

TUTTE POLYNOMIALS OF WHEELS VIA GENERATING FUNCTIONS

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ABSTRACT. We find an explicit expression of the Tutte polynomial of an n -fan. We also find a formula of the Tutte polynomial of an n -wheel in terms of the Tutte polynomial of n -fans. Finally, we give an alternative expression of the Tutte polynomial of an n -wheel and then prove the explicit formula for the Tutte polynomial of an n -wheel.

1. Introduction

The Tutte polynomial of a graph was originally defined by Tutte in 1954 as an extension of the chromatic polynomial, see [5, 6]. It is related to many difficult problems in a wide variety of areas such as linear coding, knot theory, etc., see [2]. Computing the Tutte polynomial of a graph is known to be $\#P$ hard, see [3], hence there is much interest to find their explicit expressions for families of graphs. Although, some families have simple expressions for the chromatic polynomial, the same cannot be said about the Tutte polynomial. For example, the family of complete graphs has fairly simple chromatic polynomial, see [4], whereas the only known expression for the Tutte polynomial of the same family, given in [7], is somewhat more complicated.

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An n -wheel is another family of graphs, denoted by W_n , with a simple chromatic polynomial [4]. In [1], the Tutte polynomial of an n -wheel was defined recursively and the auxiliary equation for the recurrence was used to write down an explicit formula of the Tutte polynomial of the wheel involving the roots of this equation as

$$(1.1) \quad T(W_n; x, y) = xy - x - y - 1 + 2^{-n}[(x + y + 1 + \sqrt{\beta})^n + (x + y + 1 - \sqrt{\beta})^n]$$

where $\beta = (x + y + 1)^2 - 4xy$.

The motivation of the paper is to find an alternative explicit formula of the Tutte polynomial of a wheel without using a computer program. The formula will be found using a completely different approach from the one used in [1]. Having done the computation, we show that the formula in [1], which was given without proof, can be derived from our formula. In the paper, we therefore prove this formula.

Here we provide a proof to the formula (1.1) for $T(W_n; x, y)$. We begin by finding a formula for the Tutte polynomial of an n -fan, F_n

$$\begin{aligned} T(F_n; x, y) &= \sum_{a=0}^{n-1} \sum_{j=0}^{n-1} \binom{a+j}{j} \left[\binom{n-2-j}{a-1} x + \binom{n-2-j}{a} \right] x^j y^{n-1-a-j}. \end{aligned}$$

We then define the Tutte polynomial of an n -wheel recursively in terms of the Tutte polynomial of n -fans. This allows us to find an alternative explicit formula for the Tutte polynomial of an n -wheel, which is as follows

$$\begin{aligned} T(W_n; x, y) &= \sum_{m=2}^n T(F_m; x, y) + yT(F_1; x, y) \\ &\quad + \sum_{m=2}^{n-1} \sum_{i=1}^m \sum_{k=i}^{n-m+i-1} y^k T(F_{m-i}; x, y) + \sum_{k=1}^{n-1} y^{k+1}. \end{aligned}$$

Finally, with the use of generating functions, we recover and hence prove equation (1.1).

2. Tutte polynomials of fans

In this section, we give the recursive method and the explicit expression of the Tutte polynomial of an n -fan.

An n -fan, denoted by F_n , is defined as $F_n = P_n + K_1$. It follows that $F_0 = K_1$.

Our subsequent proofs involve the graph G_n^j . It is defined as an n -fan with one bunch of j parallel edges on the outermost edge e_1 as shown in Figure 1.

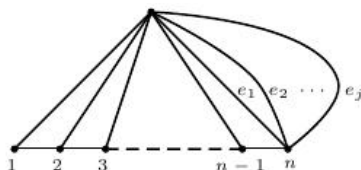


FIGURE 1. The graph G_n^j

Using the deletion and contraction method of the Tutte polynomial, we have the following recursion.

Theorem 2.1. *Let F_n be an n -fan. For $n \geq 1$, the Tutte polynomial of F_n is given by*

$$T(F_n; x, y) = x^n + \sum_{j=1}^{n-1} x^{n-1-j} T(G_j^2; x, y).$$

Proof. By successive deletion and contraction of the edges on the path P_n in F_n , we obtain the required result. \square

We need the following lemmas to get an explicit formula for Tutte polynomial of an n -fan. Lemma 2.1 gives the Tutte polynomial of the graph G_n^2 in terms of the Tutte polynomial of n -fans.

Lemma 2.2. *The Tutte polynomial of the graph G_n^2 is given by*

$$T(G_n^2; x, y) = \sum_{j=0}^n y^j T(F_{n-j}; x, y).$$

Proof. The formula is obtained by deleting and contracting one of the parallel edges then removing the loop and doing the process n times. \square

We will define $T(F_0; x, y) = T(K_1; x, y) = 1$. Lemma 2.3 gives the relationship between the Tutte polynomials of F_n and G_n^2 in terms of generating functions.

Lemma 2.3. *Let*

$$G_0(z) = \sum_{n \geq 1} T(G_n^2; x, y) z^n \text{ and } F(z) = \sum_{n \geq 1} T(F_n; x, y) z^n,$$

then

$$\begin{aligned} F(z) &= \frac{xz}{1-xz} + \frac{z}{1-xz} G_0(z), \\ G_0(z) &= \frac{1}{1-yz} F(z) + \frac{zy}{1-yz}. \end{aligned}$$

Proof. By the use of Theorem 2.1, we deduce the following

$$\begin{aligned} F(z) &= \sum_{n \geq 1} T(F_n; x, y) z^n \\ &= \sum_{n \geq 1} x^n z^n + \sum_{n \geq 2} \sum_{j=1}^{n-1} (x^{n-1-j} T(G_j^2; x, y)) z^n \\ &= \frac{xz}{1-xz} + T(G_1^2; x, y) \frac{z^2}{1-xz} + T(G_2^2; x, y) \frac{z^3}{1-xz} + \cdots \\ &= \frac{xz}{1-xz} + \frac{z}{1-xz} \sum_{j=1} T(G_j^2; x, y) z^j \\ &= \frac{xz}{1-xz} + \frac{z}{1-xz} G_0(z). \end{aligned}$$

By the use of Lemma 2.2, we deduce that

$$\begin{aligned} G_0(z) &= \sum_{n \geq 1} T(G_n^2; x, y) z^n \\ &= \sum_{n \geq 1} T(F_n; x, y) z^n + \sum_{n \geq 2} \sum_{j=1}^{n-1} (y^j T(F_{n-j}; x, y)) z^n + \sum_{n \geq 1} y^n z^n \\ &= F(z) + \frac{yz}{1-yz} + yF_1 \frac{z^2}{1-yz} + yF_2 \frac{z^3}{1-yz} + \cdots \\ &= F(z) + \frac{yz}{1-yz} + \frac{yz}{1-yz} F(z) \\ &= \frac{1}{1-yz} F(z) + \frac{yz}{1-yz}, \end{aligned}$$

as claimed. \square

We are now in a position to give the explicit formula for the Tutte polynomial of F_n .

Theorem 2.4. *Let F_n be an n -fan. The Tutte polynomial of an n -fan is,*

$$\begin{aligned} T(F_n; x, y) &= \sum_{a=0}^{n-1} \sum_{j=0}^{n-1} \binom{a+j}{j} \left[\binom{n-2-j}{a-1} x + \binom{n-2-j}{a} \right] x^j y^{n-1-a-j}. \end{aligned}$$

Proof. By the use of Lemma 2.3, we deduce the following

$$\begin{aligned} F(z) &= \frac{xz}{1-xz} + \frac{z}{1-xz} G_0(z) \\ &= \frac{xz}{1-xz} + \frac{z}{1-xz} \left[\frac{1}{1-yz} F(z) + \frac{yz}{1-yz} \right] \end{aligned}$$

which implies that

$$\begin{aligned} (2.1) \quad F(z) &= \frac{\frac{xz}{1-xz} + \frac{z^2 y}{(1-xz)(1-yz)}}{1 - \frac{z}{(1-xz)(1-yz)}} \\ &= \left(\frac{xz}{1-xz} + \frac{z^2 y}{(1-xz)(1-yz)} \right) \sum_{a \geq 0} \frac{z^a}{(1-xz)^a (1-yz)^a} \\ &= \sum_{a \geq 0} \frac{xz^{a+1}}{(1-xz)^{a+1} (1-yz)^a} + \sum_{a \geq 0} \frac{yz^{a+2}}{(1-xz)^{a+1} (1-yz)^{a+1}} \\ &= \sum_{a \geq 0} \sum_{j \geq 0} \sum_{i \geq 0} xz^{a+1} \binom{a+j}{j} (zx)^j \binom{a-1+i}{i} (yz)^i \\ &\quad + \sum_{a \geq 0} \sum_{j \geq 0} \sum_{i \geq 0} \binom{a+j}{j} \binom{a+i}{i} yz^{a+2} (zx)^j (yz)^i. \end{aligned}$$

However, we know that $F(z) = \sum_{n \geq 1} T(F_n; x, y) z^n$, thus

$$\begin{aligned} T(F_n; x, y) &= \sum_{a=0}^{n-1} \sum_{j=0}^{n-1} \binom{a+j}{j} \binom{n-2-j}{n-1-a-j} x^{1+j} y^{n-1-a-j} \\ &\quad + \sum_{a=0}^{n-1} \sum_{j=0}^{n-1} \binom{a+j}{j} \binom{n-2-j}{n-2-a-j} x^j y^{n-1-a-j} \\ &= \sum_{a=0}^{n-1} \sum_{j=0}^{n-1} \binom{a+j}{j} \left[\binom{n-2-j}{a-1} x + \binom{n-2-j}{a} \right] x^j y^{n-1-a-j}, \end{aligned}$$

as required. \square

3. Tutte polynomials of wheels

An n -wheel, denoted by W_n , is defined as $W_n = C_n + K_1$. Any edge joining K_1 to a vertex in C_n is called a spoke of W_n . The edges which are not spokes are called arc edges. It is obvious that if we delete an arc edge we get an n -fan. Since we know the explicit expression of the Tutte polynomial of an n -fan, we just need to express the Tutte polynomial of an n -wheel in terms of the Tutte polynomial of n -fans to get an explicit expression of the Tutte polynomial of an n -wheel. Recall from Section 2 the definition of the graph G_n^j is an n -fan with one bunch of j parallel edges on the outermost edge.

Lemma 3.1. *The Tutte polynomial of the graph G_n^j ,*

$$(3.1) \quad T(G_n^j; x, y) = T(F_n; x, y) + \sum_{i=1}^n \sum_{k=i}^{j+i-2} y^k T(F_{n-i}; x, y).$$

Moreover, the generating function $G_j(z) = \sum_{n \geq 1} T(G_n^j; x, y) z^n$ is given by

$$G_j(z) = F(z) + \frac{zy(1-y^{j-1})}{(1-y)(1-zy)} (F(z) + 1),$$

where $F(z)$ is given by (2.1).

Proof. We use the deletion and contraction method. Let e_1 be the outermost edge of G_n^j , the graph with j parallel edges. We start by deleting

and contracting one edge parallel to e_1 , one at a time to get (3.1). By multiplying (3.1) by z^n and summing over $n \geq 1$, we obtain

$$\begin{aligned}
 G_j(z) &= F(z) + \sum_{n \geq 1} z^n \left(\sum_{i=1}^n \frac{y^i - y^{j+i-1}}{1-y} T(F_{n-i}; x, y) \right) \\
 &= F(z) + \frac{1}{1-y} \sum_{n \geq 0} \frac{z^{n+1}y - z^{n+1}y^j}{1-zy} T(F_n; x, y) \\
 &= F(z) + \frac{zy(1-y^{j-1})}{(1-y)(1-zy)} (F(z) + T(F_0; x, y)) \\
 &= F(z) + \frac{zy(1-y^{j-1})}{(1-y)(1-zy)} (F(z) + 1),
 \end{aligned}$$

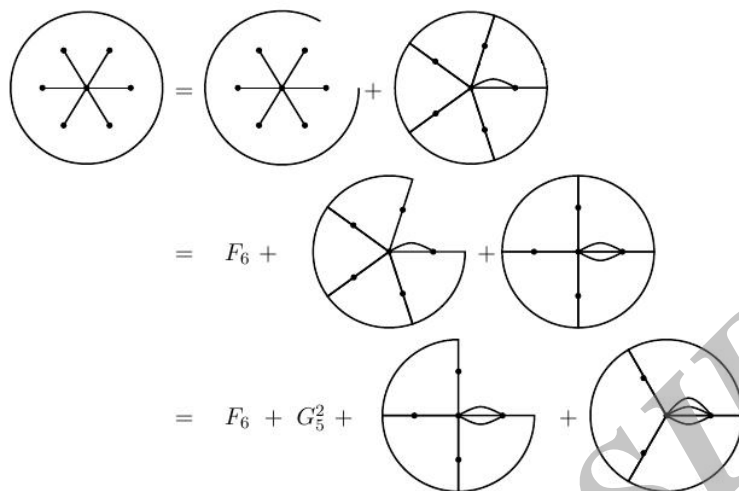
as required. \square

We can define the Tutte polynomial of an n -wheel in terms of the Tutte polynomials of an n -fan and of these G_n^j graphs as follows:

Lemma 3.2. *The Tutte polynomial of an n -wheel, W_n is*

$$\begin{aligned}
 T(W_n; x, y) &= \sum_{m=2}^n T(F_m; x, y) + yT(F_1; x, y) \\
 (3.2) \quad &+ \sum_{m=2}^{n-1} \sum_{i=1}^m \sum_{k=i}^{n-m+i-1} y^k T(F_{m-i}; x, y) + \sum_{k=1}^{n-1} y^{k+1}.
 \end{aligned}$$

Proof. Without loss of generality, we use W_6 to prove the general case. By using the deletion and contraction method, systematically deleting the arc edge which is adjacent to the spoke with parallel edges at each stage, we get



Thus, in general we have

$$\begin{aligned}
 (3.3) \quad T(W_n; x, y) &= T(F_n; x, y) \\
 &\quad + T(G_{n-1}^2; x, y) + T(G_{n-2}^3; x, y) + \cdots + T(G_2^{n-1}; x, y) \\
 &\quad + yT(G_1^n; x, y).
 \end{aligned}$$

Now, substituting Lemma 3.1 in Equation (3.3), we get

$$\begin{aligned}
 T(W_n; x, y) &= T(F_n; x, y) + \sum_{m=2}^{n-1} T(G_m^{n-m+1}; x, y) + yT(G_1^n; x, y) \\
 &= T(F_n; x, y) + \sum_{m=2}^{n-1} \left[T(F_m; x, y) + \sum_{i=1}^m \sum_{k=i}^{n-m+i-1} y^k T(F_{m-i}; x, y) \right] \\
 &\quad + y \left[T(F_1; x, y) + \sum_{k=1}^{n-1} y^k T(F_0; x, y) \right] \\
 &= \sum_{m=2}^n T(F_m; x, y) + yT(F_1; x, y) + \sum_{m=2}^{n-1} \sum_{i=1}^m \sum_{k=i}^{n-m+i-1} y^k T(F_{m-i}; x, y) \\
 &\quad + \sum_{k=1}^{n-1} y^{k+1},
 \end{aligned}$$

as claimed. \square

Now, let us recover the formula (1.1) for $T(W_n; x, y)$. In order to do that, we define $W(z) = \sum_{n \geq 1} T(W_n; x, y)z^n$ to be the generating function for the sequence $T(W_n; x, y)$. Multiplying (3.2) by z^n and summing over $n \geq 1$, using the fact that $T(F_0; x, y) = 1$, we have

$$W(z) = \frac{1}{1-z}(F(z) - xz) + \frac{xyz}{1-z} + H(z) + \frac{1}{1-y} \sum_{n \geq 2} (y^2 z^n - y^{n+1} z^n),$$

where $F(z)$ is given by (2.1) and $H(z)$ is given by

$$\begin{aligned} H(z) &= \sum_{n \geq 3} \left[\sum_{m=2}^{n-1} \sum_{i=1}^m \sum_{k=i}^{n-1-m+i} y^k T(F_{m-i}; x, y) \right] z^n \\ &= \sum_{n \geq 3} \left(\sum_{m=2}^{n-1} \sum_{i=1}^m \frac{y^i - y^{n-m+i}}{1-y} T(F_{m-i}; x, y) \right) z^n \\ &= \frac{1}{1-y} \sum_{n \geq 3} \left(\sum_{m=2}^{n-1} \sum_{i=1}^m (y^i - y^{n-m+i}) T(F_{m-i}; x, y) \right) z^n \\ &= \frac{1}{1-y} \sum_{n \geq 3} \left(\sum_{m=0}^{n-2} \left(\frac{y^{m+1}}{1-z} - \frac{y^{m+2}}{1-yz} \right) T(F_{n-2-m}; x, y) \right) z^n \\ &= \frac{1}{1-y} \left(\frac{(1-y)y^2 z^3 T(F_0; x, y)}{(1-z)(1-yz)^2} + \sum_{n \geq 1} \frac{(1-y)yz^{n+2} T(F_n; x, y)}{(1-z)(1-yz)^2} \right) \\ &= \frac{y^2 z^3}{(1-z)(1-yz)^2} + \frac{yz^2}{(1-z)(1-yz)^2} F(z). \end{aligned}$$

Hence, by (2.1) the generating function $W(z)$ can be written as

$$(3.4) \quad W(z) = xyz + \frac{(x+y+xy+x^2+y^2)z^2 - 2xy(x+y)z^3 + x^2y^2z^4}{(1-z)(1-(1+x+y)z+xyz^2)}.$$

We are now in a position to give the explicit formula for the Tutte polynomial of W_n . By (3.4) the generating function $W(z)$ can be written

as

$$\begin{aligned} W(z) &= \frac{xyz}{1-z} + \frac{z^2}{1-z} \cdot \frac{x+y+x^2+y^2+xy(1-x-y)z}{1-(1+x+y)z+xyz^2} \\ &= \frac{xyz}{1-z} + \frac{z^2}{1-z} \left(\frac{A}{1-az} + \frac{B}{1-bz} \right), \end{aligned}$$

where

$$A = -a(1-a), \quad B = -b(1-b),$$

and

$$\begin{aligned} a &= \frac{1+x+y+\sqrt{(1+x+y)^2-4xy}}{2}, \\ b &= \frac{1+x+y-\sqrt{(1+x+y)^2-4xy}}{2}. \end{aligned}$$

Hence,

$$W(z) = \sum_{n \geq 1} xyz^n + z^2 \sum_{n \geq 0} z^n \sum_{n \geq 0} (Aa^n + Bb^n)z^n,$$

which implies that the coefficient of z^n , $n \geq 1$, in $W(z)$ is given by

$$\begin{aligned} T(W_n; x, y) &= xy + \sum_{j=0}^{n-2} (Aa^j + Bb^j) \\ &= xy - a(1-a) \frac{1-a^{n-1}}{1-a} - b(1-b) \frac{1-b^{n-1}}{1-b} \\ &= xy - a(1-a^{n-1}) - b(1-b^{n-1}) = xy - a - b + a^n + b^n \\ &= xy - x - y - 1 + a^n + b^n. \end{aligned}$$

Thus we have proved the following result:

Theorem 3.3. For all $n \geq 1$,

$$T(W_n; x, y) = xy - x - y - 1 + 2^{-n}[(x+y+1+\sqrt{\beta})^n + (x+y+1-\sqrt{\beta})^n],$$

where $\beta = (x+y+1)^2 - 4xy$.

Indeed, the proof of equation (1.1) stated in [1] is now complete.

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