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ON GENERALIZATIONS OF PRIME SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ is the set of submodules of M. Suppose $n \geq 2$ is a positive integer. A proper submodule P of M is called $(n-1,n)-\phi$-prime, if whenever $a_1,\dots,a_{n-1} \in R$ and $x \in M$ and $a_1 \dots a_{n-1}x \in P \backslash \phi(P)$, then there exists $i \in \{1,\dots,n-1\}$ such that $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P$ or $a_1 \dots a_{n-1} \in (P : M)$. In this paper we study $(n-1,n)-\phi$-prime submodules ($n \geq 2$). A number of results concerning $(n-1,n)-\phi$-prime submodules are given. Modules with the property that for some ϕ, every proper submodule is $(n-1,n)-\phi$-prime, are characterized and we show that under some assumptions $(n-1,n)$-prime submodules and $(n-1,n)-\phi_m$-prime submodules coincide ($n, m \geq 2$).

1. Introduction

We assume throughout the paper that all rings are commutative with $1 \neq 0$ and modules are unital. We will denote the set of maximal ideals of R by $Max(R)$.

Suppose that M is an R-module. We will denote the set of submodules of M by $S(M)$. For an ideal I of R and a submodule N of M, let $\sqrt{I}$ denote the radical of I and $(N :_R M) = \{r \in R : rM \subseteq N\}$, which is

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clearly an ideal of R. The R-module M is called faithful if $(0 : M) = 0$. We say that N is a radical submodule of M if $\sqrt{(N :_R M)} = (N :_R M)$.

Prime ideals play a central role in commutative ring theory. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodules (see for example [11, 13, 14, 15, 19]). These have led to more information on the structure of the R-module M. A proper submodule P of M is called prime if $r \in R$ and $x \in M$, with $rx \in P$ implies that $r \in (P :_R M)$ or $x \in P$. It is easy to show that if P is a prime submodule of M, then $(P :_R M)$ is a prime ideal of R.

Anderson and Smith in [7]; defined a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Weakly prime elements were introduced by Galovich in [12], and used by the authors in [2], to study the unique factorization in rings with zero-divisors.

Nekooei in [17], extended this concept to weakly prime submodule, i. e., a proper submodule P of M with the property that whenever $r \in R$ and $x \in M$ and $0 \neq rx \in P$, then $x \in P$ or $r \in (P : M)$.

To study unique factorization domains, Bhatwadekar and Sharma in [10] defined the notation of almost prime ideal, i.e., a proper ideal I with the property that if $a, b \in R$ and $ab \in I \backslash I^2$, then either $a \in I$ or $b \in I$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Anderson and Bataineh in [6], extended these concepts to ϕ-prime ideals as follows: Let $S(R)$ be the set of ideals of R and $\phi : S(R) \to S(R) \cup \{\emptyset\}$ a function. Then a proper ideal I of R is ϕ-prime if for $x, y \in R$, $xy \in I \backslash \phi(I)$ implies $x \in I$ or $y \in I$.

Zamani in [19] extended this concept to ϕ-prime submodule. For a function $\phi : S(M) \to S(M) \cup \{\emptyset\}$, a proper submodule P of M is called ϕ-prime if whenever $r \in R$, $x \in M$ and $rx \in P \backslash \phi(P)$, then $r \in (P : M)$ or $x \in P$. Let P be a submodule of M. Since $P \backslash \phi(P) = P \backslash (P \cap \phi(P))$, without loss of generality, throughout the paper we will assume $\phi(P) \subseteq P$. For two functions $\psi_1, \psi_2 : S(M) \to S(M) \cup \{\emptyset\}$, we write $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$, for each $N \in S(M)$. For the following functions $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ the corresponding ϕ_α-prime submodules are:

$\phi_\emptyset$	$\phi(N) = \emptyset$	prime submodule
ϕ_0	$\phi(N) = 0$	weakly prime submodule
ϕ_1	$\phi(N) = N$	any module
ϕ_2	$\phi(N) = (N : M)N$	almost prime submodule
$\phi_n (n \geq 2)$	$\phi(N) = (N : M)^{n-1}N$	n-almost prime submodule
ϕ_ω	$\phi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$	ω-prime submodule

Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$. Then it is clear that $\phi_\emptyset$-prime and ϕ_0-prime submodules are prime and weakly prime submodules respectively. Zamani in [19] defined almost prime submodule by the function $\phi(N) = (N : M)N$ and ϕ_n-prime submodule by the functions $\phi_n(N) = (N : M)^n N (n \geq 2)$. In this paper if $\phi(N) = (N : M)^{n-1}N$, then we say that N is n-almost prime submodule.

We recall from [5] that a proper ideal I of R is called an n-absorbing ideal if whenever $x_1x_2 \ldots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. For $n \geq 2$, we denote an $(n-1)$-absorbing ideal I of R by $(n-1, n)$-prime. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. We say that a proper ideal I of R is $(n-1, n) - \phi$-prime if $a_1 \ldots a_n \in I \backslash \phi(I)$ $(a_1, \ldots, a_n \in R)$, implies $a_1 \ldots a_{i-1}a_{i+1} \ldots a_n \in I$, for some $i \in \{1, \ldots, n\}$.

In this paper we extend this concept to $(n-1, n) - \phi$-prime submodules.

Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and P be a proper submodule of M. We say that P is $(n-1, n)-\phi$-prime if $a_1 \ldots a_{n-1}x \in P \backslash \phi(P)$, $(a_1, \ldots, a_{n-1} \in R$ and $x \in M)$, implies $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x \in P$, for some $i \in \{1, \ldots, n-1\}$ or $a_1 \ldots a_{n-1} \in (P : M)$. If $\phi = \phi_\emptyset$, then $(n-1, n) - \phi_\emptyset$-prime submodule is called $(n-1, n)$-prime submodule. If $\phi = \phi_0$, then a $(n-1, n) - \phi_0$-prime submodule is called a $(n-1, n)$-weakly prime submodule and if $\phi = \phi_m$, then a $(n-1, n) - \phi_m$-prime submodule is called a $(n-1, n)-m$-almost prime submodule $(n, m \geq 2)$.

Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. We show (Theorem 2.1) that a $(n-1, n) - \phi$-prime submodule P that is not $(n-1, n)$-prime satisfies $(P : M)^{n-1}P \subseteq \phi(P)$. In particular, if $\phi = \phi_0$ and M is faithful, Then $(P : M)^{n-1} = 0$, and thus $(P : M) \subseteq \sqrt{0}$. Among the many results in this paper, we show (Theorem 3.8) if $0 \neq M_i$ is a F_i-vector space, for every $i \in \{1, \ldots, n\}$ and $R = F_1 \times \cdots \times F_n$ and $M = M_1 \times \cdots \times M_n$, then every proper submodule of M is $(n-1, n)$-weakly prime if and only if $dim M_i = 1$, for all i. We know that a commutative ring R is Von Neumann regular if and only if every ideal of R is idempotent. Anderson and Bataineh used this concept [6], Theorem 17, to characterize a commutative ring R that every proper ideal of R is almost prime. Recall from [3] that a submodule N of M is called idempotent if $(N : M)N = N$. An R-module M is a fully idempotent module if every submodule of M is idempotent. We use this concept to characterize modules M for which, every proper submodule is $(n-1, n) - n$-almost prime (Theorem 3.10) or every proper submodule is n-almost prime (Theorem 3.11).

It is well known that, every proper ideal of R is a product of prime ideals if and only if R is a finite direct product of Dedekind domains and SPIRs. Such rings are called ZPI-rings. Anderson and Smith [7], Theorem 7, have shown that every proper ideal of R is a product of weakly prime ideals if and only if R is a ZPI-ring or (R, m) is quasi-local with $m^2 = 0$. Also, Anderson and Bataineh [6], Theorem 22, have shown that in a Noetherian ring R every proper ideal of R is a product of almost prime ideals if and only if R is a finite direct product of Dedekind domains, SPIRs, and SPAP-rings.

Let M be a multiplication module, i.e., An R-module M with the property that for every submodule N of M there exists an ideal I of R such that $N = IM$. In this paper we give a characterization of some multiplication modules in which every proper submodule is a product of almost prime submodules.

Some of the results in this paper are inspired by [6].

2. $(n-1, n) - \phi$-prime submodules

The following theorem asserts that under some conditions $(n-1, n)-\phi$-prime submodules are $(n-1, n)$-prime, $(n \geq 2)$.

Theorem 2.1. *Let R be a commutative ring and M be an R-module. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and P be a $(n-1, n) - \phi$-prime submodule of M, that is not $(n-1, n)$-prime, then $(P : M)^{n-1}P \subseteq \phi(P)$. Hence a $(n-1, n) - \phi$-prime submodule P with $(P : M)^{n-1}P \not\subseteq \phi(P)$ is $(n-1, n)$-prime.*

Proof. Suppose that $(P : M)^{n-1}P \not\subseteq \phi(P)$; we show that P is $(n-1, n)$-prime. Let $a_1, a_2, \ldots, a_{n-1} \in R$ and $x \in M$ with $a_1a_2 \ldots a_{n-1}x \in P$. If $a_1a_2 \ldots a_{n-1}x \notin \phi(P)$, then $a_1a_2 \ldots a_{n-1} \in (P : M)$ or

$a_1a_2\dots a_{i-1}a_{i+1}\dots a_{n-1}x \in P$, for some $i \in \{1, 2, \dots, n-1\}$. Now, let $a_1a_2\dots a_{n-1}x \in \phi(P)$.

We can assume that $a_1a_2\dots a_{n-k}(P : M)^{k-1}x \subseteq \phi(P)$, for all $k \in \{1, 2, \dots, n-1\}$, because, if $a_1a_2\dots a_{n-k}(P : M)^{k-1}x \nsubseteq \phi(P)$, then there exist $r_1, \dots, r_{k-1} \in (P : M)$ such that $a_1a_2\dots a_{n-k}r_1\dots r_{k-1}x \notin \phi(P)$. Hence $a_1a_2\dots a_{n-k}(a_{n-k+1} + r_1)\dots(a_{n-1} + r_{k-1})x \in P\backslash\phi(P)$. Since P is $(n-1, n) - \phi$-prime, $a_1a_2\dots a_{i-1}a_{i+1}\dots a_{n-1}x \in P$, for some $i \in \{1, 2, \dots, n-1\}$ or $a_1a_2\dots a_{n-1} \in (P : M)$.

Likewise, we can assume that for all $\{i_1, \dots, i_{n-k}\} \subseteq \{1, 2, \dots, n-1\}$, $a_{i_1}a_{i_2}\dots a_{i_{n-k}}(P : M)^{k-1}x \subseteq \phi(P)$, $1 \le k \le n-1$. Also, we can assume that $a_1\dots a_{n-k}(P : M)^{k-1}P \subseteq \phi(P)$, for all $k \in \{1, \dots, n-1\}$, because, if $a_1\dots a_{n-k}(P : M)^{k-1}P \nsubseteq \phi(P)$, then $a_1a_2\dots a_{n-k}r_1r_2\dots r_{k-1}p_0 \notin \phi(P)$, where $p_0 \in P$ and $r_1, r_2, \dots, r_{k-1} \in (P : M)$, and so $a_1a_2\dots a_{n-k}(a_{n-k+1}+r_1)\dots(a_{n-1}+r_{k-1})(p_0+x) \in P\backslash\phi(P)$. Since P is $(n-1, n) - \phi$-prime, $a_1\dots a_{n-1} \in (P : M)$ or $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x \in P$, for some $i \in \{1, 2, \dots, n-1\}$. Likewise, we can assume that for all $\{i_1, i_2, \dots, i_{n-k}\} \subseteq \{1, 2, \dots, n-1\}$, $a_{i_1}a_{i_2}\dots a_{i_{n-k}}(P : M)^{k-1}P \subseteq \phi(P)$, $1 \le k \le n-1$. Since $(P : M)^{n-1}P \nsubseteq \phi(P)$, there exist $p_0 \in P$ and $r_1, r_2, \dots, r_{n-1} \in (P : M)$ with $r_1r_2\dots r_{n-1}p_0 \notin \phi(P)$. Then $(a_1 + r_1)(a_2+r_2)\dots(a_{n-1}+r_{n-1})(x+p_0) \in P\backslash\phi(P)$. Since P is $(n-1, n) - \phi$-prime, $a_1a_2\dots a_{n-1} \in (P : M)$ or $a_1a_2\dots a_{i-1}a_{i+1}\dots a_{n-1}x \in P$, for some $i \in \{1, 2, \dots, n-1\}$. So P is $(n-1, n)$-prime. □

Corollary 2.2. *Let R be a commutative ring, M be an R-module and P be a proper submodule of M. If P is a $(n-1, n)$-weakly prime submodule that is not $(n-1, n)$-prime, then $(P : M)^{n-1}P = 0$.*

Corollary 2.3. *Let P be a $(n-1, n) - \phi$-prime submodule where $\phi \le \phi_{n+1}$. Then P is $(n-1, n) - \omega$-prime ($n \ge 2$).*

Proof. If P is $(n-1, n)$-prime, then P is $(n-1, n) - \omega$-prime. Suppose that P is not $(n-1, n)$-prime. By Theorem 1.1, $(P : M)^{n-1}P \subseteq \phi(P) \subseteq (P : M)^nP$. Hence $\phi(P) = (P : M)^kP$, for each $k \ge n-1$. Thus P is $(n-1, n) - \omega$-prime.

□

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

let $P_1 \times M_2$ be a $(n-1,n)$-weakly prime submodule of M. Let $r_1, ..., r_{n-1} \in R_1$ and $x_1 \in M_1$, with $r_1...r_{n-1}x_1 \in P_1$. Let $0 \neq x_2 \in M_2$. Then $(r_1, 1)...(r_{n-1}, 1)(x_1, x_2) \in P_1 \times M_2 \backslash \{0\}$. By assumption, this gives that $r_1, ..., r_{n-1} \in (P_1 : M_1)$ or $r_1...r_{i-1}r_{i+1}...r_{n-1}x_1 \in P_1$, for some $i \in \{1, ..., n-1\}$. Therefore, P_1 is a $(n-1,n)$-prime submodule of M_1. So, if P_1 is a $(n-1,n)$-weakly prime submodule of M_1, then $P_1 \times M_2$ need not be a $(n-1,n)$-weakly prime submodule of M.

Next we show that, if P is a $(n-1,n)$-weakly prime submodule of M_1, then $P_1 \times M_2$ is a $(n-1,n)-\phi$-prime submodule if $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$.

Proposition 2.4. *Let R_i be a commutative ring and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Suppose that P_1 is a $(n-1,n)$-weakly prime submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$. Then $P_1 \times M_2$ is a $(n-1,n) - \phi$-prime submodule of $M_1 \times M_2$ $(n \geq 2)$.*

Proof. We have $P_1 \times M_2 \backslash \phi(P_1 \times M_2) \subseteq P_1 \times M_2 \backslash \{0\} \times M_2 = (P_1 \backslash \{0\}) \times M_2$. Let $(a_1, b_1) \ldots (a_{n-1}, b_{n-1})(x_1, x_2) = (a_1 \ldots a_{n-1}x_1, b_1 \ldots b_{n-1}x_2) \in P_1 \times M_2 \backslash \phi(P_1 \times M_2)$, where $(a_1, b_1) \ldots (a_{n-1}, b_{n-1}) \in R$ and $(x_1, x_2) \in M$.
So $(a_1 \ldots a_{n-1}x_1, b_1 \ldots b_{n-1}x_2) \in (P_1 \backslash \{0\}) \times M_2$. Then $a_1 \ldots a_{n-1}x_1 \in P_1 \backslash \{0\}$ and by the assumption on P_1 we have $a_1 \ldots a_{n-1} \in (P_1 : M_1)$ or $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x_1 \in P_1$, for some $i \in \{1, 2, \ldots, n-1\}$. This gives that $(a_1, b_1) \ldots (a_{n-1}, b_{n-1}) = (a_1 \ldots a_{n-1}, b_1 \ldots b_{n-1}) \in (P_1 \times M_2 : M_1 \times M_2)$ or $(a_1, b_1) \ldots (a_{i-1}, b_{i-1})(a_{i+1}b_{i+1}) \ldots (a_{n-1}, b_{n-1})(x_1, x_2) \in P_1 \times M_2$. Therefore, $P_1 \times M_2$ is a $(n-1,n) - \phi$-prime submodule of M. □

Corollary 2.5. *With the same notations as in Proposition 2.4, let ϕ be a function such that $\phi_\omega \leq \phi$. Then for any $(n-1,n)$-weakly prime submodule P_1 of M_1, $P_1 \times M_2$ is a $(n-1,n) - \phi$-prime submodules of M $(n \geq 2)$.*

Proof. If P_1 is a $(n-1,n)$-prime submodule of M_1, then $P_1 \times M_2$ is $(n-1,n)$-prime and so $(n-1,n) - \phi$-prime submodule of M. Suppose that P_1 is not $(n-1,n)$-prime. Then by Corollary 2.2, we have $(P_1 :$

$M_1)^{n-1}P_1 = 0$. This gives that

$$\begin{aligned} \phi_\omega(P_1 \times M_2) &= \bigcap_{i=2}^{\infty}[(P_1 \times M_2 : M)^{i-1}(P_1 \times M_2)] \\ &= \bigcap_{i=2}^{\infty}([(P_1 \times M_2 : M)^{i-1} \times R_2]P_1 \times M_2) = 0 \times M_2 \\ \Rightarrow & \quad \phi_\omega(P_1 \times M_2) = 0 \times M_2 \subseteq \phi(P_1 \times M_2). \end{aligned}$$

The result follows by Proposition 2.4. □

In the next theorem we give a characterization of $(n-1,n)-\phi$-prime submodules ($n \geq 2$).

Theorem 2.6. *Let P be a proper submodule of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:*
(i) P is $(n-1,n)-\phi$-prime.
(ii) For $a_1, \ldots, a_{n-2} \in R$ and $x \in M$ with $a_1a_2\ldots a_{n-2}x \in M \backslash P$;

$$(P : a_1 \ldots a_{n-2}x) = \bigcup_{i=1}^{n-2}(P : a_1 \ldots a_{i-1}a_{i+1}\ldots a_{n-2}x)$$
$$\cup(P : a_1 \ldots a_{n-2}M) \cup (\phi(P) : a_1 \ldots a_{n-2}x)$$

Proof. (i)⇒(ii) Let $a_1 \ldots a_{n-2}x \in M \backslash P$. Assume that $r \in (P : a_1 \ldots a_{n-2}x)$; so $ra_1a_2\ldots a_{n-2}x \in P$. If $ra_1\ldots a_{n-2}x \notin \phi(P)$, then $ra_1\ldots a_{n-2} \in (P : M)$. So $r \in (P : a_1\ldots a_{n-2}M)$ or $ra_1\ldots a_{i-1}a_{i+1}\ldots a_{n-2}x \in P$, for some $i \in \{1, 2, \ldots, n-2\}$. Hence $r \in (P : a_1\ldots a_{i-1}a_{i+1}\ldots a_{n-2}x)$. If $ra_1\ldots a_{n-2}x \in \phi(P)$, then $r \in (\phi(P) : a_1\ldots a_{n-2}x)$. So

$$(P : a_1 \ldots a_{n-2}x) \subseteq \bigcup_{i=1}^{n-2}(P : a_1 \ldots a_{i-1}a_{i+1}\ldots a_{n-2}x)$$
$$\cup(P : a_1 \ldots a_{n-2}M) \cup (\phi(P) : a_1 \ldots a_{n-2}x)$$

The other containment always holds (remember we are assuming that $\phi(P) \subseteq P$).

(ii)⇒(i) Let $a_1, \ldots, a_{n-1} \in R$ and $x \in M$ with $a_1\ldots a_{n-1}x \in P\backslash\phi(P)$. If $a_1\ldots a_{n-2}x \in P$, then there is nothing to prove.

So we can assume that $a_1\ldots a_{n-2}x \notin P$. Thus $(Pa_1\ldots a_{n-2}x) = \bigcup_{i=1}^{n-2}(P : a_1\ldots a_{i-1}a_{i+1}\ldots a_{n-2}x) \cup (P : a_1\ldots a_{n-2}M) \cup (\phi(P) : a_1\ldots a_{n-2}x)$.

Since $a_1 \dots a_{n-1}x \in P$, we have $a_{n-1} \in (P : a_1 \dots a_{n-2}x)$. But $a_{n-1} \notin (\phi(P) : a_1 \dots a_{n-2}x)$, hence $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-2}a_{n-1}x \in P$, for some $i \in \{1, 2, \dots, n-2\}$ or $a_{n-1} \in (P : a_1 \dots a_{n-2}M)$ (so $a_1 \dots a_{n-1} \in (P : M)$). Thus P is $(n-1, n) - \phi$-prime. □

Let S be a multiplicatively closed subset of R. We know [18], 9.11 (v), that each submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Also it is well known that there is a one-to-one correspondence between the set of all prime submodules P of M with $(P : M) \cap S = \emptyset$ and the set of all prime submodules of $S^{-1}M$, given by $P \to S^{-1}P$, see [16], Theorem 3.4. Furthermore, it is easy to see that if P is a weakly prime submodule of M with $S^{-1}P \neq S^{-1}M$, then $S^{-1}P$ is a weakly prime submodule of $S^{-1}M$. In the next theorem we want to generalize this fact for $(n-1, n) - \phi$-prime submodules. Let $N(S) = \{x \in M : \exists s \in S, sx \in N\}$. We know that $N(S)$ is a submodule of M containing N and $S^{-1}(N(S)) = S^{-1}N$. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $(S^{-1}\phi) : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $(S^{-1}\phi)(S^{-1}N) = S^{-1}(\phi(N(S)))$ if $\phi(N(S)) \neq \emptyset$ and $(S^{-1}\phi)(S^{-1}N) = \emptyset$ if $\phi(N(S)) = \emptyset$. Since $\phi(N) \subseteq N$, hence $(S^{-1}\phi)(S^{-1}N) \subseteq S^{-1}N$.

We next show that if $(S^{-1}(\phi(N)) \subseteq (S^{-1}\phi)(S^{-1}N)$, then $(n-1, n)-\phi$-primeness of P together with $S^{-1}P \neq S^{-1}M$ imply that $S^{-1}P$ is $(n-1, n) - (S^{-1}\phi)$-prime $(n \geq 2)$. For a submodule L of M, let $\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\}$ be defined by $\phi_L(\frac{N}{L}) = \frac{\phi(N)+L}{L}$ for $L \subseteq N$ and $\emptyset$ for $\phi(N) = \emptyset$.

Theorem 2.7. *Let M be an R-module and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let P be a $(n-1, n) - \phi$-prime submodule of M.*

(i) If $L \subseteq P$ is a submodule of M, then $\frac{P}{L}$ is a $(n-1, n) - \phi_L$-prime submodule of $\frac{M}{L}$ $(n \geq 2)$.

(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}P \neq S^{-1}M$ and $S^{-1}(\phi(P)) \subseteq (S^{-1}\phi)(S^{-1}P)$. Then $S^{-1}P$ is a $(n-1, n) - (S^{-1}\phi)$-prime submodule of $S^{-1}M$ $(n \geq 2)$.

Proof. (i) Let $a_1, \dots, a_{n-1} \in R$, $x + L \in \frac{M}{L}$ with $a_1 \dots a_{n-1}(x + L) \in \frac{P}{L} \setminus \phi_L(\frac{P}{L})$. By definition of ϕ_L, we have $a_1 \dots a_{n-1}x \in P \setminus \phi(P)$.
Since P is $(n-1, n) - \phi$-prime, we have $a_1 \dots a_{n-1} \in (\frac{P}{L} : \frac{M}{L})$ or $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}(x + L) \in \frac{P}{L}$, for some $i \in \{1, \dots, n-1\}$. Thus $\frac{P}{L}$ is $(n-1, n) - \phi_L$-prime.

(ii) Let $\frac{a_1}{s_1}, \dots, \frac{a_{n-1}}{s_{n-1}} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}M$ with $\frac{a_1}{s_1} \dots \frac{a_{n-1}}{s_{n-1}} \frac{x}{t} \in S^{-1}P \backslash (S^{-1}\phi)(S^{-1}P)$, where $a_1, \dots, a_{n-1} \in R$, $s_1, \dots, s_{n-1}, t \in S$, $x \in M$. Then by assumption, $\frac{a_1 \dots a_{n-1} x}{s_1 \dots s_{n-1} t} \in S^{-1}P \backslash S^{-1}(\phi(P))$. So there exists $u \in S$ such that $ua_1 \dots a_{n-1}x \in P \backslash \phi(P)$. Thus $\frac{a_1}{s_1} \dots \frac{a_{n-1}}{s_{n-1}} \in S^{-1}(P : M) \subseteq (S^{-1}P : S^{-1}M)$ or $\frac{a_1}{s_1} \dots \frac{a_{i-1}}{s_{i-1}} \frac{a_{i+1}}{s_{i+1}} \dots \frac{a_{n-1}}{s_{n-1}} \frac{x}{t} \in S^{-1}P$, for some $i \in \{1, \dots, n-1\}$. Hence $S^{-1}P$ is a $(n-1, n) - (S^{-1}\phi)$-prime submodule of $S^{-1}M$. □

Proposition 2.8. *Let $R = R_1 \times \cdots \times R_n$ and $M = M_1 \times \cdots \times M_n$ be an R-module, where R_i is a commutative ring and M_i is an R_i-module, for $i \in \{1, 2, \dots, n\}$. Let $P = P_1 \times \cdots \times P_n$ be a $(n-1, n) - \phi$-prime submodule of M, where P_i is a submodule of M_i and let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ and $\phi(P) = \psi_1(P_1) \times \psi_2(P_2) \times \cdots \times \psi_n(P_n)$. Then P_j is a $(n-1, n) - \psi_j$-prime submodule of M_j, for each j with $P_j \neq M_j$.*

Proof. Let $P_j \neq M_j$, $x_j \in M_j$ and $a_1, \dots, a_{n-1} \in R_j$ such that
$a_1 \dots a_{n-1} x_j \in P_j \backslash \psi_j(P_j)$. Thus
$(1, \dots, 1, a_1, 1, \dots, 1)(1, \dots, 1, a_2, 1, \dots, 1) \dots$
$(1, \dots, 1, a_{n-1}, 1, \dots, 1)(0, \dots, 0, x_j, 0, \dots, 0) =$
$(0, \dots, 0, a_1 \dots a_{n-1} x_j, 0, \dots, 0) \in P \backslash \phi(P)$.
Therefore, $(1, \dots, 1, a_1 \dots a_{n-1}, 1, \dots, 1) \in (P : M)$. So $a_1 \dots a_{n-1} \in (P_j : M_j)$ or $a_1 \dots a_{i-1} a_{i+1} \dots a_{n-1} x_j \in P_j$, for some $i \in \{1, \dots, n-1\}$. Thus P_j is $(n-1, n) - \psi_j$-prime.

□

Corollary 2.9. *Let $R = R_1 \times \cdots \times R_n$ and $M = M_1 \times \cdots \times M_n$ and $P = P_1 \times \cdots \times P_n$, where R_i is a commutative ring and M_i is an R_i-module and P_i is a submodule of M_i, for $i \in \{1, \dots, n\}$. Let P be a $(n-1, n) - \phi_m$-prime submodule of M. Then P_j is a $(n-1, n) - \phi_m$-prime submodule of M_j, for each j with $P_j \neq M_j$ $(n, m \geq 2)$.*

Proof. We have

$$\begin{aligned} \phi_m(P) &= (P : M)^{m-1}P = (P_1 : M_1)^{m-1}P_1 \times \cdots \times (P_n : M_n)^{m-1}P_n \\ &= \phi_m(P_1) \times \cdots \times \phi_m(P_n). \end{aligned}$$

So the result follows by the Proposition 2.8. □

It is clear that every $(n-1, n)$-weakly prime submodule is $(n, n+1)$-weakly prime. We show that the converse is not true in general.

Example 2.10. *a) Let* $R = \mathbf{Z}_8$ *and* $M = R$ *as R-module. By* [9], *Example 3.5 (a), every nonzero proper submodule of* M *is (2,3)-prime, and hence* $(n-1,n)$*-weakly prime, for all* $n \geq 3$. *Now consider* $N = \{\overline{0}, \overline{4}\}$. *We have* $\overline{0} \neq \overline{2}\cdot\overline{2} \in N$ *but* $\overline{2} \notin N$. *So* N *is not a weakly prime submodule.*

b) Let $R = \dfrac{\mathbf{R}[|x,y|]}{(xy, x^2-y^2, x^3, y^3)}$ *and* $M = R$ *as R-module. By* [9], *Example 3.5(b), every nonzero proper submodule of* M *is (2,3)-prime and so* $(n-1,n)$*-weakly prime, for all* $n \geq 3$. *Consider* $N = \dfrac{(xy, x^2, y^2)}{(xy, x^2-y^2, x^3, y^3)}$. *We have* $0 \neq \overline{x}^2 \in N$, *but* $\overline{x} \notin N$, *where* $\overline{x}^i = x^i + (xy, x^2-y^2, x^3, y^3)$, *for* $i = 1,2$. *So* N *is not weakly prime.*

3. $(n-1,n)-\phi_\alpha$-prime submodules

Theorem 3.1. *Let* M *be an R-module and* $0 \neq x \in M$ *such that* $Rx \neq M$ *and* $(0 :_R x) = 0$. *If* Rx *is not a* $(n-1,n)$*-prime submodule of* M, *then* Rx *is not* $(n-1,n)-m$*-almost prime submodule of* M $(n, m \geq 2)$.

Proof. Since Rx is not $(n-1,n)$-prime, there exist $a_1,\ldots,a_{n-1} \in R$ and $y \in M$ such that $a_1\ldots a_{n-1} \notin (Rx : M)$ and $a_1\ldots a_{i-1}a_{i+1}\ldots a_{n-1}y \notin Rx$, for all $i \in \{1,2,\ldots,n-1\}$. But $a_1\ldots a_{n-1}y \in Rx$. If $a_1\ldots a_{n-1}y \notin (Rx : M)^{m-1}Rx$, then by definition Rx is not $(n-1,n)-m$-almost prime. So let $a_1\ldots a_{n-1}y \in (Rx : M)^{m-1}x$. We have $(a_1\ldots a_{i-1}a_{i+1}\ldots a_{n-1})(y+x) \notin Rx$, for all $i \in \{1,2,\ldots,n-1\}$ and $a_1\ldots a_{n-1}(y+x) \in Rx$. If $a_1\ldots a_{n-1}(y+x) \notin (Rx : M)^{m-1}x$, then again by definition Rx is not $(n-1,n)-m$-almost prime. So let $a_1\ldots a_{n-1}(y+x) \in (Rx : M)^{m-1}x$, then $a_1\ldots a_{n-1}x \in (Rx : M)^{m-1}x$, which gives that $a_1\ldots a_{n-1}x = rx$, for some $r \in (Rx : M)^{m-1}$. Since $(0 :_R x) = 0$, it gives that $a_1\ldots a_{n-1} = r \in (Rx : M)^{m-1} \subseteq (Rx : M)$, which contradicts our assumption. □

Corollary 3.2. *Let* $0 \neq x \in M$, *where* M *is an R-module and* $(0 :_R x) = 0$ *and* $Rx \neq M$. *Then* Rx *is a* $(n-1,n)$*-prime submodule of* M *if and only if* Rx *is a* $(n-1,n)-m$*-almost prime submodule of* M $(n, m \geq 2)$.

Corollary 3.3. *Let the assumptions be as in the Corollary 3.2. Then* Rx *is* $(n-1,n)$*-almost prime if and only if* Rx *is* $(n-1,n)-m$*-almost prime* $(n, m \geq 2)$.

Proof. Let Rx be $(n-1,n)-m$-almost prime. So Rx is $(n-1,n)$-almost prime. Conversely, let Rx be $(n-1,n)$-almost prime. By Corollary 3.2

(for $m = 2$) Rx is $(n-1, n)$-prime. So again by Corollary 3.2 Rx is $(n-1, n) - m$-almost prime. □

We give an example of a $(n-1, n) - n$-almost prime submodule that is not $(n-2, n-1) - (n-1)$-almost prime ($n \geq 3$).

Example 3.4. *Let K be a field and $R = K[|x|]$. We know that $m = (x)$ is the unique maximal ideal of R. Put $\overline{R} = \dfrac{R}{m^n}$ and $M = \overline{R}$ as $\overline{R}$-module. We have $\overline{m}^n = 0$. Let N be a proper submodule of M. We have $(N^{n-1})^n = N^{n^2-n} \subseteq \overline{m}^{n^2-n} \subseteq \overline{m}^n = 0$. Suppose that $a_1, \ldots, a_{n-1} \in \overline{R}$, $a_n \in M$ and $0 \neq a_1 \ldots a_n \in N^{n-1}$. Since $\overline{m}^n = 0$, there exists $i \in \{1, \ldots, n\}$ such that $a_i \notin \overline{m}$ and hence a_i is a unit. Thus $a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n \in N^{n-1}$ and N^{n-1} is $(n-1, n) - n$-almost prime. We now show that $\overline{m}^{n-1}$ is not $(n-2, n-1) - (n-1)$-almost prime. Since $(\overline{m}^{n-1})^{n-1} = \overline{m}^{2n-2} = 0$, we have $0 \neq \overline{x}^{n-1} \in \overline{m}^{n-1}$. Hence $\overline{x}^{n-1} \in \overline{m}^{n-1} \backslash (\overline{m}^{n-1})^{n-1}$. But $\overline{x}^{n-2} \notin \overline{m}^{n-1}$. So $\overline{m}^{n-1}$ is not $(n-2, n-1) - (n-1)$-almost prime.*

Theorem 3.5. *Let M be an R-module and a be an element of R such that $aM \neq M$. Suppose that $(0 :_M a) \subseteq aM$. Then aM is $(n-1, n)$-almost prime submodule of M if and only if it is $(n-1, n)$-prime ($n \geq 2$).*

Proof. $\Leftarrow$) is clear.
$\Rightarrow$) Suppose that aM be $(n-1, n)$-almost prime. Let $b_1, \ldots, b_{n-1} \in R$ and $x \in M$ with $b_1 \ldots b_{n-1} x \in aM$. If $b_1 \ldots b_{n-1} x \notin (aM : M)aM$, then $b_1 \ldots b_{n-1} \in (aM : M)$ or $b_1 \ldots b_{i-1} b_{i+1} \ldots b_{n-1} x \in aM$, for some $i \in \{1, 2, \ldots, n-1\}$. So suppose that $b_1 \ldots b_{n-1} x \in (aM : M)aM$. Now $(b_1 + a) b_2 \ldots b_{n-1} x \in aM$. If $(b_1 + a) b_2 \ldots b_{n-1} x \notin (aM : M)aM$, then, aM is$(n-1, n)-$almost prime, then $b_1 \ldots b_{n-1} \in (aM : M)$ or $b_1 \ldots b_{i-1} b_{i+1} \ldots b_{n-1} x \in aM$, for some $i \in \{1, 2, \ldots, n-1\}$. So assume that $(b_1 + a) b_2 \ldots b_{n-1} x \in (aM : M)aM$. Then $b_1 \ldots b_{n-1} x \in (aM : M)aM$ gives that $a b_2 \ldots b_{n-1} x \in (aM : M)aM$. Hence there exist $r \in (aM : M)$ and $y \in M$ such that $a b_2 \ldots b_{n-1} x = ray$ and so $b_2 \ldots b_{n-1} x - ry \in (0 :_M a)$. This gives that $b_2 \ldots b_{n-1} x \in (0 :_M a) + aM \subseteq aM$. □

Lemma 3.6. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i is a commutative ring, for all $i \in \{1, 2, \ldots, n\}$. If P is a $(n-1, n)-$weakly prime ideal of R, then either $P = 0$ or $P = P_1 \times P_2 \times \cdots \times P_{i-1} \times R_i \times P_{i+1} \times \cdots \times P_n$ for some $i \in \{1, 2, \ldots, n\}$ and if $P_j \neq R_j$ (for $j \neq i$), then P_j is $(n-1, n)-$prime in R_j.*

Proof. Let $P = P_1 \times P_2 \times \cdots \times P_n$ be a $(n-1,n)-$weakly prime ideal of R. So there exists $(0,\ldots,0,0) \neq (a_1,a_2,\ldots,a_n) \in P$ and hence

$$(a_1,1,1,\ldots,1)(1,a_2,1,\ldots,1)\ldots(1,1,\ldots,1,a_n) = (a_1,a_2,\ldots,a_n) \in P.$$

Since P is $(n-1,n)-$weakly prime; we have $(a_1,a_2,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n) \in P$, for some $i \in \{1,2,\ldots,n\}$. Hence $(0,0,\ldots,0,1,0,0,\ldots,0) \in P$. So $P = P_1 \times P_2 \times \cdots \times P_{i-1} \times R_i \times P_{i+1} \times \cdots \times P_n$. If $P_j \neq R_j$ (for $j \neq i$), we claim that P_j is $(n-1,n)-$prime. Suppose that $i < j$. Let $b_1b_2\ldots b_n \in P_j$, we have

$$\begin{aligned} & 0 \neq (0,0,\ldots,0,1,0,\ldots,0,b_1b_2\ldots b_n,0,\ldots,0) \\ = \; & (0,\ldots,0,1,0,\ldots,0,b_1,0,\ldots,0)(0,\ldots,0,1,0,\ldots,0,b_2,0,\ldots,0) \\ & \ldots(0,\ldots,0,1,0,\ldots,0,b_n,0,\ldots,0) \in P. \end{aligned}$$

Since P is $(n-1,n)-$weakly prime, we have

$$(0,0,\ldots,0,1,0,\ldots,0,b_1b_2\ldots b_{k-1}b_{k+1}\ldots b_n,0,\ldots,0) \in P$$

for some $k \in \{1,2,\ldots,n\}$. So $b_1b_2\ldots b_{k-1}b_{k+1}\ldots b_n \in P_j$. Thus P_j is $(n-1,n)-$prime. The proof $j < i$ is similar. □

Proposition 3.7. *Let $R = R_1 \times \cdots \times R_n$, where R_i is a commutative ring, for all $i \in \{1,\ldots,n\}$ and every proper ideal of R is $(n-1,n)$-weakly prime. Let $M = M_1 \times \cdots \times M_n$, where $0 \neq M_j$ is an R_j-module, for all $j \in \{1,\ldots,n\}$. If $0 \neq P$ is a $(n-1,n)$-weakly prime submodule of M such that $(P : M) \neq 0$, then $P = P_1 \times P_2 \times \cdots \times P_{i-1} \times M_i \times P_{i+1} \times \cdots \times P_n$ for some $i \in \{1,\ldots,n\}$ and if $P_j \neq M_j$ (for $j \neq i$), then P_j is a $(n-1,n)$-prime submodule of M_j.*

Proof. Let $P = P_1 \times \cdots \times P_n$, where P_i is a submodule of M_i, for $i \in \{1,\ldots,n\}$. $0 \neq (P : M) = (P_1 : M_1) \times \cdots \times (P_n : M_n)$ is a nonzero proper ideal of R. So it is $(n-1,n)$-weakly prime, by assumption. We have by Lemma 3.6 $(P_i : M_i) = R_i$, for some $i \in \{1,\ldots,n\}$ and so $P_i = M_i$. Thus $P = P_1 \times \cdots \times P_{i-1} \times M_i \times P_{i+1} \times \cdots \times P_n$. If $P_j \neq M_j$ (for $j \neq i$). We claim that P_j is $(n-1,n)$-prime. Suppose that $i < j$. Let $a_1,\ldots,a_{n-1} \in R_j$ and $x \in M_j$ such that $a_1\ldots a_{n-1}x \in P_j$. There exists $0 \neq y \in M_i$. We have

$$\begin{aligned} & 0 \neq (0,\ldots,0,y,0,\ldots,0,a_1\ldots a_{n-1}x,0,\ldots,0) \\ & = (0,\ldots,0,1,0,\ldots,0,a_1,0,\ldots,0)\ldots(0,\ldots,0,1,0,\ldots,0,a_{n-1},0,\ldots,0) \\ & \quad \times(0,\ldots,0,y,0,\ldots,0,x,0,\ldots,0) \in P. \end{aligned}$$

Since P is $(n-1,n)$-weakly prime, we have $a_1\dots a_{n-1}\in(P_j:M_j)$ or $a_1\dots a_{k-1}a_{k+1}\dots a_{n-1}x\in P_j$, for some $k\in\{1,\dots,n-1\}$. Thus P_j is $(n-1,n)$-prime. The proof $j<i$ is similar. □

It was shown by Anderson and Smith [7], Theorem 8, that every proper ideal of R is weakly prime if and only if R is a direct product of two fields or (R,M) is quasi-local with $M^2=0$. Now we extend this result to $(n-1,n)$-weakly prime modules.

Theorem 3.8. *Let $R=F_1\times\cdots\times F_n$, where F_i is a field and $0\neq M_i$ is a F_i-vector space, for all $i\in\{1,\dots,n\}$ and $M=M_1\times\cdots\times M_n$. Every proper submodule of M is $(n-1,n)$-weakly prime if and only if $dimM_i=1$, for all i.*

Proof. ($\Leftarrow$) Let $dimM_i=1$, for each $i\in\{1,\dots,n\}$ and $N=N_1\times\cdots\times N_n$ be a proper submodule of M, where N_i is a submodule of M_i. So $N_j=0$, for at least one $j\in\{1,\dots,n\}$ (because N is a proper submodule). It is easy to show that N is $(n-1,n)$-weakly prime.

($\Rightarrow$) Suppose that every proper submodule of M is $(n-1,n)$-weakly prime. We claim that $dimM_i=1$, for all $i\in\{1,\dots,n\}$. Let $dimM_i>1$, for some $i\in\{1,\dots,n\}$. So there exists a proper submodule $0\neq N_i$ of M_i. We have by assumption that $P=0\times\cdots\times0\times N_i\times0\times\cdots\times0$ is $(n-1,n)$-weakly prime. Let $0\neq x_i\in N_i$ and $0\neq x_j\in M_j$ (for each $j\neq i$). We have

$$\begin{aligned}0 &\neq (0,\dots,0,x_i,0\dots,0)=(0,\dots,0,1,0,\dots,0)(x_1\dots x_n)\\ &= (a_{11},\dots,a_{i-11},1,a_{i1},\dots,a_{n-11})(a_{12},\dots,a_{i-12},1,a_{i2},\dots,a_{n-12})\\ \dots&\ (a_{1n-1},\dots,a_{i-1n-1},1,a_{in-1},\dots,a_{n-1n-1})(x_1,x_2,\dots,x_n)\in P,\end{aligned}$$

Where x_i and ones are in the $i'th$ place and $a_{jj}=0$ and $a_{jk}=1$, for each $k\neq j$ and $j\in\{1,\dots,n-1\}$. Since $M_j\neq0$, for each $j\neq i$, we have $(0:M_j)=0$. Since N_i is a proper submodule of M_i, we have $(N_i:M_i)=0$. Thus $(P:M)=(0,\dots,0)$. So $(a_{11},\dots,a_{i-11},1,a_{i1},\dots,a_{n-11})\dots$ $(a_{1n-1},\dots,a_{i-1n-1},1,a_{in-1},\dots,a_{n-1n-1})\notin(P:M)$, where the number 1 has appeared in the $i'th$ place and for each $j\neq i$ $(i<j)$, we have
$(0,\dots,0,1,0,\dots,0,1,0,\dots,0)(x_1,\dots,x_n)$
$=(0,\dots,0,x_i,0,\dots,0,x_j,0,\dots,0)\notin P$, where the first one is in the $i'th$ place and the second one is in the $j'th$ place, which is impossible. The proof for $j<i$ is similar. □

It was shown by Anderson and Smith [7], Theorem 8, that every proper ideal of R is weakly prime if and only if R is a direct product of two fields or (R, M) is quasi-local with $M^2 = 0$. Now we extend this result to $(n-1, n)$-weakly prime modules.

Theorem 3.9. *Let $R = F_1 \times \cdots \times F_n$, where F_i is a field and $0 \neq M_i$ is a F_i-vector space, for all $i \in \{1, \ldots, n\}$ and $M = M_1 \times \cdots \times M_n$. Every proper submodule of M is $(n-1, n)$-weakly prime if and only if $dim M_i = 1$, for all i.*

Proof. $(\Leftarrow)$ Let $dim M_i = 1$, for each $i \in \{1, \ldots, n\}$ and $N = N_1 \times \cdots \times N_n$ be a proper submodule of M, where N_i is a submodule of M_i. So $N_j = 0$, for at least one $j \in \{1, \ldots, n\}$ (because N is a proper submodule). It is easy to show that N is $(n-1, n)$-weakly prime.

$(\Rightarrow)$ Suppose that every proper submodule of M is $(n-1, n)$-weakly prime. We claim that $dim M_i = 1$, for all $i \in \{1, \ldots, n\}$. Let $dim M_i > 1$, for some $i \in \{1, \ldots, n\}$. So there exists a proper submodule $0 \neq N_i$ of M_i. We have by assumption that $P = 0 \times \cdots \times 0 \times N_i \times 0 \times \cdots \times 0$ is $(n-1, n)$-weakly prime. Let $0 \neq x_i \in N_i$ and $0 \neq x_j \in M_j$ (for each $j \neq i$). We have

$$\begin{aligned} 0 &\neq (0, \ldots, 0, x_i, 0 \ldots, 0) = (0, \ldots, 0, 1, 0, \ldots, 0)(x_1 \ldots x_n) \\ &= (a_{11}, \ldots, a_{i-11}, 1, a_{i1}, \ldots, a_{n-11})(a_{12}, \ldots, a_{i-12}, 1, a_{i2}, \ldots, a_{n-12}) \\ &\ldots (a_{1n-1}, \ldots, a_{i-1n-1}, 1, a_{in-1}, \ldots, a_{n-1n-1})(x_1, x_2, \ldots, x_n) \in P, \end{aligned}$$

Where x_i and ones are in the $i'th$ place and $a_{jj} = 0$ and $a_{jk} = 1$, for each $k \neq j$ and $j \in \{1, \ldots, n-1\}$. Since $M_j \neq 0$, for each $j \neq i$, we have $(0 : M_j) = 0$. Since N_i is a proper submodule of M_i, we have $(N_i : M_i) = 0$. Thus $(P : M) = (0, \ldots, 0)$. So $(a_{11}, \ldots, a_{i-11}, 1, a_{i1}, \ldots, a_{n-11}) \ldots (a_{1n-1}, \ldots, a_{i-1n-1}, 1, a_{in-1}, \ldots, a_{n-1n-1}) \notin (P : M)$, where the number 1 has appeared in the $i'th$ place and for each $j \neq i$ $(i < j)$, we have $(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)(x_1, \ldots, x_n)$
$= (0, \ldots, 0, x_i, 0, \ldots, 0, x_j, 0, \ldots, 0) \notin P$, where the first one is in the $i'th$ place and the second one is in the $j'th$ place, which is impossible. The proof for $j < i$ is similar. □

Recall from [3] that a submodule N of M is called idempotent if $N = (N : M)N$. We know that a commutative ring R is Von Neumann regular if and only if every ideal of R be idempotent. Ansari-Toroghy and Farshadifar in [8] defined a fully idempotent module i.e., an R-module M with the property that every submodule of M is idempotent.

Lemma 3.10. *Let $n \geq 1$ be a natural number. An R-module M is regular if and only if $(N : M)^n N = N$, for every submodule N of M.*

Proof. (⇒) Let M be a regular R-module and N be a submodule of M. We have $(N : M)N = N$. So $(N : M)^n N = N$.

(⇐) Let N be a submodule of M. We have $N = (N : M)^n N \subseteq (N : M)N \subseteq N$. So $N = (N : M)N$ and M is regular. □

Theorem 3.11. *Let $R = R_1 \times \cdots \times R_n$, where R_i is a commutative ring and $0 \neq M_i$ be an R_i-module, for all $i \in \{1, \ldots, n\}$. Let $M = M_1 \times \cdots \times M_n$. Every proper submodule of M is $(n-1, n) - n$-almost prime if and only if M is a regular R-module ($n \geq 2$).*

Proof. (⇐) Let M be a regular R-module and N be a proper submodule of M. So $(N : M)^{n-1}N = N$. Since $N \backslash (N : M)^{n-1}N = \emptyset$, we have N is $(n-1, n) - n$-almost prime.

(⇒) Let every proper submodule of M be $(n-1, n) - n$-almost prime. We show that M_i is regular, for all $i \in \{1, \ldots, n\}$, hence M is regular. Suppose that M_1 is not regular, so there exists a submodule N_1 of M_1 such that $(N_1 : M_1)^{n-1}N_1 \neq N_1$. By hypothesis $N_1 \times 0 \times \cdots \times 0$ must be $(n-1, n) - n$-almost prime. But, since $(N_1 : M_1)^{n-1}N_1 \neq N_1$, there exists $x_1 \in N_1 \backslash (N_1 : M_1)^{n-1}N_1$. Let $0 \neq x_i \in M_i$, for all $i \geq 2$. We have

$$\begin{aligned} (x_1, 0, \ldots, 0) &= (1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \ldots (1, 1, \ldots, 1, 0) \\ (x_1, \ldots, x_n) &\in (N_1 \times 0 \times \cdots \times 0) \backslash (N_1 \times 0 \times \cdots \times 0 : M)^{n-1} \end{aligned}$$
$$(N_1 \times 0 \times \cdots \times 0).$$

Since $N_1 \times 0 \times \cdots \times 0$ is $(n-1, n) - n$-almost prime, we have $1 \in (N_1 : M_1)$ or $x_i \in (0)$, for some $i \geq 2$, which is impossible. So M_1 is regular. Likewise, M_i is regular, for all $i \in \{2, \ldots, n\}$. □

Theorem 3.12. *Let $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times \cdots \times M_m$, where R_i is a commutative ring and $0 \neq M_i$ is an R_i-module, for all $i \in \{1, \ldots, m\}$. Then every proper submodule of M is n-almost prime if and only if M is regular ($n, m \geq 2$).*

Proof. (⇐) Let M be a regular module and N be a proper submodule of M. So $(N : M)^{n-1}N = N$, hence N is n-almost prime.

(⇒) Suppose that every proper submodule of M is n-almost prime. So every proper submodule of M is almost prime. We show that M_i is regular, for all $i \in \{1, 2, \ldots, m\}$, hence M is regular. If M_1 is not regular, then there exists a proper submodule N_1 of M_1 such that $(N_1 : M_1)N_1 \neq N_1$. So there exists $x_1 \in N_1 \backslash (N_1 : M_1)N_1$. Let $0 \neq x_i \in M_i$, for all $i \geq 2$. We have

$$\begin{aligned}(x_1, 0, \dots, 0) &= (1, 0, \dots, 0)(x_1, x_2, \dots, x_m) \in (N_1 \times 0 \times \cdots \times 0)\setminus \\ &\quad (N_1 \times 0 \times \cdots \times 0 : M)(N_1 \times 0 \times \cdots \times 0).\end{aligned}$$

By hypothesis $N_1 \times 0 \times \cdots \times 0$ must be almost prime. But $(1, 0, \dots, 0) \notin (N_1 \times 0 \times \cdots \times 0 : M)$ and $(x_1, \dots, x_m) \notin N_1 \times 0 \times \cdots \times 0$. So M_1 is regular. Likewise, M_i is regular, for all $i \in \{2, \dots, m\}$. □

Corollary 3.13. *Let $m, n \geq 2$ be natural numbers and $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times \cdots \times M_m$, where R_i is a commutative ring and M_i is an R_i-module, for all $i \in \{1, \dots, m\}$. Then, every proper submodule of M is n-almost prime if and only if every proper submodule of M is $(n+1)$-almost prime.*

Proof. ($\Leftarrow$) It is clear.
($\Rightarrow$) Let every proper submodule of M be n-almost prime. We get that M is regular by Theorem 3.11. But $n + 1 > 2$, hence by Theorem 3.11, we obtain that every proper submodule of M is $(n+1)$-almost prime. □

4. Multiplication modules and $(n-1, n) - \phi_\alpha$-prime submodules

Let R be a commutative ring and M an R-module. We know that M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Note that $I \subseteq (N : M)$, hence $N = IM \subseteq (N : M)M \subseteq N$, so that $N = (N : M)M$. We use this concept in the next results.

Theorem 4.1. *(i) Let R be a commutative ring and M_1, M_2 be two R-modules. Let P be a $(n-1, n)$-weakly prime submodule of M_1. Then $Q = P \times M_2$ is a $(n-1, n) - \phi$-prime submodule of $M = M_1 \times M_2$, for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$ $(n \geq 2)$.*

(ii) Let R be a commutative ring, M be an R-module and P be a finitely generated faithful multiplication submodule of M. Suppose that P is $(n-1, n) - \phi$-prime, where $\phi \leq \phi_{n+1}$ and $(P : M)$ is a finitely generated ideal of R. Then either P is $(n-1, n)$-weakly prime or $(P : M)^{n-1}P \neq 0$ and M decomposes as $M_1 \times M_2$, where $M_2 = (P : M)^{n-1}M$ and $P = Q \times M_2$, where Q is $(n-1, n)$-weakly prime. Hence P is $(n-1, n) - \phi$-prime, for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$.

Proof. (i) If P is $(n-1, n)$-prime then Q is $(n-1, n)$-prime, hence is $(n-1, n) - \phi$-prime, for all ϕ. Suppose that P is not $(n-1, n)$-prime.

Then by Corollary 2.2, $(P:M)^{n-1}P=0$. We have $(Q:_R M)=(P:_R M_1)$. So $\phi_\omega(Q)=\bigcap_{i=1}^{\infty}(Q:M)^{i-1}Q=\bigcap_{i=1}^{\infty}(P:M_1)^{i-1}(P\times M_2)=0\times\bigcap_{i=1}^{\infty}(P:M_1)^{i-1}M_2$. Thus $Q\backslash\phi_\omega(Q)=P\times M_2\backslash 0\times\bigcap_{i=1}^{\infty}(P:M_1)^{i-1}M_2=(P\backslash\{0\})\times M_2\backslash\bigcap_{i=1}^{\infty}(P:M_1)^{i-1}M_2$. Thus

$$\begin{aligned}
& a_1\dots a_{n-1}(x,y)\in Q\backslash\phi_\omega(Q)\\
\Rightarrow\ & a_1\dots a_{n-1}x\in P\backslash\{0\}\\
\Rightarrow\ & a_1\dots a_{n-1}\in(P:M_1)=(Q:M)\\
& \text{or}\ \ a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x\in P\ \ \text{for some}\ \ i\in\{1,\dots,n\}\\
\Rightarrow\ & a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}(x,y)\in Q.
\end{aligned}$$

So Q is $(n-1,n)-\phi_\omega$-prime and hence $(n-1,n)-\phi$-prime.

(ii) If P is $(n-1,n)$-prime, then P is not $(n-1,n)$-weakly prime. So we can assume that P is not $(n-1,n)$-prime. Then $(P:M)^{n-1}P\subseteq\phi(P)$; and hence $(P:M)^{n-1}P\subseteq\phi_{n+1}(P)=(P:M)^nP$. So $(P:M)^{n-1}P=(P:M)^{2(n-1)}P$. Thus by [1], Theorem 3.1, we have $(P:M)^{n-1}=(P:M)^{2(n-1)}$. Hence $(P:M)^{n-1}$ is idempotent. Since $(P:M)^{n-1}$ is finitely generated, $(P:M)^{n-1}=(e)$ for some idempotent element $e\in R$. Suppose $(P:M)^{n-1}P=0$. So $\phi(P)=0$ and hence P is $(n-1,n)$-weakly prime. Assume that $(P:M)^{n-1}P\neq 0$. Put $M_2=(P:M)^{n-1}M=(e)M$ and $M_1=(1-e)M$; hence M decomposes as $M_1\times M_2$. Let $Q=(1-e)P$, so $P=Q\times M_2$. We show that Q is $(n-1,n)$-weakly prime. Let $a_1,\dots,a_{n-1}\in R$ and $x\in M_1$ and $0\neq a_1\dots a_{n-1}x\in Q$; so $a_1\dots a_{n-1}(x,0)=(a_1\dots a_{n-1}x,0)\in Q\times M_2=P$. We have $(P:M)^{n-1}P=\{0\}\times M_2$ and $\phi(P)\subseteq(P:M)^{n-1}P$. Hence $P\backslash(P:M)^{n-1}P\subseteq P\backslash\phi(P)$. Since P is $(n-1,n)-\phi$-prime, so

$$\begin{aligned}
& (a_1,\dots,a_{n-1}x,0)\in P\backslash\phi(P)\\
\Rightarrow\ & a_1\dots a_{n-1}(x,0)\in P\backslash\phi(P)\\
\Rightarrow\ & a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x\in Q\ \ \text{for some}\ \ i\in\{1,\dots,n-1\}
\end{aligned}$$

or $a_1,\dots,a_{n-1}\in(P:M)=(Q:M_1)$. Hence Q is $(n-1,n)$-weakly prime. $\square$

Theorem 4.2. *Let (R, m) be a quasi local ring with $m^n = 0$. If M is a multiplication R-module, Then every proper submodule of M is $(n-1, n)$-weakly prime $(n \geq 2)$.*

Proof. Since every multiplication module over a quasi-local ring is cyclic [1], Theorem 2.8, there exists $y \in M$ such that $M = Ry$. Suppose that N is a proper submodule of M. Let $r_1, \ldots, r_{n-1} \in R$ and $x \in M$ and $0 \neq r_1 \ldots r_{n-1}x \in N$. There exists $s \in R$ such that, $x = sy$ and hence $0 \neq r_1 \ldots r_{n-1}sy \in N$. Since $m^n = 0$, we have $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}x \in N$, for some $i \in \{1, \ldots, n-1\}$ or $r_1 \ldots r_{n-1} \in (N : M)$. Thus N is $(n-1, n)$-weakly prime. □

The converse of Theorem 4.2, is not true in general. For example let M be a vector space over the field F with $dimM \geq 2$. We know that M is not a multiplication module. Every proper submodule of M is prime and so is $(n-1, n)$-weakly prime $(n \geq 2)$.

In the following lemma, we will characterize the almost prime submodules of a finitely generated faithful multiplication module.

Lemma 4.3. *Let R be a commutative ring and M be a finitely generated faithful multiplication R-module and let P be an ideal of R.*

(i) If PM is a n–almost prime submodule of M, then P is a n-almost prime ideal of R $(n \geq 2)$.

(ii) If P is an almost prime ideal of R and for every $Q \in Max(R)$ with $P \subset Q$; $P \cap Q^2 = 0$ and $\bigcap_{n \geq 1} Q^n = 0$, then PM is an almost prime submodule of M.

Proof. (i) Suppose that PM is a n-almost prime submodule of M and $r, s \in R$ with $rs \in P \backslash P^n$. Since $N = PM = (N : M)M$, we have by [1], Theorem 3.1, $P = (N : M) = (PM : M)$. So $(PM : M)^{n-1}PM = P^nM$. If $rsM \subseteq P^nM$, then $(rs) \subseteq P^n$ by [1], Theorem 3.1, which is impossible. So $(r)[(s)M] \subseteq PM$ and $(r)[(s)M] \not\subseteq P^nM$. Thus we have by [19], Theorem 2.11, $(r) \subseteq P$ or $(s)M \subseteq PM$. So $r \in P$ or $s \in P$ by [1], Theorem 3.1, and hence P is n-almost prime.

(ii) Suppose that P is an almost prime ideal of R. If $PM = P$, then there exists $a \in P$, such that $(1 - a)M = 0$. Hence $a = 1$, which is impossible. So $PM \neq M$. Let $a \in R$ and $x \in M$ with $ax \in PM \backslash P^2M$. If $a \in P$, then $a \in (PM : M)$. Thus we may assume that $a \notin P$. Put $K = \{r \in R | rx \in PM\}$. If $K = R$, then $x \in PM$. Let $K \neq R$. So there exists $Q \in Max(R)$ with $K \subseteq Q$. Since $a \in K \subseteq Q$, we have $P \subset Q$. Since M is multiplication, by [1], Theorem 1.2, $M = \{m \in M | \exists q \in Q, (1-q)m = 0\}$ or there exists $q \in Q$ and $m \in M$ such that $(1-q)M \subseteq Rm$. If $M = \{m \in M | \exists q \in Q, (1-q)m = 0\}$, then $(1-q)x = 0$ and so $(1-q) \in K \subseteq Q$, a contradiction. Now assume that there exists $m \in M$ and $q \in Q$, such that $(1-q)M \subseteq Rm$. Hence $(1-q)x = sm$, for some $s \in R$. We have $ax \in PM$ and so $(1-q)ax \in (1-q)PM \subseteq Pm$. Therefore there exists $p \in P$ such that $(1-q)ax = pm$ and so $asm = pm$. Again, $[(1-q)ann(m)]M = 0$ and hence $(1-q)ann(m) = 0$. Therefore $(1-q)(as-p) = 0$. So $(1-q)as = (1-q)p \in P$. If $(1-q)p \notin P^2$, then $(1-q)as \in P \backslash P^2$. Since $a \notin P$ and P is almost prime, so $(1-q)s \in P$. Since $(1-q)p \notin P^2$, $(1-q)s \in P \backslash P^2$. Therefore $(1-q) \in P \subset Q$, (a contradiction), or $s \in P$. If $s \in P$, then $(1-q)x \in PM$ and so $(1-q) \in K \subseteq Q$, a contradiction.

It follows that $(1-q)p \in P^2$. Since $P \subseteq Q$ we have $p \in Q \cap P^2 = 0$. So $(1-q)ax = pm = 0$ and hence $ax = qax = q^2ax = \dots$. So $ax \in \bigcap_{n\geq 1}(Q^nM) = (\bigcap_{n\geq 1} Q^n)M = 0$ by [1], Theorem 1.6, which is impossible. □

Anderson and Bataineh [6], Theorem 22, have shown that in a Noetherian ring R every proper ideal of R is a product of almost prime ideals if and only if R is a finite direct product of Dedekind domains, SPIRs, and SPAP-rings. Recall that R is an SPAP-ring, if (R, m) is quasi-local and satisfies the following two conditions: (i) for each $x \in m \backslash m^2$, $(x^2) = m^2$ and (ii) $m^3 = 0$.

Let M be a multiplication R-module and N_1 and N_2 be submodules of M. There exist ideals I_1 and I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Ameri in [4] defined the product of N_1 and N_2 by $N_1N_2 = I_1I_2M$. We use this notion and extend the result in the above paragraph to some modules.

Theorem 4.4. *Let R be a commutative Noetherian ring and for each almost prime ideal P of R and each $Q \in Max(R)$ with $P \subset Q$; $Q^2 \cap P = 0$ and $\bigcap_{n\geq 1} Q^n = 0$. Let M be a finitely generated faithful multiplication R-module. Every proper submodule of M is a product of almost prime submodules if and only if R is a finite direct product of Dedekind domains, SPIRs, and SPAP-rings.*

Proof. ($\Rightarrow$) Suppose that every proper submodule of M is a product of almost prime submodules. Let I be a proper ideal of R. We have IM is a proper submodule of M by [1], Theorem 3.1. Since $N = IM = (N : M)M$, so $I = (N : M)$. By hypothesis $N = (I_1M) \dots (I_nM)$, where I_iM is an almost prime submodule of M, for all $i \in \{1, \dots, n\}$. So $IM = (I_1 \dots I_n)M$. Thus $I = I_1I_2 \dots I_n$ by [1], Theorem 3.1. Now I_i is an almost prime ideal of R, for all $i \in \{1, \dots, n\}$; by Lemma 4.3 (i). So every proper ideal of R is a product of almost prime ideals. Therefore R is a finite direct product of Dedekind domains, SPIRs, and SPAP-rings; by [6], Theorem 22.

($\Leftarrow$) Let R be a finite direct product of Dedekind domains, SPIRs, and SPAP-rings. So every proper ideal of R is a product of almost prime ideals, by [6], Theorem 22. Let N be a proper submodule of M. So $N = IM$, for some proper ideal I of R. Thus $I = I_1I_2 \dots I_n$, where I_i is an almost prime ideal of R, for each $i \in \{1, \dots, n\}$. Now we have I_iM is an almost prime submodule of M by Lemma 4.3 (ii) and $N = I_1 \dots I_nM = I_1M \dots I_nM$. □

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