \in \mathbb{F}_n$*;*

(*b*) $Tx = \alpha x D + \beta x \boldsymbol{J}$, *for some* $\alpha, \beta \in \mathbb{F}$ *and invertible matrix* $D \in \boldsymbol{GD}_n$.

Corollary 2.9. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves rgs-row majorization if and only if there exist $A_1, \ldots, A_n \in \boldsymbol{M}_{n,m}$, $b_1, \ldots, b_n \in \cup_{i=1}^n \text{Span}\{e_i\}$, invertible matrices $B_1, \ldots, B_n \in \boldsymbol{GD}_m$, and $S \in \boldsymbol{M}_n$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i A_1 = \cdots = e_i A_n = 0$ and for all $X = [x_1 / \ldots / x_n] \in \boldsymbol{M}_{n,m}$,*

$$TX = \sum_{j=1}^{n} \text{tr}(x_j) A_j + [b_1 X B_1 / \ldots / b_n X B_n] + S X \boldsymbol{J}.$$

Corollary 2.10. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T strongly preserves rgs-row majorization if and only if there exist*

$B_1, \dots, B_n \in$ $\boldsymbol{GD}_m$, $S \in \boldsymbol{M}_n$ and $b_1, \dots, b_n \in \cup_{i=1}^n \mathrm{Span}\{e_i\}$ such that $D(D + mS)$ is invertible and

$$TX = [b_1 X B_1 / \dots / b_n X B_n] + SX\boldsymbol{J},$$

where $D = [b_1 / \dots / b_n]$.

3. Rgw and lgw-majorization on $\mathbf{M}_{n,m}$

In this section, we begin to study the structure of linear preservers of rgw and lgw-majorization on $\mathbf{M}_{n,m}$, and then the linear operators $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserving or strongly preserving rgw-row (lgw-column) majorization will be characterized.
In the following theorems we state some results from [2].
Proposition 3.1. [2, Theorem 2.3] *Let $T : \mathbb{F}_m \to \mathbb{F}_m$ be a linear operator. Then, T preserves $\prec_{rgw}$ if and only if one of the following statements holds:*

(i) $Tx = \alpha x B$, for some $\alpha \in \mathbb{F}$ and some invertible $B \in \boldsymbol{GR}_n$;

(ii) $Tx = \alpha x B$, for some $\alpha \in \mathbb{F}$ and some $B \in \boldsymbol{GR}_n$ such that $\{x : xB = 0\} = \{x : tr\,(x) = 0\}$.

Proposition 3.2. [2, Lemma 2.6] *Let $A \in \boldsymbol{M}_n$ and α be a nonzero scalar in $\mathbb{F}$. Then $A = \gamma\ \boldsymbol{I}$ for some $\gamma \in \mathbb{F}$ if and only if we have*

$$\alpha x R A + yR \prec_{rgw} \alpha x A + y,\ \forall x, y \in \mathbb{F}_m, \forall R \in \boldsymbol{GR}_m.$$

Lemma 3.3. *Let $A \in \boldsymbol{GR}_m$ be invertible and $0 \neq \alpha \in \mathbb{F}$. Define $T_1 : \mathbb{F}_m \to \mathbb{F}_m$ by $T_1 x = \alpha x A$ and suppose $T_2 : \mathbb{F}_m \to \mathbb{F}_m$ is a linear preserver of $\prec_{rgw}$ such that*

$$T_1 x R + T_2 y R \prec_{rgw} T_1 x + T_2 y,$$

for all $x, y \in \mathbb{F}_m$ and $R \in \boldsymbol{GR}_m$. Then there exists $\lambda \in \mathbb{F}$ such that $T_2 x = \lambda x A$.

Proof. Since T_2 preserves $\prec_{rgw}$, T_2 is of form (i) or (ii) in Proposition 3.1. Assume that T_2 is of form (ii), then $T_2 x = \mathrm{tr}(x) a$ for some nonzero $a \in \mathbb{F}_m$. Let $x = -\frac{1}{\alpha} a A^{-1}$, and set $y := e_1$. Then we have

$$\alpha x R A + \mathrm{tr}(yR) a \prec_{rgw} \alpha x A + \mathrm{tr}(y) a,$$

for all $R \in \mathbf{GR}_m$. It follows that

$$\alpha(-\frac{1}{\alpha} a A^{-1}) R A + \mathrm{tr}(e_1 R) a \prec_{rgw} \alpha(-\frac{1}{\alpha} a A^{-1}) A + \mathrm{tr}(e_1) a = -a + a = 0.$$

So $-aA^{-1}RA + a = 0$, for all $R \in \mathbf{GR}_m$. Thus $aR = a$, for all $R \in \mathbf{GR}_m$, and hence $a = 0$, a contradiction. Therefore, $T_2 x = \beta x A_2$, for some

$\beta \in \mathbb{F}$ and invertible matrix $A_2 \in \mathbf{GR}_m$. Now, by Proposition 3.2, $T_2 x = \lambda x A$, for some $\lambda \in \mathbb{F}$. $\square$

For every $i, j \in \mathbb{N}_n$ consider the embedding $E^j : \mathbb{F}_m \to \mathbf{M}_{n,m}$ and the projection $E_i : \mathbf{M}_{n,m} \to \mathbb{F}_m$, where $E^j(x) = e_j x$ and $E_i(A) = e_i A$. It is easy to prove that for every linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$, $TX = T[x_1/\cdots/x_n] = \left[\sum_{j=1}^{n} T_1^j x_j/\cdots/\sum_{j=1}^{n} T_n^j x_j\right]$, where x_i is the i^{th} row of X and $T_i^j = E_i \circ T \circ E^j$. If $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves rgw-majorization, then its easy to see that $T_i^j : \mathbb{F}_m \to \mathbb{F}_m$ preserves rgw-majorization.

Now, we find the possible structure of linear operators preserving $\prec_{rgw}$ on $\mathbf{M}_{n,m}$.

Theorem 3.4. *If a linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ preserves rgw-majorization, then there exist $\mathcal{A} \in \boldsymbol{M}_n(\mathbb{F}_m)$, $b_1, \ldots, b_n \in \mathbb{F}_n$, and invertible matrices $A_1, \ldots, A_n \in \boldsymbol{GR}_m$, such that*

$$TX = m\mathcal{A}\overline{X} + [b_1 X A_1/\ldots/b_n X A_n], \ \forall X \in \boldsymbol{M}_{n,m}.$$

Proof. For every $p \in \mathbb{N}_n$, one of the following cases holds:

Case 1: there exists $q \in \mathbb{N}_n$ such that $T_p^q x = \alpha x A_p$ for some $0 \neq \alpha \in \mathbb{F}$ and invertible $A_p \in \mathbf{GR}_m$. We show that for all $j \in \mathbb{N}_n$, $T_p^j x = \lambda_p^j x A_p$, for some $\lambda_p^j \in \mathbb{F}$. For $x, y \in \mathbb{F}_m$ put $X = e_p x + e_j y$. It is clear that $XR \prec_{rgw} X$, for all $R \in \mathbf{GR}_m$, therefore $TXR \prec_{rgw} TX$, for all $R \in \mathbf{GR}_m$ and hence,

$$T_p^q xR + T_p^j yR \prec_{rgw} T_p^q x + T_p^j y, \ \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.$$

Use Lemma 3.3 to conclude that $T_p^j x = \lambda_p^j x A_p$, for some $\lambda_p^j \in \mathbb{F}$. Put

$$b_p := (\lambda_p^1, \ldots, \lambda_p^n) \in \mathbb{F}_n$$

and $\mathcal{A}_{(p)} = 0 \in \mathbb{F}_n(\mathbb{F}_m)$.

Case 2: For every $q \in \mathbb{N}_n$, T_p^q is of form (ii) in Proposition 3.1. Then $T_p^q x = \mathrm{tr}(x) a_p^q$ for some $a_p^q \in \mathbb{F}_m$. Put $\mathcal{A}_{(p)} = [a_p^1 \ldots a_p^n] \in \mathbb{F}_n(\mathbb{F}_m)$ and $b_p = 0 \in \mathbb{F}_m$. Now, Let $\mathcal{A} = [\mathcal{A}_{(1)}/\ldots/\mathcal{A}_{(n)}]$. Then

$$\begin{aligned} TX &= T[x_1/\dots/x_n] \\ &= \left[\sum_{j=1}^{n} T_1^j x_j/\dots/\sum_{j=1}^{n} T_n^j x_j\right] \\ &= [b_1XA_1/\dots/b_nXA_n] + m\mathcal{A}\overline{X}, \end{aligned}$$

where $\mathcal{A} \in \mathbf{M}_n(\mathbb{F}_m)$, $b_1, \dots, b_n \in \mathbb{F}_n$, and $A_1, \dots, A_n \in \mathbf{GR}_m$ are invertible matrices. □

Corollary 3.5. *Let* $\{b_1, \dots, b_n\} \subset \mathbb{F}_n$ *and* $\dim(\mathrm{Span}\{b_1, \dots, b_n\}) \geq 2$. *Assume that* $A_1, \dots, A_n \in \boldsymbol{GR}_m$ *are invertible and define* $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ *by* $TX = [b_1XA_1/\dots/b_nXA_n]$. *If* T *preserves* $\prec_{rgw}$, *then there exist* $B \in \boldsymbol{M}_m$ *and invertible* $A \in \boldsymbol{GR}_m$ *such that* $TX = BXA$.

Proof. Without loss of generality we can assume that $\{b_1, b_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$, $R \in \mathbf{GR}_m$ be arbitrary. Then $XR \prec_{rgw} X$, and hence $TXR \prec_{rgw} TX$. It follows that
$[b_1XRA_1/\dots/b_nXRA_n] \prec_{rgw} [b_1XA_1/\dots/b_nXA_n]$
$\Rightarrow b_1XRA_1 + b_2XRA_2 \prec_{rgw} b_1XA_1 + b_2XA_2$
$\Rightarrow b_1XR + b_2XR(A_2A_1^{-1}) \prec_{rgw} b_1X + b_2X(A_2A_1^{-1})$.

Since $\{b_1, b_2\}$ is linearly independent, for every $x, y \in \mathbb{F}^n$, there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $b_1B_{x,y} = x$ and $b_2B_{x,y} = y$. Put $X = B_{x,y}$ in the above relation. Thus,

$$xR + yR(A_2A_1^{-1}) \prec_{rgw} x + y(A_2A_1^{-1}), \forall R \in \mathbf{GR}_m, \forall x, y \in \mathbb{F}_m.$$

Then by Proposition 3.2, $(A_2A_1^{-1}) = \alpha\ \mathbf{I}$ and hence $A_2 = \alpha A_1$, for some $0 \neq \alpha \in \mathbb{F}$. For every $i \geq 3$, if $b_i = 0$ we can choose $A_i = A_1$; if $b_i \neq 0$ then $\{b_1, b_i\}$ or $\{b_2, b_i\}$ is linearly independent. By the same argument as above, we conclude that $A_i = \gamma_i A_1$, for some $0 \neq \gamma_i \in \mathbb{F}$, or $A_i = \lambda_i A_2$, for some $0 \neq \lambda_i \in \mathbb{F}$.
Define $A = A_1$. Then for every $i \geq 2$, $A_i = \alpha_i A$, for some $\alpha_i \in \mathbb{F}$ and we get

$$TX = [b_1XA/(r_2b_2)XA/\dots/(r_nb_n)XA] = BXA,$$

where $B = [b_1 \mid r_2b_2/\dots/r_nb_n]$, for some $r_2, \dots, r_m \in \mathbb{F}$. □

If $A \in \mathbf{GR}_m$ is invertible and $B \in \ \mathbf{M}_n$, it is easy to see that $X \mapsto BXA$ is a linear preserver of $\prec_{rgw}$. But the following example shows that there exist linear preservers of $\prec_{rgw}$ which are not of this form.

Example 3.6. Let $T : \boldsymbol{M}_2 \to \boldsymbol{M}_2$ be such that
$TX = \begin{pmatrix} x_{11} & x_{12} \\ -x_{11} - x_{12} & x_{11} + x_{22} \end{pmatrix}$ where $X = [x_{ij}]$.
We show that T preserves $\prec_{rgw}$ but T is not of the form $X \mapsto MXA$.
Let $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, and suppose that $X \prec_{rgw} Y$. If $y_{11} + y_{12} = 0$, so $x_{11} + x_{12} = 0$, and $TX \prec_{rgw} TY$. Let $y_{11} + y_{12} \neq 0$. Without loss of generality assume that $y_{11} + y_{12} = 1$. Since $X \prec_{rgw} Y$, there exists $R \in \boldsymbol{GR}_2$, such that $X = YR$. Let $R = \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}$ and $y = (\lambda, 1-\lambda)$. Put $S := \begin{pmatrix} \alpha & 1-\alpha \\ \alpha - 1 & 2-\alpha \end{pmatrix}$, where $\alpha = \lambda(a-b) + b - \lambda + 1$. Therefore $S \in \boldsymbol{GR}_2$ and $TYS = TX$. So $TX \prec_{rgw} TY$. By a straightforward calculation one may show that T is not of the form $X \mapsto BXA$.

The proof of the following lemma is similar to the proof of Lemma 2.5.

Lemma 3.7. *Let $a \in \mathbb{F}_n$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [aX/\dots/aX]$ preserves $\prec_{rgw}^{row}$ if and only if $a \in \cup_{i=1}^{n} \mathrm{Span}\{e_i\}$.*

The structure of linear preservers and strong linear preservers of rgw-row majorization is characterized as follows:

Theorem 3.8. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves rgw-row majorization if and only if there exist $\mathcal{A} \in \boldsymbol{M}_n(\mathbb{F})$, $b_1, \dots, b_n \in \cup_{i=1}^{n} \mathrm{Span}\{e_i\}$, and invertible matrices $A_1, \dots, A_n \in \boldsymbol{GR}_m$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $\mathcal{A}_{(i)} = 0$, where $\mathcal{A} = [\mathcal{A}_{(1)}/\dots/\mathcal{A}_{(n)}]$ and*

$$TX = m\mathcal{A}\overline{X} + [b_1XA_1/\dots/b_nXA_n]\,. \tag{3.1}$$

Proof. The fact that the condition (3.1) is sufficient for T to be a linear preserver of $\prec_{rgw}^{row}$ is easy to prove. So, we prove the necessity of the condition. Therefore, assume that T preserves $\prec_{rgw}^{row}$. For every $i, j \in \mathbb{N}_n$, $T_i^j : \mathbb{F}_m \to \mathbb{F}_m$ preserves $\prec_{rgw}$. Then, each T_i^j is of the form (i) or (ii) in Proposition 3.1. Let

$$\mathbf{I} = \{k \in \mathbb{N}_n : \exists l \in \mathbb{N}_n \text{ such that } T_k^l \text{ is of the form } (ii) \text{ with } \alpha_k^l \neq 0\}.$$

We show that if $k \in \mathbf{I}$, then T_k^j is of form (ii) of Proposition 3.1, with the same invertible matrix $A_k \in \mathbf{GR}_m$, for every $j \in \mathbb{N}_n$. Suppose $k \in \mathbf{I}$, then there exist $l \in \mathbb{N}_n$, $0 \neq \alpha_k^l \in \mathbb{F}$ and invertible matrix $A_k \in \mathbf{GR}_m$

such that $T_k^l x = \alpha_k^l x A_k$. Set $X = e_l x + e_j y$. It is clear that $XR \prec_{rgw}^{row} X$ and hence $TXR \prec_{rgw}^{row} TX$ for all $R \in \mathbf{GR}_m$. This implies that

$$T_k^l xR + T_k^j yR \prec_{rgw} T_k^l x + T_k^j y,\ \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.$$

So by Proposition 3.2, there exists $\alpha_k^j \in \mathbb{F}$ such that $T_k^j x = \alpha_k^j x A_k$. Set $b_k := (\alpha_k^1, \ldots, \alpha_k^n)$ if $k \in \mathbf{I}$, and $b_k = 0$ if $k \notin \mathbf{I}$.
If $k \notin \mathbf{I}$, then T_k^j is of form (i) of Proposition 3.1, for every $j \in \mathbb{N}_n$ and hence $T_k^j x = m a_k^j \overline{x}$ where $a_k^j \in \mathbb{F}_m$. If $k \in \mathbf{I}$, put $a_k^j = 0$ for every $j \in \mathbb{N}_n$. For $k \in \mathbb{N}_n$ define $\mathcal{A}_{(k)} = [a_k^1 \ldots a_k^n]$.
It is clear that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $\mathcal{A}_{(i)} = 0$. Let $\mathcal{A} = [\mathcal{A}_{(1)}/\ldots/\mathcal{A}_{(n)}]$. Then $TX = [\sum_{j=1}^n T_1^j x_j/\ldots/\sum_{j=1}^n T_n^j x_j] = m\mathcal{A}\overline{X} + [b_1 X A_1/\ldots/b_n X A_n]$. To complete the proof we must apply Lemma 3.7 to conclude that $b_i \in \mathrm{Span}\{e_i\}$ for every $i \in \mathbb{N}_n$.

□

Theorem 3.9. *A linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ is a strong linear preserver of rgw-row majorization if and only if there exist invertible matrices $A_1, \ldots, A_n \in \boldsymbol{GR}_m$ and $b_1, \ldots, b_n \in \cup_{i=1}^n \mathrm{Span}\{e_i\}$ such that $B := [b_1/\ldots/b_n]$ is invertible and*

$$TX = [b_1 X A_1/\ldots/b_n X A_n].$$

Proof. Assume that there exists a $k \in (\mathbb{N}_n \backslash \mathbf{I})$. Without loss of generality let $1 \in (\mathbb{N}_n \setminus \mathbf{I})$, so $b_1 = 0$. Set $V := \mathrm{Span}\{b_2, \ldots, b_n\}$, then $\dim V \leqslant n-1$. It follows that $\dim V^\perp \geq 1$ and there exists $0 \neq s \in V^\perp$. Set $X := [s \mid -s \mid 0 \mid \ldots \mid 0]$. Therefore X is nonzero and for every $i \in \mathbb{N}_n$, $b_i X = 0$ so $TX = 0$, which is a contradiction. Then $I = \mathbb{N}_n$ and $TX = [b_1 X A_1/\ldots/b_n X A_n]$.
Now, we show that B is invertible. If B is not invertible, set $V := \mathrm{Span}\{b_1, \ldots, b_n\}$. So $\dim V \leqslant n-1$. Therefore $\dim V^\perp \geq 1$ and there exists $0 \neq s \in V^\perp$. Set $X := [s \mid -s \mid 0 \mid \ldots \mid 0]$. Then X is nonzero and $TX = 0$, which is a contradiction. □

In the remainder of this section we characterize linear operators that preserve or strongly preserve lgw or lgw-column majorization. We begin with a theorem of [5].

Theorem 3.10. *[5, Theorem 2.4] A linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ preserves lgw-majorization if and only if one of the following assertions holds:*

(*i*) *There exists $R \in \boldsymbol{M}_n$ such that* $\mathrm{Ker}(R) = \mathrm{Span}\{e\}$, $e \notin \mathrm{Im}(R)$, *and $Tx = Rx$ for every $x \in \mathbb{F}^n$;*

(ii) There exist an invertible matrix $R \in \boldsymbol{GR}_n$ and $\alpha \in \mathbb{F}$ such that $Tx = \alpha Rx$ for every $x \in \mathbb{F}^n$.

Corollary 3.11. *A linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ preserves lgw-majorization if and only if one of the following assertions holds:*
(i) there exists an invertible matrix $D \in \boldsymbol{GR}_n$, such that $Tx = \left(D - \frac{1}{n}\ \boldsymbol{J}\right) x$ for every $x \in \mathbb{F}^n$;
(ii) There exist an invertible matrix $R \in \boldsymbol{GR}_n$ and $\alpha \in \mathbb{F}$ such that $Tx = \alpha Rx$ for every $x \in \mathbb{F}^n$.

Proof. Let $R \in \mathbf{M}_n$. We show that $\mathrm{Ker}(R) = \mathrm{Span}\{e\}$ and $e \notin \mathrm{Im}(R)$ if and only if $R = (D - \frac{1}{n}\mathbf{J})$ for some invertible matrix $D \in \mathbf{GR}_n$.

First, Let $R = (D - \frac{1}{n}\mathbf{J})$ for some invertible matrix $D \in \mathbf{GR}_n$. It is clear that $\mathrm{Span}\{e\} \subset \mathrm{Ker}(R)$. If $x \in \mathrm{Ker}(R)$, then $Dx = \frac{1}{n}\mathrm{tr(x)}e$ and $x \in \mathrm{Span}\{e\}$. Therefore $\mathrm{Ker}(R) = \mathrm{Span}\{e\}$. Assume that $e \in \mathrm{Im}(R)$, then $(D - \frac{1}{n}\mathbf{J})x = e$ for some $x \in \mathbb{R}^n$. It implies that $Dx = \left(\frac{1}{n}\mathrm{tr(x)} + 1\right) e$ and hence $x \in \mathrm{Span}\{e\}$, which is a contradiction. So $e \notin \mathrm{Im}(R)$.

Conversely. Let $\mathrm{Ker}(R) = \mathrm{Span}\{e\}$ and $e \notin \mathrm{Im}(R)$. Put $D := R + \frac{1}{n}\mathbf{J}$. Since $Re = 0$, $D \in \mathbf{GR}_n$. It is enough to show that D is invertible. If $Dx = 0$ then $Rx = \left(-\frac{1}{n}\mathrm{tr(x)}\right) e$. If $\mathrm{tr(x)} \neq 0$, then $e \in \mathrm{Im}(R)$ which is a contradiction, so $\mathrm{tr(x)} = 0$ and $Rx = 0$. Therefore $x \in \mathrm{Span}\{e\}$, which implies that $x = 0$. □

Lemma 3.12. *Let $A \in \boldsymbol{GR}_n$ be invertible. Then the following conditions are equivalent:*
(a) $A = \alpha\ \boldsymbol{I} + \beta \boldsymbol{J}$, for some $\alpha, \beta \in \mathbb{R}$;
(b) $Dx + ADy \prec_{lgw} x + Ay$, for all $D \in\ \boldsymbol{GD}_n$ and for all $x, y \in \mathbb{R}^n$.

Proof. $(a \Rightarrow b)$ If $A = \alpha\mathbf{I} + \beta\mathbf{J}$, it is easy to show that $Dx + ADy \prec_{lgw} x + Ay$, for all $D \in\ \mathbf{GD}_n$ and for all $x, y \in \mathbb{F}^n$.

$(b \Rightarrow a)$ The matrix A is invertible, so condition (b) can be written as follows:

$$Dx + ADA^{-1}y \prec_{lgw} x + y,\ \forall D \in\ \mathbf{GD}_n,\ \forall x, y \in \mathbb{F}^n.$$

Put $x = e - e_i$ and $y = e_i$ in the above relation. Thus, $[e - (D - ADA^{-1})e_i] \prec_{lgw} e$, for every $i \in \mathbb{N}_n$. So $(D - ADA^{-1})e_i = 0$, for every

$i \in \mathbb{N}_n$, and $DA = AD$, for every $D \in \mathbf{GD}_n$. Therefore, $A = \alpha\mathbf{I} + \beta\,\mathbf{J}$, for some $\alpha, \beta \in \mathbb{F}$.

□

Theorem 3.13. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator that preserves lgw-majorization. Then, there exist invertible matrices $A_1, \ldots, A_m \in \boldsymbol{GR}_n$, $b_1, \ldots, b_m \in \mathbb{F}^m$ and $S \in \boldsymbol{M}_m$ such that*

$$TX = [A_1 X b_1 \mid \ldots \mid A_m X b_m] + \boldsymbol{J}XS.$$

Proof. Suppose that T preserves lgw-majorization. It is easy to prove that $T_i^j : \mathbb{F}^n \to \mathbb{F}^n$ preserves lgw-majorization. Then by Corollary 3.11, for every $i, j \in \mathbb{N}_m$, $T_i^j x = (\alpha_i^j A_i^j - \frac{1}{n}\gamma_i^j\ \mathbf{J})x$, for some invertible matrices $A_i^j \in \mathbf{GR}_n$, $\alpha_i^j \in \mathbb{F}$ *and* $\gamma_i^j \in \{0,1\}$. Then

$$\begin{aligned} TX &= T[x_1|\ldots|x_m] \\ &= \left[\sum_{j=1}^{m} T_1^j x_j \middle| \ldots \middle| \sum_{j=1}^{m} T_m^j x_j\right] \\ &= \left[\sum_{j=1}^{m}(\alpha_1^j A_1^j - \frac{1}{n}\gamma_1^j J)x_j \middle| \ldots \middle| \sum_{j=1}^{m}(\alpha_m^j A_m^j - \frac{1}{n}\gamma_m^j \mathbf{J})x_j\right]. \end{aligned}$$

For every $x, y \in \mathbb{F}^n$, define $X = E^j(x) + E^q(y) \in \mathbf{M}_{n,m}$. If $\alpha_i^q = 0$ for every $i \in \mathbb{N}_m$, then put $A_i^q = I$. Now, suppose that there exists some $p \in \mathbb{N}_m$ such that $\alpha_p^q \neq 0$. Then for every $D \in \mathbf{GD}_n$, $DX \prec_{lgw} X$, and hence $[\alpha_1^q A_1^q Dx + \alpha_1^j A_1^j Dy|\ldots|\alpha_m^q A_m^q Dx + \alpha_m^j A_m^j Dy] \prec_{lgw}$ $[\alpha_1^q A_1^q x + \alpha_1^j A_1^j y|\ldots|\alpha_m^q A_m^q x + \alpha_m^j A_m^j y]$
$\Rightarrow \alpha_p^q A_p^q Dx + \alpha_p^j A_p^j Dy \prec_{lgw} \alpha_p^q A_p^q x + \alpha_p^j A_p^j y$
$\Rightarrow Dx + (A_p^q)^{-1} A_p^j D(\frac{\alpha_p^j}{\alpha_p^q} y) \prec_{lgw} x + (A_p^q)^{-1} A_p^j (\frac{\alpha_p^j}{\alpha_p^q} y)$.

So by Lemma 3.12, $(A_p^q)^{-1} A_p^j = \lambda_p^j \mathbf{I} + \beta_p^j \mathbf{J}$. Set $A_p := A_p^q$, then $A_p^j = \lambda_p^j A_p + \beta_p^j \mathbf{J}$. Therefore for some $\mu_i^j \in \mathbb{F}$ we have

$$TX = \left[A_1 \sum_{j=1}^{m} \mu_1^j x_j \middle| \ldots \middle| A_p \sum_{j=1}^{m} \mu_p^j x_j \middle| \ldots \middle| A_m \sum_{j=1}^{m} \mu_m^j x_j\right] + \mathbf{J}XS,$$

where

$$S = \begin{pmatrix} -\frac{1}{n}\gamma_1^1 + \beta_1^1 \dots & -\frac{1}{n}\gamma_m^1 + \beta_m^1 \\ \vdots & \vdots \\ -\frac{1}{n}\gamma_1^m + \beta_1^m \dots & -\frac{1}{n}\gamma_m^m + \beta_m^m \end{pmatrix}.$$

Now, For every $i \in \mathbb{N}_m$, define

$$b_i = \begin{pmatrix} \mu_i^1 \\ \mu_i^2 \\ \vdots \\ \mu_i^m \end{pmatrix}.$$

Then,

$$TX = [A_1 X b_1 \mid \dots \mid A_m X b_m] + \mathbf{J} XS.$$

□

Corollary 3.14. *Let T satisfy the condition of Theorem 3.13 and let* $\mathrm{rank}[\mathrm{b_1} \mid \dots \mid \mathrm{b_m}] \geq 2$. *Then $TX = AXR + \boldsymbol{J}XS$, for some $R, S \in \boldsymbol{M}_m$, and invertible matrix $A \in \boldsymbol{GR}_n$.*

Proof. Without loss of generality we can assume that $\{b_1, b_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$, $D \in \mathbf{GD}_n$ be arbitrary. Then $DX \prec_{lgw} X$ and hence, $TDX \prec_{lgw} TX$. It follows that
$[A_1 DXb_1 \mid \dots \mid A_m DXb_m] \prec_{lgw} [A_1 Xb_1 \mid \dots \mid A_m Xb_m]$
$\Rightarrow A_1 DXb_1 + A_2 DXb_2 \prec_{lgw} A_1 Xb_1 + A_2 Xb_2$
$\Rightarrow DXb_1 + (A_1^{-1} A_2) DXb_2 \prec_{lgw} Xb_1 + (A_1^{-1} A_2) Xb_2.$

Since $\{b_1, b_2\}$ is linearly independent, for every $x, y \in \mathbb{R}^n$, there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $B_{x,y} b_1 = x$ and $B_{x,y} b_2 = y$. Put $X := B_{x,y}$ in the above relation. Thus,

$$DB_{x,y}b_1 + (A_1^{-1}A_2)DB_{x,y}b_2 \prec_{lgw} B_{x,y}b_1 + (A_1^{-1}A_2)B_{x,y}b_2 \Rightarrow$$
$$Dx + (A_1^{-1}A_2)Dy \prec_{lgw} x + (A_1^{-1}A_2)y, \forall D \in \mathbf{GD}_n.$$

Then by Lemma 3.12, $A_1^{-1}A_2 = \alpha\ \mathbf{I} + \beta \mathbf{J}$ and hence $A_2 = \alpha A_1 + \beta\ \mathbf{J}$, for some $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$. For every $i \geq 3$, if $b_i = 0$ we can choose $A_i = A_1$. If $b_i \neq 0$ then $\{b_1, b_i\}$ or $\{b_2, b_i\}$ is linearly independent. Then by the same argument as above, $A_i = \gamma_i A_1 + \delta_i \mathbf{J}$, for some $\gamma_i, \delta_i \in \mathbb{F}$, $\gamma_i \neq 0$, or $A_i = \lambda_i A_2 + \mu_i \mathbf{J}$, for some $\lambda_i, \mu_i \in \mathbb{F}$, $\lambda_i \neq 0$.
Define $A := A_1$. Then for every $i \geq 2$, $A_i = \alpha_i A_2 + \beta_i \mathbf{J}$, for some $\alpha_i, \beta_i \in \mathbb{F}$ and hence

$$TX = [AXb_1 \mid AX(r_2 b_2) \mid \dots \mid AX(r_m b_m)] + \mathbf{J}XS = AXR + \mathbf{J}XS,$$

where, $R = [b_1 \mid r_2b_2 \mid \ldots \mid r_mb_m]$, for some $r_2, \ldots, r_m \in \mathbb{F}$ and S is as in Theorem 3.13. $\square$

Lemma 3.15. *Let $b_1, \ldots, b_m \in \mathbb{F}^m$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [Xb_1| \ldots \mid Xb_m]$ preserves $\prec_{lgw}^{column}$ if and only if $b_j \in \cup_{i=1}^{n} \mathrm{Span}\{e_i\}$, for every $j \in \mathbb{N}_m$.*

The following theorems give the structure of linear and strong linear preserver of $\prec_{lgw}^{column}$ on $\mathbf{M}_{n,m}$. Since the proofs are similar to the proofs of Theorems 2.6 and 2.7, we leave the proofs to the readers.

Theorem 3.16. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves $\prec_{lgw}^{column}$ if and only if there exist invertible matrices $A_1, \ldots, A_m \in \boldsymbol{GR}_n$, $b_1, \ldots, b_m \in \cup_{i=1}^{m} \mathrm{Span}\{e_i\}$ and $D \in \boldsymbol{M}_m$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $A_1e_i = \ldots = A_me_i = 0$ and for all $X = [x_1 \mid \ldots \mid x_n] \in \boldsymbol{M}_{n,m}$, $TX = [A_1Xb_1 \mid \ldots \mid A_mXb_m] + \boldsymbol{J}XD$.*

Theorem 3.17. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T strongly preserves lgw-column majorization if and only if there exist invertible matrices $A_1, \ldots, A_m \in \boldsymbol{GR}_n$ and $b_1, \ldots, b_m \in \cup_{i=1}^{m} \mathrm{Span}\{e_i\}$ such that $B := [b_1 \mid \ldots \mid b_m]$ is invertible and*

$$TX = [A_1Xb_1 \mid \ldots \mid A_nXb_m].$$

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