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APPLICATIONS OF EPI-RETRACTABLE AND CO-EPI-RETRACTABLE MODULES

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ABSTRACT. A module M is called epi-retractable if every submodule of M is a homomorphic image of M. Dually, a module M is called co-epi-retractable if it contains a copy of each of its factor modules. In special case, a ring R is called co-pli (respectively, co-pri) if ${}_RR$ (respectively, R_R) is co-epi-retractable. It is proved that if R is a left principal right duo ring, then every left ideal of R is an epi-retractable R-module. A co-pli strongly prime ring R is a simple ring. A left self-injective co-pli ring R is left Noetherian if and only if R is a left perfect ring. It is shown that a cogenerator ring R is a pli ring if and only if it is a co-pri ring. Moreover, if R is a left perfect ring such that every projective R-module is co-epi-retractable, then R is a quasi-Frobenius ring.

1. Introduction

Throughout the paper all rings are associative with non-zero identity elements and modules are unitary left modules. Let R be a ring. The ring R is said to be a *pli* (respectively, *pri*) if each left (respectively, right) ideal of R is principal. Ghorbani and Vedadi [5] generalized this concept to modules, an R-module M is called *epi-retractable* if every submodule of M is a homomorphic image of M. Therefore, R is a

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pli (respectively, pri) ring if and only if ${}_RR$ (respectively, R_R) is epi-retractable. Ghorbani [4] introduced dual notions: An R-module M is called *co-epi-retractable* if it contains a copy of any of its factor modules. It is clear that a left R-module M is co-epi-retractable if and only if for each submodule $N \subseteq M$, there exists an endomorphism $f : M \to M$ such that $N = \text{Ker } f$. A ring R is called *co-pli* (respectively, *co-pri*) if ${}_RR$ (respectively, R_R) is a co-epi-retractable module. It was shown that a ring R is co-pli (respectively, co-pri) if and only if each of its left (respectively, right) ideals is the left (respectively, right) annihilator of an element of R (see [4, Proposition 1.6]).

In section 2, conditions are found under which an epi-retractable module M is Hopfian and uniform. Also we show that a self-generator module ${}_RM$ with principal left ideal endomorphism ring $\text{End}_R(M)$ is an epi-retractable module.

In section 3, we prove that a self-injective co-epi-retractable module ${}_RM$ is a Noetherian module if and only if its endomorphism ring, $\text{End}_R(M)$, is a left perfect ring. A co-epi-retractable strongly prime module M is a strongly coprime module. In particular, a co-pli strongly prime ring R is a simple ring. In [4], Ghorbani shows that if R is a pli ring such that R_R is self-cogenerator, then R is a co-pri ring. We show that if R is a cogenerator ring, then R is a pli ring if and only if it is a co-pri ring. In [5], Ghorbani and Vedadi proved that a right (respectively, left) hereditary ring R is a pri (respectively, pli) ring if and only if every free right (respectively, left) R-module is epi-retractable. We prove that over a left hereditary ring R the following statements are equivalent:

(a) R is a semisimple ring.
(b) R is a pli ring.
(c) R is a co-pli ring.
(d) Every injective R-module is epi-retractable.
(e) Every free R-module is epi-retractable.
(f) Every free R-module is co-epi-retractable.

As before, ${}_RM$ is a non-zero left module over the ring R, its endomorphism ring $\text{End}_R(M)$ will act on the right side of ${}_RM$, in other words, ${}_RM_{\text{End}_R(M)}$ will be studied mainly. For the convenience of the readers, we recall in this section some definitions of modules that will be used in the sequel. Let M be a left R-module. we say that $N \in R$-Mod is *subgenerated* by M if N is a submodule of an M-generated module (see

the [13]). The category of M-subgenerated modules is denoted by $\sigma[M]$. When N is a submodule of M, we write $N \ll M$ and $N \trianglelefteq M$ to denote respectively the condition that N is a superfluous (or small) submodule or that N is an essential submodule in M. Let K be a submodule of M. If for any $f \in \mathrm{End}_R(M)$, $(K)f \subseteq K$, K is called a *fully invariant submodule* of M. An R-module M is called a *duo module* provided that every submodule of M is fully invariant. A ring R is called *left (right) duo ring* if every left (right) ideal of R is an ideal of R. A left or right self-injective ring R is called *quasi-Frobenius ring* if it is left or right Noetherian, (see Nicholson and Yousif [9]).
An R-module M is said to satisfy the $(*)$*-property* if every non-zero endomorphism of M is a monomorphism (see [12]). Note that a ring R is domain if and only if ${}_RR$ satisfies the $(*)$-property. An R-module M is said to satisfy the $(**)$*-property* if every non-zero endomorphism of M is an epimorphism (see [14]). In special case, ${}_RR$ is simple if and only if ${}_RR$ satisfies the $(**)$-property.

2. **Epi-retractable modules**

We begin our investigation of epi-retractable modules by recalling an important Lemma from 28.1 part (2) and (4) of [13]:

Lemma 2.1. *Let M be an R-module and $S = End_R(M)$.*
(1) *For any submodule $K \subseteq M$,*

$$Ker(r.ann_S(K)) = K$$

if and only if M is a self-cogenerator module.
(2) *If M is self-injective, then for every finitely generated right ideal $I \subseteq S$,*

$$r.ann_S(Ker\ I) = I.$$

Definition 2.2. Recall that ${}_RM$ is
– *Hopfian* (respectively, *co-Hopfian*) if every surjective (respectively, injective) homomorphism of M is an isomorphism.
– *co-compressible* if M is an epimorphic image of each of its non-zero factor modules.
– *uniform* if each of its non-zero submodules is essential in M.

Recall that R is called *reversible* if for $a, b \in R$, $ab = 0$ implies that $ba = 0$, see Cohn [3].

Proposition 2.3. *Let M be an epi-retractable module with $S = End_R(M)$. Then the following statements hold:*
(1) If S is reversible, then M is Hopfian.
(2) If ${}_RM$ is a self-injective module and S is a right Noetherian ring, then S is a co-pri ring.
(3) If M is co-compressible, then every factor module of M is epi-retractable.
(4) If ${}_RM$ is a duo module with the $(*)$-property, then M is uniform.

Proof. (1) Let $f : M \to M$ be an epimorphism. Since M is epi-retractable, there exists $g \in \mathrm{End}_R(M)$ such that Ker $f = (M)g$. Hence $gf = 0$. By reversibility of S, $fg = 0$. Since f is epimorphism, we have

$$\text{Ker } f = (M)g = (M)fg = 0.$$

So the proof is complete.
(2) Let I be a right ideal of S. Since M is epi-retractable, there exists $f \in S$ such that Ker $I = (M)f$. Thus $I = \text{r.ann}_S(\text{Ker } I) = \text{r.ann}_S(f)$, by part 2 of Lemma 2.1. Consequently S is a co-pri ring.
(3) Let $N \subseteq L$ be submodules of M. We show that there exists an epimorphism from M/N to L/N. Since M is co-compressible there exists epimorphism $f : M/N \to M$. On the other hand there exists an epimorphism $g : M \to L$, because M is epi-retractable. Consequently $fg\pi_N : M/N \to L/N$ is an epimorphism, where $\pi_N : L \to L/N$ denotes the canonical projection.
(4) Let A and B be two non-zero submodules of M with $A \cap B = 0$. Since M is epi-retractable, there exist $f, g \in S$ such that $(M)f = A$ and $(M)g = B$. Then $(M)gf = (B)f \subseteq A \cap B = 0$. Consequently $B = \text{Im } g \subseteq \text{Ker } f = 0$, a contradiction. □

Remark 2.4. Recall that the endomorphism rings of the quasi-cyclic group $\mathbb{Z}(p^\infty)$ and the group of p-adic integers ${\mathbb{Q}_p}^*$ are isomorphic commutative rings. On the other hand $\mathbb{Z}(p^\infty)$ is not Hopfian, so by part (1) of Proposition 2.3, we can see that $\mathbb{Z}(p^\infty)$ cannot be an epi-retractable $\mathbb{Z}$-module.

Corollary 2.5. *Let R be a pli ring. Then the following statements hold:*
(1) If ${}_RR$ is co-compressible, then every factor ring of R is a pli ring.
(2) If R is a left duo domain, then R is a uniform ring.

The following Lemma is needed.

Lemma 2.6. *If ${}_RM$ is a self-generator module, then for any $f \in End_R(M)$, $M l.ann_S(f) = Ker\ f$.*

Proof. We can easily see that $M\text{l.ann}_S(f) \subseteq \text{Ker } f$. Conversely, consider an arbitrary element $x \in \text{Ker } f$. Since M is self-generator, $\text{Ker } f = \text{Tr}(M, \text{Ker } f)$. Thus $x = \sum_{i=1}^{n}(m_i)g_i$ for some elements $m_i \in M$ and $g_i \in \text{Hom}_R(M, \text{Ker } f)$. But $g_i \in \text{l.ann}_S(f)$ for each $i = 1, 2, ..., n$. Then $x \in M\text{l.ann}_S(f)$. This shows that $M\text{l.ann}_S(f) = \text{Ker } f$. □

Proposition 2.7. *Let $_RM$ be a self-generator module and $S = End_R(M)$.*
(1) *If S is a pli ring, then M is epi-retractable.*
(2) *If S is a co-pli ring, then M is co-epi-retractable.*

Proof. (1) Let K be an R-submodule of M. Since S is a pli ring, there exists $f \in S$ such that $\text{Hom}_R(M, K) = Sf$. Now since M is self-generator, we have $K = \text{Tr}(M, K) = M\text{Hom}_R(M, K) = (M)f$. Consequently M is epi-retractable.
(2) Let K be an R-submodule of M. Then there exists $f \in S$ such that $\text{Hom}_R(M, K) = \text{l.ann}_S(f)$, because S is a co-pli ring. Since M is self-generator and by Lemma 2.6, we have $K = M\text{Hom}_R(M, K) = M\text{l.ann}_S(f) = \text{Ker } f$, which implies that M is co-epi-retractable. □

Proposition 2.8. *If R is a left principal right duo ring, then any left ideal of R is an epi-retractable R-module.*

Proof. Let $J \leq I$ be left ideals of R. If $I = Rx$ and $J = Ry$, then $y \in Rx \subseteq xR$, because R is right duo. Hence there exists $z \in R$ such that $y = xz$. Define $f : I \to J$, by $(x)f = xz$. Obviously f is epimorphism, and so I is epi-retractable. □

We need the following Lemma.

Lemma 2.9. [10, Lemma 2.1] *Let $M = \oplus_{i\in I}M_i$ be a direct sum of submodules M_i $(i \in I)$ and let N be a fully invariant submodule of M. Then $N = \oplus_{i\in I}(N \cap M_i)$.*

In was shown in [5] that a direct summand of an epi-retractable module need not be epi-retractable.

Proposition 2.10. *Let $M = \oplus_{i\in I}M_i$ be a duo module. Then M is epi-retractable if and only if each M_i is epi-retractable.*

Proof. (⇒). By [5, Proposition 2.11 part (i)].
(⇐). Let each M_i be epi-retractable and N be a submodule of M. Then for any $i \in I$ there exists $f_i \in \text{End}_R(M_i)$ such that $(M_i)f_i = N \cap M_i$. Thus $(M) \oplus_{i\in I} f_i = (\oplus_{i\in I}M_i) \oplus_{i\in I} f_i = \oplus_{i\in I}(N \cap M_i) = N$. □

Proposition 2.11. *Let M be an epi-retractable module with $S = End_R(M)$. Then the following statements are equivalent:*
(a) *M is a simple module.*
(b) *S is a division ring.*
(c) *M satisfies the $(**)$-property.*
(d) *M satisfies the $(*)$-property and $Soc(M) \neq 0$.*

Proof. $(a) \Rightarrow (b)$. By Schur's Lemma, M is simple implies S is a division ring.
$(b) \Rightarrow (c)$. This is clear.
$(c) \Rightarrow (a)$. Let K be a non-zero submodule of M. Since M is epi-retractable, there exists a homomorphism $f : M \to M$ such that $\text{Im } f = K$. Because K is non-zero and M satisfies the $(**)$-property, $K = \text{Im } f = M$. Therefore M is simple.
$(a) \Rightarrow (d)$ holds trivially.
$(d) \Rightarrow (a)$. Because $\text{Soc}(M) \neq 0$, there exists a simple submodule $K \subseteq M$. Since M is epi-retractable $K = \text{Im } f$ for some homomorphism $f : M \to M$. By (d) we have $M \simeq K$ that is simple. □

Corollary 2.12. *Let R be a pli ring. Then the following statements are equivalent:*
(1) *${}_RR$ is simple.*
(2) *R is a division ring.*
(3) *R is a domain and $Soc({}_RR) \neq 0$.*

A submodule U of R-module N is called *M-rational* in N if for any $U \subseteq V \subseteq N$, $\text{Hom}_R(V/U, M) = 0$. M is called *polyform* if any essential submodule is rational in M. The dual notions are: A submodule X of N is called *M-corational* in N if for any $Y \subseteq X \subseteq N$, $\text{Hom}_R(M, X/Y) = 0$. M is called *copolyform* if any superfluous submodule is corational in M. A ring R is called *Von Neumann regular* if for any $a \in R$ there is an element $b \in R$ with $aba = a$. Note that R is Von Neumann regular if and only if every left principal ideal is a direct summand in R (see [13, 3.10]).

Proposition 2.13. *If M is a finitely cogenerated epi-retractable module with $S = End_R(M)$, then the following statements are equivalent:*
(a) *M is copolyform.*
(b) *$Rad(M) = 0$.*
(c) *M is semisimple.*
(d) *S is a semisimple ring.*
(e) *S is a Von Neumann regular ring.*

Proof. $(a) \Rightarrow (b)$. Let M be a copolyform module. Assume K be a non-zero superfluous submodule of M. Since M is epi-retractable, there exists an epimorphism $f : M \to K$. Thus $\mathrm{Hom}_R(M, K) \neq 0$, a contradiction, because M is copolyform. Hence M has no non-zero superfluous submodule, i.e., $\mathrm{Rad}(M) = 0$.
$(b) \Rightarrow (a)$. Since $\mathrm{Rad}(M) = 0$, M has no non-zero superfluous submodule. Thus M is copolyform.
$(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$. By [13, 21.6 part (6)] and [13, 20.8], respectively.
$(c) \Rightarrow (b)$. See [1, Proposition 9.16].
$(d) \Rightarrow (e)$. This is obvious.
$(e) \Rightarrow (c)$. Let K be a submodule of M, then $K = \mathrm{Im}\ f$ for some $f \in S$. Now apply [13, 37.7 part (2)]. □

3. **Co-epi-retractable modules**

A ring R is said to be *right Bezout* if every finitely generated right ideal of R is principal. Also, a ring R is *left strongly prime* if for every left ideal $I \subseteq R$ there is a monomorphism $R \hookrightarrow I^k$ for some $k \in \mathbb{N}$ (Handelman-Lawrence [6]). This notion was extended to left modules in Beidar-Wisbauer [2]. A module M is called *strongly prime* if for any non-zero fully invariant submodule $K \subseteq M$, $M \in \sigma[K]$. Dually, M is called *strongly coprime* if for any proper fully invariant submodule K of M, $M \in \sigma[M/K]$.
An R-module M is called *hollow* if each of its proper submodules is superfluous in M.

Proposition 3.1. *Let M be a co-epi-retractable module with $S = End_R(M)$. Then:*
(1) *If S is reversible, then M is co-Hopfian.*
(2) *If ${}_RM$ is a self-injective module, then S is a right Bezout ring.*
(3) *If ${}_RM$ is a strongly prime module, then M is a strongly coprime module.*
(4) *If ${}_RM$ is a duo module with the $(**)$-property, then M is hollow.*

Proof. (1) Let $f : M \to M$ be a monomorphism. Since M is co-epi-retractable, there exists $g \in \mathrm{End}_R(M)$ such that $\mathrm{Im}\ f = \mathrm{Ker}\ g$. Hence $fg = 0$. By reversibility of S, $gf = 0$. Since f is monomorphism, we have $(M)g = 0$. So $\mathrm{Im}\ f = \mathrm{Ker}\ g = M$. Consequently M is co-Hopfian.

(2) Let I be a finitely generated right ideal of S. Because ${}_RM$ is co-epi-retractable, then there exists $f \in S$ such that Ker I = Ker f = Ker fS. Since M is self-injective,

$$I = \text{r.ann}_S(\text{Ker } I) = \text{r.ann}_S(\text{Ker } fS) = fS,$$

by part (2) of Lemma 2.1. Then S is a right Bezout ring.
(3) Let K be a proper fully invariant submodule of M. Since M is co-epi-retractable, then there exists a non-zero submodule L of M such that, $M/K \simeq L$. Since M is strongly prime, M is subgenerated by L, i.e., $M \in \sigma[L]$. Thus $M \in \sigma[M/K]$ and then M is strongly coprime.
(4) Let A and B be two proper submodules of M with $A + B = M$. Since M is co-epi-retractable, there exist $f, g \in S$ such that Ker $f = A$ and Ker $g = B$. Then $M = (M)f = (A + B)f =$ (Ker $g)f \subseteq$ Ker g. Thus $g = 0$, a contradiction. □

A co-Hopfian module need not be co-epi-retractable:

Remark 3.2. We can easily see that there does not exist an endomorphism $f \in \text{End}_{\mathbb{Z}}(\mathbb{Q})$ such that $\mathbb{Z}$ = Ker f. Then ${}_{\mathbb{Z}}\mathbb{Q}$ is not co-epi-retractable. But we know that ${}_{\mathbb{Z}}\mathbb{Q}$ is co-Hopfian.

The following theorem gives some information about self-injective co-epi-retractable modules.

Theorem 3.3. *Let ${}_RM$ be a self-injective co-epi-retractable module with $S = End_R(M)$. Then the following statements are equivalent:*
(a) ${}_RM$ is a Noetherian module.
(b) S satisfies dcc for cyclic right ideals.
(c) S is a left perfect ring.

Proof. $(a) \Rightarrow (b)$. A descending chain of cyclic right ideals $f_1S \supseteq f_2S \supseteq \cdots$ yields an ascending chain of submodules $\text{Ker} f_1S \subseteq \text{Ker} f_2S \subseteq \cdots$. By assumption, there is some $n \in \mathbb{N}$ such that $\text{Ker} f_iS = \text{Ker} f_nS$ for all $i \geq n$. With applying $\text{r.ann}_S(-)$ to this module, and by part (2) of Lemma 2.1, we have $f_iS = f_nS$ for all $i \geq n$. This shows that S satisfies dcc for cyclic right ideals.
$(b) \Rightarrow (a)$. Let $K_1 \subseteq K_2 \subseteq \cdots$ be an ascending chain of submodules of M. Because M is co-epi-retractable, each K_i is of the form $\text{Ker} f_i = \text{Ker} f_iS$ for some $f_i \in S$. With applying $\text{r.ann}_S(-)$ to this chain, we see that $f_1S \supseteq f_2S \supseteq \cdots$. But S satisfies dcc for cyclic right ideals, thus there is some n such that $f_iS = f_nS$ for all $i \geq n$, and then $K_i = K_n$ for all $i \geq n$. Therefore M is left Noetherian.
$(b) \Leftrightarrow (c)$. This follows from [13, 43.9]. □

Corollary 3.4. *Let R be a co-pli ring. Then the following statements hold:*
(1) *If R is a reversible ring, then ${}_RR$ is co-Hopfian.*
(2) *If R is a left self-injective ring, then R is a right Bezout ring.*
(3) *If R is a left self-injective, then the following are equivalent:*
  (a) *R is a left Noetherian ring.*
  (b) *R satisfies dcc for cyclic right ideals.*
  (c) *R is a left perfect ring.*
(4) *If R is a strongly prime ring, then R is a simple ring.*

We note that over a left perfect ring R, every left R-module has a projective cover.

Proposition 3.5. *Let R be a left perfect ring such that every projective R-module is co-epi-retractable. Then R is a quasi-Frobenius ring.*

Proof. By [8, Remark 15.10], we need to show that every injective R-module is projective. Now, let M be an injective R-module and let P be the projective cover of ${}_RM$. Then, by our assumption, $M \hookrightarrow P$. Because ${}_RM$ is injective, then M is isomorphic to a direct summand of P, and so is projective. □

Proposition 3.6. *Let ${}_RM$ be a self-cogenerator, and set $S = End_R(M)$.*
(1) *If S is a co-pri ring, then:*
  (i) *M is epi-retractable.*
  (ii) *M is left Noetherian if and only if S is right Artinian.*
(2) *If S is a pri ring, then M is co-epi-retractable.*

Proof. (1) (i). Let K be an R-submodule of M. Since S is a co-pri ring, there exists $f \in S$ such that $\mathrm{r.ann}_S(K) = \mathrm{r.ann}_S(f)$. Since M is self-cogenerator, we have

$$K = \mathrm{Ker}(\mathrm{r.ann}_S(K)) = \mathrm{Ker}(\mathrm{r.ann}_S((M)f)) = (M)f,$$

by part (1) of Lemma 2.1. Thus M is epi-retractable.
(ii)($\Rightarrow$). Consider the ascending chain $I_1 \supseteq I_2 \supseteq \cdots$ of right ideals of S. Since S is co-pri, then for each i there exists $f_i \in S$ such that $I_i = r.ann_S(f_i) = r.ann_S((M)f_i)$. With applying $\mathrm{Ker}(-)$ to this chain and by part (1) of Lemma 2.1 we get the descending chain $(M)f_1 \subseteq (M)f_2 \subseteq \cdots$ of submodules of M. Because M is left Noetherian, there exists some $n \in \mathbb{N}$ such that $(M)f_i = (M)f_n$ for all $i \geqslant n$. So we have $\mathrm{r.ann}_S(f_i) = \mathrm{r.ann}_S(f_n)$ for all $i \geqslant n$. Thus S satisfies dcc for its right ideals.

($\Leftarrow$). Let $K_1 \subseteq K_2 \subseteq \cdots$ be an ascending chain of submodules of M. Then $\text{r.ann}_S(K_1) \supseteq \text{r.ann}_S(K_2) \supseteq \cdots$. But S satisfies dcc on its right ideals, thus there is some n such that $\text{r.ann}_S(K_i) = \text{r.ann}_S(K_n)$ for all $i \geq n$. With applying $\text{Ker}(-)$ to this module, and by part (1) of Lemma 2.1, we have $K_i = K_n$ for all $i \geq n$. Therefore M is left Noetherian.
(2) Let K be an R-submodule of M. Since S is pri ring, then there exists $f \in S$ such that $\text{r.ann}_S(K) = fS$. Since M is self-cogenerator, we deduce that

$$K = \text{Ker}(\text{r.ann}_S(K)) = \text{Ker}\ (fS) = \text{Ker}\ f,$$

by part (1) of Lemma 2.1. Thus M is co-epi-retractable. □

Example 3.7. By Example 3.7 of [7], $\text{End}_Z(\mathbb{Q}/\mathbb{Z}) \simeq \prod_p \mathbb{Q}_p^*$. Since for any prime number p, Q_p^* is a commutative principal ideal domain, then $\text{End}_Z(\mathbb{Q}/\mathbb{Z})$ is a commutative principal ideal ring. Consequently by part (2) of Proposition 3.6, the cogenerator $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is co-epi-retractable.

Corollary 3.8. *Let $_RR$ be a self-cogenerator module.*
(1) *If R is a co-pri ring, then:*
  (i) *R is a pli ring.*
  (ii) *R is left Noetherian if and only if R is right Artinian.*
(2) *If R is a pri ring, then R is a co-pli ring.*

A *cogenerator ring* is a ring R for which both $_RR$ and R_R are cogenerators. Quasi-Frobenius rings are examples of cogenerator rings (see Lam [8, 15.11 part (1)]).

Theorem 3.9. *A cogenerator ring R is a pli ring if and only if it is a co-pri ring.*

Proof. By Corollary 3.8 and [4, Corollary 2.6]. □

Proposition 3.10. *Let $_RM$ be a hollow copolyform module. Then M is co-epi-retractable if and only if $M/N \simeq M$, for all proper submodules N of M.*

Proof. ($\Leftarrow$). By definition.
($\Rightarrow$). Let N be a proper submodule of M. Then there exists a nonzero submodule K of M such that $M/N \simeq K$. If K is proper, then since M is hollow copolyform, $\text{Hom}_R(M, M/N) = \text{Hom}_R(M, K) = 0$, a contradiction. Thus $M/N \simeq M$. □

It is well known that a module M is semisimple if and only if each of its submodules is essentially closed.

Proposition 3.11. *The following are equivalent for a nonsingular R-module M:*
(a) ${}_RM$ *is semisimple.*
(b) ${}_RM$ *is co-epi-retractable.*

Proof. $(a) \Rightarrow (b)$. This is clear.
$(b) \Rightarrow (a)$. Let N be a submodule of ${}_RM$. By assumption, there exists a submodule K of M such that $M/N \simeq K$. Because M/N is nonsingular, then N is an essentially closed submodule of M. So M is semisimple. □

An R-module M is called subisomorphic to an R-module M' if there exist monomorphisms $f : M \to M'$ and $g : M' \to M$.

Proposition 3.12. *Let M be an R-module. Then the following statements are equivalent:*
(a) *M is a co-epi-retractable.*
(b) *M is subisomorphic to a co-epi-retractable module.*
(c) *There exists a monomorphism $\varphi : M \to K$ for some co-epi-retractable submodule K of M.*

Proof. $(a) \Rightarrow (b)$. This is clear.
$(b) \Rightarrow (c)$. Suppose that there exist a co-epi-retractable module M' and monomorphisms $\alpha : M \to M'$, $\beta : M' \to M$. Set $K := \text{Im}\ \beta \simeq M'$. Then, $\alpha\beta : M \to K$ is a monomorphism for co-epi-retractable submodule K of M.
$(c) \Rightarrow (a)$. Let L be any submodule of M. By our assumption, for submodule $K' := (L)\varphi$ of K, there exists a monomorphism $\theta : K/K' \to K$. Consider inclusion map $i_K : K \to M$ and monomorphism $\overline{\varphi} : M/L \to K/K'$ with $(m + L)\overline{\varphi} = (m)\varphi + K'$. Then $\overline{\varphi}\theta i_K : M/L \to M$ is a monomorphism, proving that M is co-epi-retractable. □

Proposition 3.13. *Let $M = \oplus_{i\in I} M_i$ be a duo module. Then M is co-epi-retractable if and only if each M_i is co-epi-retractable.*

Proof. $(\Rightarrow)$. By [4, Proposition 1.1 part (4)].
$(\Leftarrow)$. Let each M_i be epi-retractable and N be a submodule of M. By Lemma 2.9, submodule N can be written as $N = \oplus_{i\in I}(N \cap M_i)$. On the other hand, for any $i \in I$ there exists a monomorphism $f_i : M_i/(N \cap M_i) \to M_i$. Thus the homomorphism

$$f : \bigoplus_{i\in I}[M_i/(N\cap M_i)] \to \oplus_{i\in I} M_i,\ (m_i+N\cap M_i)_{i\in I} \mapsto \sum_{i\in I}(m_i+N\cap M_i)f_i$$

is injective. Moreover we have the monomorphism

$$g : M/N \to \bigoplus_{i\in I}[M_i/(N\cap M_i)],\ \sum_{i\in I} m_i + N \mapsto (m_i + N\cap M_i)_{i\in I}.$$

Consequently the homomorphism $gf : M/N \to M$ is injective, as desired. □

Proposition 3.14. *Let M be a co-epi-retractable module with $S = End_R(M)$. Then the following statements are equivalent:*
(a) M is a simple module.
(b) S is a division ring.
(c) M satisfies the $(*)$-property.
(d) M satisfies the $(**)$-property and $Rad(M) = 0$.

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are obvious.
$(c) \Rightarrow (a)$. Let K be a proper submodule of M. Since M is co-epi-retractable, there exists a homomorphism $f : M \to M$ such that $\text{Ker } f = K$. Because K is proper and M satisfies the $(*)$-property, $K = \text{Ker } f = 0$. Consequently M is simple.
$(a) \Rightarrow (d)$. Straightforward.
$(d) \Rightarrow (a)$. There exists a maximal submodule $N \subset M$, because $\text{Rad}(M) \neq M$. Since M is co-epi-retractable $N = \text{Ker } f$ for some homomorphism $f : M \to M$. By the $(**)$-property we have $M/N \simeq M$, which implies that M is simple. □

Corollary 3.15. *Let R be a co-pli ring. Then the following statements are equivalent:*
(1) ${}_RR$ is simple.
(2) R is a division ring.
(3) R is a domain.

A ring R is called *left hereditary* if all of its left ideals are projective. Moreover, R is left hereditary if and only if every submodule of every projective R-module is projective if and only if quotients of injective R-modules are injective (see [8, Corollary 2.26] and [8, Theorem 3.22]).

Proposition 3.16. *Let R be a left hereditary ring. Then every projective co-epi-retractable R-module is semisimple.*

Proof. Assume that R is a left hereditary ring and K is a submodule of a co-epi-retractable projective R-module M. Since M is co-epi-retractable, M/K is isomorphic to a submodule of M. Thus M/K is projective, and we can lift $I_{M/K}$ to $f \in \text{Hom}_R(M/K, M)$ with $f\pi_K = I_{M/K}$.

Hence $M = \text{Im } f \oplus \text{Ker } \pi_K = \text{Im } f \oplus K$. Consequently M is semisimple. □

Proposition 3.17. *Let M be a projective module over a left hereditary ring R. Then the following statements are equivalent:*
(a) *M is semisimple.*
(b) *M is epi-retractable.*
(c) *M is co-epi-retractable.*
(d) *In $\sigma[M]$ every injective module is epi-retractable.*

Proof. $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$ and $(a) \Rightarrow (d)$ are trivial.
$(b) \Rightarrow (a)$. By [11, 3.1 part (2)].
$(c) \Rightarrow (a)$. See 3.16.
$(d) \Rightarrow (a)$. According to [11, 3.1 part (1)], the M-injective hull $\widehat{M}$ of M in $\sigma[M]$ is semisimple. Then M is also semisimple. □

The following result generalizes [5, Proposition 2.5].

Corollary 3.18. *Let R be a left hereditary ring. Then the following statements are equivalent:*
(a) *R is a semisimple ring.*
(b) *R is a pli ring.*
(c) *R is a co-pli ring.*
(d) *Every injective R-module is epi-retractable.*
(e) *Every free R-module is epi-retractable.*
(f) *Every free R-module is co-epi-retractable.*

Proposition 3.19. *If M is a co-epi-retractable module with $S = End_R(M)$. Then the following are equivalent:*
(a) *M is polyform.*
(b) *M is semisimple.*
(c) *S is a Von Neumann regular ring.*

Proof. $(a) \Rightarrow (b)$. Let L be an essential submodule of M. Since M is co-epi-retractable, there exists a monomorphism $M/L \hookrightarrow M$. Because M is polyform, we have $\text{Hom}_R(M/L, M) = 0$. Then $L = M$. Consequently $\text{Soc}(M) = \bigcap_{L \trianglelefteq M} L = M$, i.e. M is semisimple.
$(b) \Rightarrow (a)$. Is trivial.
$(b) \Rightarrow (c)$. By [13, 37.7 part (2)].
$(c) \Rightarrow (b)$. Let K be a submodule of M, then $K = \text{Ker } f$ for some $f \in S$. Now apply [13, 37.7 part (2)]. □

Corollary 3.20. *If R is a co-pli ring, then the following statements are equivalent:*
(a) ${}_RR$ *is polyform.*
(b) R *is a semisimple ring.*
(c) R *is a Von Neumann regular ring.*

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