

## POS-GROUPS WITH SOME CYCLIC SYLOW SUBGROUPS

R. SHEN, W. SHI\* AND J. SHI

Communicated by Ali Reza Ashrafi

**ABSTRACT.** A finite group  $G$  is said to be a POS-group if for each  $x$  in  $G$  the cardinality of the set  $\{y \in G \mid o(y) = o(x)\}$  is a divisor of the order of  $G$ . In this paper we study the structure of POS-groups with some cyclic Sylow subgroups.

### 1. Introduction

Throughout the paper  $G$  denotes a finite group,  $o(x)$  the order of a group element  $x$ , and  $|X|$  the cardinality of a set  $X$ . Denote by  $\pi(G) = \{p \mid p \text{ is a prime divisor of } |G|\}$ . As in [4], the order subset (or, order class) of  $G$  determined by an element  $x \in G$  is defined to be the set  $OS(x) = \{y \in G \mid o(y) = o(x)\}$ . Clearly, for every  $x \in G$ ,  $OS(x)$  is a disjoint union of some conjugacy classes in  $G$ . The group  $G$  is said to have perfect order subsets (in short,  $G$  is called a POS-group) if  $|OS(x)|$  is a divisor of  $|G|$  for all  $x \in G$ . In [4], Finch and Jones first classified abelian POS-groups. Afterwards they continued the study of nonabelian POS-groups and gave some non-solvable POS-groups (see [5],[6]). Recently, Das gave some properties of POS-groups in [2], and Shen classified POS-groups of order  $2m$  with  $(2, m) = 1$  (see [14]). In this note we study POS-groups with some cyclic Sylow subgroups. In

---

MSC(2010): Primary: 20D45; Secondary: 20D06, 11Y50.

Keywords: perfect order subset, POS-group, Frobenius group.

Received: 13 April 2012, Accepted: 21 August 2012.

\*Corresponding author

© 2013 Iranian Mathematical Society.

section 2, POS-groups with cyclic Sylow 2-subgroups are studied. It is proved that if Sylow 2-subgroups of a POS-group  $G$  are cyclic, then 3 divides  $|G|$  or  $G$  has a self-centralized Sylow 2-subgroup. In the next section, we investigate the structure of POS-groups with cyclic Sylow 2-subgroups of order 4. Finally POS-groups with two prime divisors are studied. If  $S$  is a subset of  $G$ , denote by  $f_S(m)$  the number of elements of order  $m$  in  $S$ . Let  $U(n)$  be the unit group of the ring  $Z/nZ$ . Denote by  $\text{ord}_n(q)$  the order of  $q$  in the group  $U(n)$ . First of all, we consider POS-groups with cyclic Sylow 2-subgroups.

## 2. Cyclic Sylow 2-subgroups

In this part, we study POS-groups with cyclic Sylow 2-subgroups, and prove that if Sylow 2-subgroups of a POS-group  $G$  are cyclic, then 3 divides  $|G|$  or  $G$  has a self-centralized Sylow 2-subgroup. A celebrated theorem of Frobenius asserts that if  $n$  is a positive divisor of  $|G|$  and  $X = \{g \in G \mid g^n = 1\}$ , then  $n$  divides  $|X|$  (see, for example, Theorem 9.1.2 of [9]). This result is used in the sequel frequently. First, we cite some lemmas.

**Lemma 2.1.** (Theorem 1, [10]). *If every element of a finite group  $G$  has order which is a power of a prime number and  $G$  is solvable, then  $|\pi(G)| \leq 2$ .*

Recall that  $G$  is a 2-Frobenius group if  $G = ABC$ , where  $A$  and  $AB$  are normal subgroups of  $G$ ,  $AB$  and  $BC$  are Frobenius groups with kernels  $A$  and  $B$ , and complements  $B$  and  $C$  respectively. Recall in addition that  $G$  is a  $C_{pp}$ -group if the centralizer of every non-trivial  $p$ -element is a  $p$ -group. The following lemma is due to Gruenberg and Kegel (see Corollary of [15]).

**Lemma 2.2.** *Let  $G$  be a solvable  $C_{pp}$ -group, then  $G$  is a  $p$ -group, a Frobenius group or a 2-Frobenius group.*

**Lemma 2.3.** (Theorem 3, [16]). *Let  $G$  be a finite group. Then the number of elements whose orders are multiples of  $n$  is either zero, or a multiple of the largest divisor of  $|G|$  that is prime to  $n$ .*

Next we give the following main result.

**Theorem 2.4.** *If the Sylow 2-subgroups of a POS-group  $G$  are cyclic, then 3 is a divisor of  $|G|$ , or  $G$  has a self-centralized Sylow 2-subgroup.*

*Proof.* Suppose that  $P_2$  is a Sylow 2-subgroup of  $G$  and  $|P_2| = 2^n$ . Since Sylow 2-subgroups of  $G$  are cyclic,  $G$  is 2-nilpotent. Let the normal 2-complement of  $G$  be  $H$ . Set  $C_G(P_2) = P_2 \times N$ , where  $N \leq H$ . Next we will prove that 3 is a divisor of  $|N|$  provide  $P_2$  is not self-centralizing. If  $N$  has an element of order  $m$ , then  $f_G(2^nm) = |H/N| \cdot 2^{n-1} \cdot f_N(m)$  is a divisor of  $2^n \cdot |H|$ . So  $f_N(m)$  divides  $2|N|$ . Note that  $|N|$  is odd. It follows that  $4 \nmid f_N(m)$ . Since  $\phi(m)$ , the Euler's totient function, divides  $f_N(m)$ , then we have every order of element of  $N$  is a prime power, and thus  $|\pi(N)| \leq 2$  by Lemma 2.1.

Case I.  $\pi(N) = \{p, q\}$ . Set  $|N| = p^a q^b$ . By Lemma 2.2, we have that  $N$  is a Frobenius or 2-Frobenius group. If  $N$  is Frobenius, without loss of generality, we assume that the order of the kernel of  $N$  has divisor  $q$ . As  $p$ -subgroups are cyclic, then  $f_N(p) = (p-1)q^b$  is a divisor of  $2|N| = 2p^a q^b$ . So  $p-1 = 2$ , then  $p = 3$ . If  $N$  is 2-Frobenius, we set  $N = ABC$ , where  $A$  and  $AB$  are normal subgroups of  $N$ ,  $AB$  and  $BC$  are Frobenius groups with kernels  $A$  and  $B$ , and complements  $B$  and  $C$ , respectively. Now let  $|A| = p^{a_1}$  and  $|C| = p^{a_2}$ . Then  $f_N(q) = (q-1)p^{a_1}$  divides  $2|N| = 2p^a q^b$ . Since  $p^a \mid f_N(q)$  by Lemma 2.3, it follows that  $q = 2p^{a_2} + 1$ . Clearly,  $q > p$ . In addition,  $f_N(p) = f_A(p) + (p-1)q^b |A : C_N(c)|$ , where  $c$  is an element of order  $p$  of  $C$ . Since  $p-1$  and  $q^b$  are both divisors of  $f_A(p)$  and  $(p-1, q) = 1$ , we have  $(p-1)q^b \mid f_A(p)$ . Then  $p-1 \mid 2p^a$ , so  $p = 3$ .

Case II.  $\pi(N) = \{p\}$ . Set  $|N| = p^a$ . Then by the above discussion we see that  $f_N(p) \mid 2p^a$ . Since  $p \nmid f_N(p)$ , we have  $p = 3$ .  $\square$

Note that indeed there exist POS-groups of Theorem 2.4 whose Sylow 2-subgroups are self-centralized and  $3 \nmid |G|$ . The following is an example of a POS group of order 400 whose Sylow 2-subgroups are cyclic and self-centralizing.

**Example 2.5.** Let  $G = \langle a, b \mid a^{25} = b^{16} = 1, a^b = a^{-1}, [a, b^2] = 1 \rangle$ . Then  $G$  is a POS-group with a cyclic Sylow 2-subgroup of order  $2^4$ .

Finch and Jone formulated a question in [4] whether the order of every POS-group with more than one prime divisor has a divisor 3. Although this question has a negative answer, it seems that orders of most POS-groups have the divisor 3. We put the following problem.

**Problem 2.6.** Classify POS-groups whose order has no divisor 3.

### 3. Cyclic Sylow 2-subgroups of order 4

In this section, we deal with the POS-groups with cyclic Sylow 2-subgroups of order 4. We completely classify such POS-groups whose order has no divisor 3. First, we determine the number of prime divisors of these groups.

**Proposition 3.1.** *Let  $G$  be a POS-group with cyclic Sylow 2-subgroups of order 4. Then  $|\pi(G)| \leq 6$ .*

*Proof.* Let  $\sigma(G) = \max\{|\pi(o(g))| \mid g \in G\}$ , and  $H$  be the normal 2-complement of  $G$ . Clearly,  $\sigma(H) \leq 2$ . So we have  $|\pi(H)| \leq 5$  by Theorem 1.4(b) of [11]. Therefore,  $|\pi(G)| \leq 6$ .  $\square$

Although such POS-groups have an upper bound of the number of prime divisors, is 6 the actual bound?

**Lemma 3.2.** *Let  $|G| = 2^n p^m$  and the Sylow  $p$ -subgroup  $P$  be normal in  $G$ . If all Sylow subgroups of  $G$  are cyclic and  $G$  has no element of order  $2p^m$ , then  $G$  is a Frobenius group.*

*Proof.* Since all Sylow subgroups of  $G$  are cyclic, we may see that  $G = \langle a, b \mid a^{p^m} = b^{2^n} = 1, a^b = a^r \rangle$  such that  $r^{2^n} \equiv 1 \pmod{p^m}$  and  $(2^n(r-1), p^m) = 1$ . By the above condition, we get that  $2^n$  is the order of  $r$  in  $U(p^m)$ . In fact, otherwise if the order  $\text{ord}_{p^m}(r) (= o(r))$  of  $r$  is less than  $2^n$ , then

$$a^{b^{o(r)}} = a^{r^{o(r)}} = a^1 = a,$$

hence  $b^{o(r)} \in C_G(P)$ . On the other hand, since  $C_G(P) = 1$ ,  $b^{o(r)} = 1$ , which contradicts that  $o(b) = 2^n$ . It is easy to see that the order  $\text{ord}_{p^i}(r)$  is also  $2^n$  for  $1 \leq i \leq n-1$ . So the centralizer of every nontrivial  $p$ -element is  $P$ , and then  $G$  is a Frobenius group.  $\square$

To complete the proof of Theorem 3.6 and 4.3, we need some conclusions of prime number. We call  $r_m(a)$  a *primitive* prime divisor of  $a^m - 1$  if  $r_m(a) \mid a^m - 1$  but  $r_m(a)$  doesn't divide  $a^i - 1$  for every  $i < m$ . Clearly, for primitive prime divisor  $p = r_m(a)$ , the formula  $m \mid p-1$  always holds. Let  $\Phi_n(x)$  be the  $n^{\text{th}}$  cyclotomic polynomial. It is well known that  $x^n - 1$  may be decomposed to the product of all cyclotomic polynomials whose digit is some divisor of  $n$ , that is,  $x^n - 1 = \prod_{d \mid n} \Phi_d(x)$ . The existence of primitive prime divisor is due to Zsigmondy (see [17]), and the primitive prime divisor is closely connected with the cyclotomic polynomial as follows.

**Lemma 3.3.** *Primitive prime divisors of  $a^m - 1$  exist except if  $m = 6$  and  $a = 2$ , or  $m = 2$  and  $a = 2^k - 1$ .*

**Lemma 3.4.** *Suppose that  $q^n - 1$  has at least one primitive prime divisor and  $n \geq 3$ . Then  $\Phi_n(q) = (P(n), \Phi_n(q)) \cdot Z_n(q)$ , where  $P(n)$  is the largest prime divisor of  $n$  and  $Z_n(q)$  the largest divisor of  $q^n - 1$  which contains all primitive prime divisors.*

*Proof.* By page 207 of [13] and Lemma 2.1 of [3], we have  $Z_n(q) \mid \Phi_n(q)$  and  $\Phi_n(q) \mid Z_n(q) \cdot P(n)$ , and then  $\Phi_n(q) = (P(n), \Phi_n(q)) \cdot Z_n(q)$ .  $\square$

**Lemma 3.5.** (Lemma 5, [12]). *Suppose that  $p$  is an odd primitive prime divisor of  $q^k - 1$ . Then  $p \mid \Phi_f(q)$  if and only if  $f = kp^j$  for some  $j \geq 0$ .*

We also make some preliminaries on the  $p$ -adic expansion of any integer. Let  $p$  be a prime. So any positive integer can be written in a base  $p$  expansion in the form

$$\sum_{i=0}^n a_i p^i,$$

where all  $a_i$  are integers in  $\{0, 1, \dots, p-1\}$ . Moreover for a given positive integer  $m$ , the coefficients  $a_i$  in such  $p$ -adic expansion of  $m$  are determined uniquely.

Next we give the structure of POS-groups with cyclic Sylow 2-subgroups of order 4.

**Theorem 3.6.** *Let  $G$  be a POS-group with a cyclic Sylow 2-subgroup of order 4. Then 3 is a divisor of  $|G|$ , or  $G$  is one of the following groups:*

- (a) *the cyclic group of order 4;*
- (b) *Frobenius groups  $Z_{5^m} : Z_4$ ;*
- (c) *quasi-dihedral groups  $\langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^{-1} \rangle$ .*

*Proof.* Let  $H$  be the normal 2-complement. If  $4 \nmid p-1$  for every  $p \in \pi(H)$ , then  $3 \in \pi(H)$  since the smallest prime number in  $\pi(H)$  is a Fermat prime. Next we assume that  $\pi(H)$  has a prime  $p$  such that  $4 \mid p-1$ . Then  $H$  is a  $C_{pp}$ -group, and hence  $H$  is Frobenius or 2-Frobenius by Lemma 2.2. Note that  $\pi(H)$  has only such prime  $p$  (see [11]). We should consider the following three cases.

Case I.  $H$  is Frobenius. Let  $H = K : L$  with the kernel  $K$  and the complement  $L$ . If  $L$  is a  $p$ -group, then  $L$  is cyclic. Since  $f_H(p) = (p-1)|K|$  divides  $4|K| \cdot |L|$ , we have  $p = 5$ . In addition, since  $K$  is nilpotent,  $K$  has at least two prime divisors. If  $|\pi(K)| = 1$ , clearly

$3 \in \pi(H)$ . Next let  $\pi(K) = \{p_1, p_2\}$ . Suppose that  $P_i$  is a Sylow  $p_i$ -subgroup of  $K$  for  $i = 1, 2$ . Then  $f_H(p_i)$  divides  $4|P_1| \cdot |P_2| \cdot |L|$ . If  $3 \nmid |H|$ , then we assume that  $5 < p_1 < p_2$ . Note that  $p_i \nmid f_H(p_i)$ , so we have  $p_1 = 2 \cdot 5^k + 1$  with  $k \geq 1$ . Set  $p_2 = 2 \cdot 5^{k_1} p_1^{k_2} + 1$ . Let  $u$  be an element of order 4. By Theorem 2.4,  $u$  is a fixed-point-free automorphism of  $H$ . Next we will prove that  $K$  is abelian. Otherwise, we assume that  $H_0 = K_0 : L_0$  is the minimal counterexample. Thus  $K_0$  is a  $p_1$ -group. Set  $\Phi(K_0)$  the Frattini subgroup of  $K_0$ . Clearly  $\Phi(K_0) > 1$ . Since  $\Phi(K_0)$  is a characteristic subgroup of  $K_0$ ,  $K_0/\Phi(K_0) : L_0$  has a fixed-point-free automorphism of order 4. So  $K_0/\Phi(K_0)$  is abelian, and then  $K_0$  is abelian, a contradiction. Thus  $K$  is abelian. Let  $|G| = 4 \cdot 5^m p_1^a p_2^b$ . Since  $K$  is abelian, we may assume that  $f_H(p_i) = p_i^{s_i} - 1$  for  $i = 1, 2$ . In addition, since  $f_H(p_i)$  divides  $|G|$ , we get a Diophantine equation

$$p_i^{s_i} - 1 = 2^u 5^j p_1^s p_2^t \quad (3.1)$$

where  $j, s, t \geq 0$  and  $1 \leq u \leq 2$ . Next we will prove that  $s_i = 1$  for  $i = 1, 2$ . The following two subcases, should be studied.

Subcase I.I  $i = 1$ . Then clearly  $s = 0$ . In view of Lemma 3.4,  $p_1^{s_1} - 1$  has a primitive prime divisor except if  $p_1$  is a Mersenne prime and  $s_1 = 2$ . If  $t = 0$ , since  $\pi(p_1 - 1) = \{2, 5\}$ , we have  $s_1 = 1$  or 2 by Lemma 3.4. Now if  $s_1 = 2$ , then  $p_1$  is a Mersenne prime, say  $2^l - 1$ . So  $s_1^2 - 1 = 2^{l+1}(2^{l-1} + 1) = 2^u 5^j$ , then  $l = 1$  since  $u \leq 2$ , a contradiction. When  $t > 0$ , the equation (3.1) becomes

$$p_1^{s_1} - 1 = 2^u 5^j p_2^t. \quad (3.2)$$

Similarly, since  $\pi(p_1 - 1) = \{2, 5\}$ , then  $s_1$  is equal to 1 or a prime by Lemma 3.4. Since  $p_2$  is a primitive prime divisor of  $p_1^{s_1} - 1$ ,  $s_1 | p_2 - 1 = 2 \cdot 5^{k_1} p_1^{k_2}$ . So  $s_1 = 1, 2, 5$  or  $p_1$ . If  $s_1 = 2$ , it is easy to see  $8 | p_1^2 - 1 = f_H(p_1)$ , a contradiction. If  $s_1 = 5$ , by Lemmas 3.5 and 3.6, then (3.2) becomes

$$\frac{p_1^5 - 1}{5(p_1 - 1)} = p_2^t. \quad (3.3)$$

If  $k_2 > 0$ , then (3.3) becomes

$$16 \cdot 5^{4k-1} + 8 \cdot 5^{3k} + 8 \cdot 5^{2k} + 4 \cdot 5^k + 1 = (2 \cdot 5^{k_1} (2 \cdot 5^k + 1)^{k_2} + 1)^t. \quad (3.4)$$

(3.4) is the expansion of  $p_2^t$ , if  $k > 0$  in base 5. The left hand side modulo  $5^{2k}$  is  $4 \cdot 5^k + 1$ . Clearly,  $t < 5$ . So  $k = k_1$  and  $t = 2$ . Moreover, the 5-adic expansion of the left term of (3.4) is  $3 \cdot 5^{4k} + 5^{4k-1} + 5^{3k-1} + 3 \cdot 5^{3k} + 5^{2k+1} + 3 \cdot 5^{2k} + 4 \cdot 5^k + 1$ . But the largest digit of one of the right term is more than or equal to  $2k(k_2 + 1)$ , so  $2k(k_2 + 1) \leq 4k$ , then  $k_2 = 1$ . It follows

that the right term of (3.4) is equal to  $16 \cdot 5^{4k} + 16 \cdot 5^{3k} + 12 \cdot 5^{2k} + 4 \cdot 5^k + 1$ , a contradiction.

If  $k_2 = 0$ , then (3.3) becomes

$$16 \cdot 5^{4k-1} + 8 \cdot 5^{3k} + 8 \cdot 5^{2k} + 4 \cdot 5^k + 1 = (2 \cdot 5^{k_1} + 1)^t. \quad (3.5)$$

(3.5) is the expansion of  $p_2^t$  in base (5) if  $k_2 = 0$ , then we may see  $k = k_1$ . Comparing the largest digits of the formulas of 5-adic expansion of both sides of (3.5), we got that  $t \leq 4$ . It is easy to check that for every such  $t$  the equation (3.5) does not hold.

If  $s_1 = p_1$ , then, by Lemmas 3.5 and 3.6, (3.2) becomes

$$\frac{p_1^{p_1} - 1}{p_1 - 1} = p_2^t. \quad (3.6)$$

If  $k_2 = 0$ , then  $p_2$  is a primitive prime divisor of  $p_1^{p_1} - 1$  since  $p_1 < p_2$ . So  $p_1 \mid p_2 - 1 = 2 \cdot 5^{k_1}$ , and then  $p_1 = 2$  or  $5$ . By (3.6), we have  $p_2^t = 3$  or  $781$ . Since  $781 = 11 \cdot 71$  has two prime divisors, it follows that  $p_2 = 3$ , which contradicts that  $p_2 > 5$ .

When  $k_2 > 0$ , we extend the number of (3.6) into the  $p_1$ -adic expansion, in which the first and second digits of the left and right are  $p_1 + 1$  and  $l \cdot p_1^{k_2} + 1$  and  $p_1 > l \equiv 2t \cdot 5^k \pmod{p_1}$ . So  $k_2 = 1$  and  $2t \cdot 5^k \equiv 1 \pmod{p_1}$ . It follows that  $p_1 \mid 2t \cdot 5^k - 1 + p_1 = 2 \cdot 5^k(t+1)$ , and then  $t+1 \equiv 0 \pmod{p_1}$ . On the other hand,  $t < p_1$ , we have  $t = p_1 - 1$ . Thus (3.6) is changed into

$$\frac{p_1^{p_1} - 1}{p_1 - 1} = (2 \cdot 5^k p_1 + 1)^{p_1-1}. \quad (3.7)$$

It is not hard to see that the largest digit of right hand of (3.7) is more than one of left term in light of the form of the  $p_1$ -adic expansion of both side of (3.7), a contradiction. Thus  $s_1 = 1$ .

Subcase I.II.  $i = 2$ . Then (3.1) becomes

$$p_2^{s_2} - 1 = 2^u 5^j p_1^s. \quad (3.8)$$

Clearly,  $s_2 \neq 2$  (otherwise  $8 \mid p_2^2 - 1$ ). If  $k_1$  and  $k_2$  are both more than 0, since  $\pi(p_2 - 1) = \{2, 5, p_1\}$ , we have that  $s_2 = 1$  by Lemma 3.4. If  $k_1 = 0$ , then 5 is a primitive prime divisor of  $p_2^{s_2} - 1$ . So  $s_2 \mid 5 - 1 = 4$ , hence  $s_2 = 1$  or  $4$ . But if  $s_2 = 4$ , gives that  $8 \mid p_2^4 - 1$ , a contradiction.

$k_2 = 0$ , gives that  $s_2 \mid p_1 - 1 = 2 \cdot 5^k$ . So  $s_2 = 5$ . Next we may only consider following Diophantine equation by Lemmas 3.5 and 3.6 that is

$$\frac{p_2^5 - 1}{p_2 - 1} = 5p_1^s. \quad (3.9)$$

Using the same method as one of (3.3), we can get that the (3.9) also has no solution. Thus  $s_2$  is also equal to 1. Therefore,  $K$  is cyclic. Since  $5^m$  divides  $f_H(p_i)$ , we have  $p_1 = 2 \cdot 5^m + 1$  and  $p_2 = 2 \cdot 5^m \cdot p_1^{k_2} + 1$  with  $k_2 \geq 0$ . Now note that  $f_H(p_1^a p_2^b) = \phi(p_1^a p_2^b) = 4 \cdot 5^{2m} p_1^{k_2+a-1} p_2^{b-1}$  divides  $|G| = 4 \cdot 5^m p_1^a p_2^b$ , a contradiction.

If  $K$  is a  $p$ -group, then  $f_H(q)$  divides  $4|L|$  for every  $q \in \pi(L)$ . But since  $4 \nmid q - 1$  and the smallest prime in  $\pi(L)$  is a Fermat one, we have  $3 \in \pi(L)$ .

Case II.  $H$  is 2-Frobenius. Similarly, assume that  $H = ABC$ , where  $A, B, C$  are the same as above. Clearly, the commutator subgroup  $H' = AB$ . If  $3 \nmid |H|$ , by Theorem 2.4,  $H$  admits a fixed-point-freely automorphism of order 4. Then  $H'$  is nilpotent (see Exercises 1, Chap. 10, [7]), a contradiction.

Case III.  $H$  is a  $p$ -group. Certainly,  $p = 5$ . By Proposition 2.8 of [2], the Sylow 5-subgroup, that is  $H$ , is cyclic. Let  $|H| = 5^m$ . By Theorem 2.4 we have that the Sylow 2-subgroup of  $G$  is self-centralized. So  $G$  is not cyclic. Then  $G = \langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^r \rangle$  such that  $r^4 \equiv 1 \pmod{5^m}$  and  $(r - 1, 5^m) = 1$ . If the centralizer of  $a$  is  $\langle a \rangle$ , then  $G$  is Frobenius by Lemma 3.3. If  $|C_G(a)| = 2 \cdot 5^m$ , then  $r^2 \equiv 1 \pmod{5^m}$ . Since  $(r - 1, 5^m) = 1$ , we have  $r \equiv -1 \pmod{5^m}$ . Therefore,  $G = \langle a, b \mid a^{5^m} = b^4 = 1, a^b = a^{-1} \rangle$ .  $\square$

#### 4. POS-groups with two prime divisors

In this section, assume that  $|G| = 2^n p^m$  with  $p$  an odd prime number. If  $G$  is a POS-group, then  $p$  is a Fermat prime, say  $2^{2^k} + 1$ . By Proposition 3.1 of [2], we know that if  $2^n < (p - 1)^3$ , i. e.,  $n < 3 \cdot 2^k$ , then the Sylow  $p$ -subgroup is cyclic and normal. Certainly there exists a POS-group with non-normal cyclic Sylow subgroups, such as  $SL_2(3)$  of order 24. In this section, we give the structure of  $G$  with cyclic Sylow  $p$ -subgroups. First we cite some lemmas.

**Lemma 4.1.** (Theorem 1, [1]). *Let  $G$  be a 2-group of order  $2^n$  and  $\exp(G) = 2^e > 2$ . Then the number of elements of order  $2^i$  is a multiple of  $2^i$  for  $2 \leq i \leq e$  except in the following cases:*

- (a) the cyclic 2-group;



- (b) the dihedral 2-group  $\langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{-1} \rangle$ ;
- (c) the semi-dihedral 2-group  $\langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1} \rangle$ ;
- (d) the generalized quaternion 2-group  $\langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, a^b = a^{-1} \rangle$ .

**Lemma 4.2.** *Let  $G$  be a finite group with a normal subgroup  $N$ . If  $x \in G \setminus N$  has order  $m$ , then  $f_{Nx}(m) = f_{Ny}(m)$  for all cosets  $Ny$  which are  $G/N$ -conjugate to  $Nx$ .*

*Proof.* Suppose that  $Ny$  is  $G/N$ -conjugate to  $Nx$ , so  $Ny = Ng^{-1}xg$  for some  $g \in G$ . Then the map  $\varphi : Nx \mapsto Ny$ , defined by  $nx \mapsto g^{-1}nxg$ , induces a bijection between the subset of elements of order  $m$  in  $Nx$  and the corresponding subset of  $Ny$ .  $\square$

In the following we give the structure of POS-groups with two prime divisors and cyclic Sylow  $p$ -subgroups.

**Theorem 4.3.** *Let  $G$  be a POS-group with a cyclic Sylow  $p$ -subgroup  $P$  and  $|\pi(G)| = 2$ . Then  $G$  is a Frobenius group  $Z_{p^m} : Z_{2^{2k}}$ , where  $p = 1 + 2^{2k}$  is a Fermat prime and  $m > 0$  arbitrary, or satisfies one of the following conditions:*

- (a)  $p = 3$ ,  $C_G(P) \cong P \times Z_2 \times Z_2$  and  $N_G(P) \cong P : (Z_2 \times Z_4)$ ;
- (b) the number of elements of order 2 of  $G$  is 1;
- (c)  $G$  is  $p$ -nilpotent.

*Proof.* Suppose that  $P = \langle x \rangle$  is a Sylow  $p$ -subgroup and the number of Sylow  $p$ -subgroups is  $|G : N_G(P)| = 2^t$ . By Zassenhaus's theorem we may let  $N = N_G(P) = P : K$  and  $C = C_G(P) = P \times U$ , where  $K$  and  $U$  are 2-subgroups of  $G$ . Since  $C_G(x^g) = C_G(x)^g$  and  $C_G(x) = C_G(\langle x \rangle)$ , we have that the number  $f_G(2p^m)$  of elements of order  $2p^m$  is

$$2^t \phi(p^m) f_{C_G(x)}(2) = 2^{t+2k} p^{m-1} f_{C_G(x)}(2),$$

where  $\phi$  is Euler totient function. Let  $|G| = 2^n p^m$ , where  $p = 2^{2k} + 1$  is Fermat. Since  $f_G(2p^m)$  is a divisor of  $|G| = 2^n p^m$  and  $f_{C_G(x)}(2)$  is odd or 0, it leads to  $f_{C_G(x)}(2) = 0, 1$  or  $p$ . Now note that

$$K/U \cong N/C \lesssim \text{Aut}(P) \cong Z_{2^{2k} p^{m-1}},$$

so  $U/K$  is cyclic. If  $K/U = 1$ , that is  $N = C$ , then  $G$  is  $p$ -nilpotent by the well-known Burnside's theorem. If  $K/U \neq 1$ , then  $N/U \cong P : K/U$  has no element of order  $2p^m$ . In fact, otherwise we may choose an element  $y \in K \setminus U$  such that  $xUy^U = xU$ , and then  $x^{-1}x^y \in U$ . Since

$\langle x \rangle = P \triangleleft G$ , we have  $x^{-1}x^y \in P$ . So  $x^{-1}x^y \in U \cap P = 1$ , then  $x^y = x$ . Hence  $y \in U$ , a contradiction. Therefore  $N/U$  is a Frobenius group by Lemma 3.3. If  $U \neq 1$ , that is  $f_U(2) \neq 0$ , then we have

$$f_G(2) = f_U(2) + f_{N \setminus U}(2) + f_{G \setminus N}(2) = f_U(2) + p^m f_{yU}(2) + f_{G \setminus N}(2)$$

by Lemma 4.2, where  $o(yU) = 2$ . If  $f_{yU}(2) \neq 0$ , then  $f_G(2) > p^m$ , which contradicts that  $f_G(2) \mid p^m$ . So  $f_G(2) = f_U(2) + f_{G \setminus N}(2) = f_{\bigcup_{g \in G} U^g}(2) + f_{G \setminus \bigcup_{g \in G} K^g}(2)$ . If  $f_U(2) = 1$ , denote by  $z$  the unique element of order 2 in  $U$ , then  $f_{\bigcup_{g \in G} U^g}(2) = |G : C_G(z)|$ . So we may assume that  $f_{\bigcup_{g \in G} U^g}(2) = 2^s$ . Now we choose any element  $a$  of order 2 in  $G \setminus \bigcup_{g \in G} K^g$ , we see that  $p^m \nmid |C_G(a)|$ , hence  $p \mid f_{G \setminus \bigcup_{g \in G} K^g}(2)$ . We set  $f_{G \setminus \bigcup_{g \in G} K^g}(2) = p \cdot l$ . Now use class equation of  $G$ , we can obtain  $f_G(2) = 2^s + p \cdot l \mid p^m$ . So  $s = l = 0$ , and then  $f_G(2) = 1$ . Next we consider the case of  $f_U(2) = p$ . Now assume that  $|U| = 2^u$ .

If  $f_U(4) = 0$ , then  $U$  is an elementary abelian 2-group. So  $f_U(2) = 2^u - 1 = p$ . It follows that  $u = 2$  and  $k = 0$ . By Lemma 2.3, we have  $2^n \mid f_G(p^m) + f_G(2p^m) = 2^{t+2^k} p^{m-1}(1+p) = 2^{t+3} p^{m-1}$ , and then  $n \leq t + 3$ . So  $|K| = 2^{n-t} \leq 2^3$ . It is easy to see that  $K$  is  $Z_2 \times Z_4$ .

If  $f_U(4) \neq 0$ , then  $f_G(4p^m) = 2^{t+2^k} p^m f_U(4) \mid 2^n p^m$ . So  $f_U(4) \mid 2^{n-t-2^k} p$ . On the other hand, since

$$4 \mid 1 + f_U(2) + f_U(4) = 2 + 2^{2^k} + f_U(4),$$

and  $f_U(4) \mid 2p$ . By Lemma 4.1, we have that  $U$  is a dihedral or semi-dihedral 2-group. If  $U$  is a semi-dihedral one, then  $f_U(2) = 1 + 2^{u-2}$  and  $f_U(4) = 2 + 2^{u-2}$ , which contradicts that  $f_U(2) = p$  and  $f_U(4) \mid 2p$ . If  $U$  is a dihedral one, then  $f_U(2) = 2^{u-1} + 1 = p$  and  $f_U(2^i) = \phi(2^i) = 2^{i-1}$  for  $1 \leq i \leq u-1$ . Certainly,  $u = 1 + 2^{2^k}$  since  $2^{u-1} + 1 = p$ , we get that

$$|U| = 2^{2^k+1}. \quad (4.1)$$

Moreover, since  $K/U$  is a cyclic 2-group, we have that there exists  $L \triangleleft K$  such that  $L/U \cong Z_2$ . By the above discussion, we may see that  $L$  has no element of order 2. In addition,  $L \setminus U$  has an element of order 4. In fact, otherwise  $f_L(4) = 2$ , by Lemma 4.1,  $L$  is a cyclic, dihedral, semi-dihedral or generalized quaternion 2-group, which have an element of order 4 in  $L \setminus U$ , a contradiction. So  $f_G(4) > 2$ . Next we discuss the number of elements of order 4 in  $G$ . Clearly,

$$f_G(4) = f_{\bigcup_{g \in G} U^g}(4) + f_{G \setminus \bigcup_{g \in G} U^g}(4).$$

Note that two elements of order 4 in  $U$  are conjugate, so the elements of order 4 in  $\bigcup_{g \in G} U^g$  make one conjugacy class of  $G$ . Thus

$$f_{\bigcup_{g \in G} U^g}(4) = |G : C_G(w)|,$$

where  $w$  is of order 4 in  $U$ . Since  $p^m \mid |C_G(w)|$ , we may set  $f_{\bigcup_{g \in G} U^g}(4) = 2^s$ . In addition, obviously  $p \mid f_{G \setminus \bigcup_{g \in G} K^g}(4)$ . Assume that  $f_{G \setminus \bigcup_{g \in G} K^g}(4) = p \cdot l$ . Then we have  $f_G(4) = 2^s + p \cdot l \mid 2^n p^m$ , hence  $f_G(4) = 2^j$  for  $1 \leq j \leq n$ . Since  $f_G(2) \mid p^m$ , we may set  $f_G(2) = p^i$ . By Frobenius's theorem, we have

$$4 \mid 1 + f_G(2) + f_G(4) = 1 + 2^j + p^i.$$

Since  $f_G(4) > 2$ , we have  $4 \mid 1 + p^i$ , and thus  $k = 0$  and  $i$  is odd. By Lemma 2.3 we have  $2^n \mid f_G(p^m) + f_G(2p^m) + \cdots + f_G(2^{u-1}p^m) = 2^{t+1}p^{m-1}(1 + p + f_U(4)) + \cdots + f_U(2^{u-1}) = 2^{t+2}p^{m-1}(1 + 2^{u-2})$ , and so  $n \leq t + 2$ . It leads to

$$|K| = 2^{n-t} \leq 2^2. \quad (4.2)$$

By (4.1) we may get  $U = K$ , which contradicts that  $U \neq K$ .

If  $U = 1$ , then  $N = P : K$  is a Frobenius group and  $K$  is cyclic. Then the number  $f_N(2)$  of order 2 in  $N$  is equal to  $p^m f_K(2)$ . Since  $f_K(2) \geq 1$ , we have  $f_G(2) \geq f_N(2) \geq p^m$ . On the other hand,  $f_G(2) \mid p^m$ . Hence  $f_G(2) = p^m$ . By Lemma 2.3, we have  $2^n \mid f_G(p^m)$ , and  $f_G(p^m) = 2^t \cdot \phi(p^m)$ . Hence  $t = n - 2^k$ . Next we use induction to prove that  $f_G(2^i) = 2^{i-1}p^m$  for  $1 \leq i \leq 2^k$ . Clearly when  $i = 1$ , it is true. Assume that it is true for  $i = j - 1$ . Now we deal with the case of  $i = j$ . Since

$$2^j \mid 1 + f_G(2) + \cdots + f_G(2^j) = 1 + p^m + \cdots + 2^{j-2}p^m + 2^{j-1}p^m + f_{G \setminus N}(2^j)$$

and  $p^m \equiv 1 \pmod{2^j}$ , we have  $2^j \mid f_{G \setminus N}(2^j)$ . On the other hand, since  $f_G(2^j) = 2^{j-1}p^m + f_{G \setminus N}(2^j) \mid 2^n p^m$ , we have  $2^j \nmid f_G(2^j)$ . Hence  $f_G(2^j) = 2^{j-1}p^m$ . Since every Sylow 2-subgroup of  $G$  has at most one subgroup of order 2 of  $N$  (otherwise the generated subgroup by some two elements of order 2 of  $N$  is a Frobenius group, which is not a 2-group), we have that the number of Sylow 2-subgroups is  $p^m$ , and then the intersection of every pair Sylow 2-subgroups is trivial. Thus  $G$  is a Frobenius group. It leads to  $t = 0$ , that is  $n = 2^k$ , hence  $G \cong Z_{p^m} : Z_{2^{2^k}}$  is a Frobenius group.  $\square$

Note that there exist groups satisfying the condition (a) of Theorem 4.3. We give an example as follows.

**Example 4.4.**  $\langle a, b, c \mid a^{3^m} = b^2 = c^4 = 1, a^b = a, a^{c^2} = a, a^c = a^{-1}, [b, c] = 1 \rangle$  is a POS-group of order  $8 \cdot 3^m$  with cyclic Sylow 3-subgroups.

Using the GAP software [8], it seems that the group satisfying the condition (a) has not been found except those of Example 4.4. We put the following conjecture.

**Conjecture 4.5.** POS-groups satisfying the condition (a) of Theorem 4.3 are those of Example 4.4.

It is not hard to determine groups satisfying the condition (b) of Theorem 4.3. Since the number of elements of order 2 of  $G$  is 1, so is for a Sylow 2-subgroup  $P_2$  of  $G$ . Thus  $P_2$  is cyclic or a generalized quaternion group. Those groups were classified by Zassenhaus in [18]. The Table 1 lists all such groups with two prime divisors.

TABLE 1.

Type	Order	Generators	Relations	Conditions
I	$2^n p^m$		cyclic group	
II	$2^n p^m$	$a, b$	$a^{p^m} = b^{2^n} = 1,$ $a^b = a^r$	$(r-1, p^m) = 1,$ $r^{2^n} \equiv 1 \pmod{p^m}$
III	$2^{n+1} p^m$	$a, b, c$	$b^{2^{n-1}} = c^2,$ $a^c = a^s, b^c = b^{-1}$	$n \geq 2,$ $s^2 \equiv 1 \pmod{p^m}$

We now give the following result. Note that  $p = 2^{2^k} + 1$  is a Fermat prime.

**Theorem 4.6.** Let  $G$  be a POS-group with a cyclic Sylow  $p$ -subgroup and  $|\pi(G)| = 2$ . If the number of elements of order 2 of  $G$  is 1, then  $G$  is one of the following groups:

- (a) cyclic groups  $Z_{2^n 3^m}$ ;
- (b) groups  $\langle a, b \mid a^{p^m} = b^{2^n} = 1, a^b = a^r \rangle$ , where  $\text{ord}_{p^m}(r) \geq 2^k$ ;
- (c) groups  $\langle a, b \mid a^{p^m} = b^{2^{2^k+1}} = 1, b^{2^{2^k}} = c^2, a^b = a^r, a^c = a^{-1}, b^c = b^{-1} \rangle$ , where  $\text{ord}_{p^m}(r) \geq 2^k$ .

*Proof.* Clearly if  $G$  is cyclic, then  $G \cong Z_{2^n 3^m}$ . Let  $\text{ord}_{p^m}(r) = o(r)$ . Assume that  $G$  is of Type II. Since  $a^b = a^r$  and  $o(r) = \text{ord}_{p^t}(r)$  for

$1 \leq t \leq m$ , we have  $(a^{p^i})^{b^{o(r)}} = (a^{p^i})^{r^{o(r)}} = a^{p^i}$  with  $1 \leq i \leq m-1$ . So  $G$  has an element of order  $2^{n-o(r)}p^{m-i}$  and the number of these elements is  $\phi(2^{n-o(r)}p^{m-i}) = 2^{2^k+n-o(r)-1}p^{m-i-1}$ . Then  $2^k + n - o(r) - 1 \leq n$ , that is  $o(r) \geq 2^k - 1$ . Clearly  $o(r) | 2^{2^k}$ , thus  $o(r) \geq 2^k$ . For other order  $2^i$  of the element  $x_i \in G$  for  $1 \leq i \leq n$ , the number of these elements is  $\phi(2^i) \cdot |G : N_G(\langle x_i \rangle)|$ , which is a divisor of  $|G|$ .

Assume that  $G$  is of Type III. If  $s \equiv 1 \pmod{p^m}$ , then  $f_G(4) = 2|\langle a, b \rangle : N_{\langle a, b \rangle}(\langle x \rangle)| + (2^n p^m - p^m + 1)$ , where  $x \in \langle a, b \rangle$  is of order 4. We may assume that  $|\langle a, b \rangle : N_{\langle a, b \rangle}(\langle x \rangle)| = p^t$ . Then  $f_G(4) = 2p^t + (2^n - 1)p^m + 1$  is a divisor of  $2^{n+1}p^m$ . So  $t = 0$  and  $p = 3$ , we may get a Diophantine equation

$$1 + (2^n - 1) \cdot 3^{m-1} = 2^i. \quad (4.3)$$

Clearly,  $n | i$ . So

$$\frac{2^i - 1}{2^n - 1} = 3^{m-1}. \quad (4.4)$$

By Lemma 3.4, we have that 3 is a primitive prime divisor of  $2^2 - 1$ . It follows that  $i = 2$  or 6. Thus  $n = 1$  and  $m = 2$ , or  $n = 3$  and  $m = 3$ . Since  $n \geq 2$ , we have  $n = 3$  and  $m = 3$ , that is  $G = \langle a, b, c | a^{2^7} = b^8 = 1, a^b = a^r, b^4 = c^2, a^c = a, b^c = b^{-1} \rangle$ . Using the GAP [8], we checked  $r = 1$  or  $-1$ ,  $G$  is not a POS-group.

If  $s \equiv -1 \pmod{p^m}$ , then all elements of  $G \setminus \langle a, b \rangle$  are of order 4. So  $f_G(4) = 2p^t + 2^n p^m$ , where  $2p^t$  is the number of elements of order 4 in  $\langle a, b \rangle$ . Then we can get an equation as follows, that is

$$2p^t + 2^n p^m = 2^i p^j. \quad (4.5)$$

Clearly,  $i = 1$ . Then (4.5) becomes  $p^t + 2^{n-1}p^m = p^j$ . So  $j > t$ . Thus the (4.5) becomes

$$1 + 2^{n-1}p^{m-t} = p^{j-t}. \quad (4.6)$$

Since  $j - t > 0$ , we have  $m = t$ . So the (4.6) becomes

$$2^{n-1} = p^{j-t} - 1. \quad (4.7)$$

Since 2 is a primitive prime divisor of  $p - 1$ , by Lemma 3.4 we have  $j - t = 1$  in (4.7). Then  $n = 2^k + 1$ . Since  $\langle a, b \rangle$  is same as one of Type II,  $f_G(2^{n-o(r)}p^{m-i}) = \phi(2^{n-o(r)}p^{m-i}) = 2^{2^k+n-o(r)-1}p^{m-i-1}$ . So  $2^k + n - o(r) - 1 \leq n + 1$ , then  $o(r) \geq 2^k$ . Thus  $G = \langle a, b | a^{p^m} = b^{2^{2^k+1}} = 1, a^b = a^r, b^{2^{2^k}} = c^2, a^c = a^{-1}, b^c = b^{-1} \rangle$ .  $\square$

For the remain part (c) of Theorem 4.3, we may get the following result.

**Theorem 4.7.** *Let  $G$  be a POS-group with a cyclic Sylow  $p$ -subgroup  $P$  and  $|\pi(G)| = 2$ . If  $G$  is  $p$ -nilpotent, then  $G \cong Z_{2^n 3^m}$ ,  $D_8 \times Z_5^m$ ,  $Q_{2^{2k+2}} \times Z_p^m$ , where  $p = 2^{2^k} + 1$  a Fermat prime, or satisfies the condition that  $p = 3$  and  $N_G(P) = C_G(P) \cong Z_2 \times Z_2 \times P$ .*

*Proof.* By the proof of Theorem 4.3 we see that  $N_G(P) = C_G(P)$ . Let  $N = N_G(P) = P \times U$ ,  $|G : N| = 2^t$  and  $P_2$  be the Sylow 2-subgroup. Then  $f_G(2p^m) = 2^{t+2^k} p^{m-1} f_U(2)$ , so  $f_U(2) = 1$  or  $p$ . We divide into two cases.

*Case (a).*  $f_U(2) = 1$ .

Then  $U$  is cyclic or a generalized quaternion group. If  $U$  is cyclic, then

$$f_G(2^{n-t} p^m) = 2^{t+2^k} p^{m-1} 2^{n-t-1} = 2^{2^k+n-1} \mid 2^n p^m,$$

and so  $2^k \leq 1$ , that is  $p = 3$ . Since we may assume that  $f_{\bigcup_{g \in G} U^g}(4) = 2^i$ . Also

$$f_G(4) = f_{\bigcup_{g \in G} U^g}(4) + f_{P_2 \setminus \bigcup_{g \in G} U^g}(4).$$

Since  $3 \mid f_{P_2 \setminus \bigcup_{g \in G} U^g}(4)$ , we have  $f_G(4)$  is a power of 2, say  $2^j$ . Let  $f_G(2) = 3^h$  for  $h \geq 0$ . Now assume that  $P$  acts on the set  $\Omega$  of elements of order 2 of  $G$ , we have

$$f_G(2) = 3^h \equiv |C_\Omega(P)| = 1 \pmod{3}.$$

Thus  $h = 0$ , i.e.  $f_G(2) = 1$ . On the other hand, by Frobenius's theorem we can get that  $4 \mid 1 + f_G(2) + f_G(4) = 2 + 2^j$ , and then  $f_G(4) = 2$ . By Lemma 4.1,  $P_2$  is also cyclic. Therefore,  $G$  is 2-nilpotent, that is  $G \cong Z_{2^n 3^m}$ .

If  $U$  is a generalized quaternion group, then  $f_U(4) = 2^{n-t-1} + 2$ . So

$$f_G(4p^m) = 2^{t+2^k+1} p^{m-1} (2^{n-t-2} + 1) \mid 2^n p^m,$$

then  $p = 2^{n-t-2} + 1$ . Thus  $n - t = 2^k + 2$ . Let  $f_G(2) = p^h$  for  $h \geq 0$ . Similarly, assume that  $P$  acts on the set  $\Omega$  of elements of order 2 of  $G$ , we have

$$f_G(2) = p^h \equiv |C_\Omega(P)| = 1 \pmod{p}.$$

Thus  $h = 0$ , i.e.,  $f_G(2) = 1$ . By Lemma 4.1,  $P_2$  is also a generalized quaternion group. Choose an element  $x$  of order  $2^{n-1}$ , then  $\langle x \rangle$  is characteristic in  $P_2$ . So  $\langle x \rangle$  is normal in  $G$ . Apply the  $N/C$ -theorem, we

have  $N_G(\langle x \rangle)/C_G(x) \lesssim \text{Aut}(\langle x \rangle)$ , then  $C_G(x) = \langle x \rangle \times P$ . It leads to  $x \in U$ , so  $P_2 = U$ . Therefore  $G \cong Q_{2^{2k+2}} \times Z_{p^m}$ , where  $p = 2^{2k} + 1$ .

*Case (b).*  $f_U(2) = p$ .

Similarly, if  $f_U(4) \neq 0$ , then  $f_U(4) \mid 2^n p$ . On the other hand, since  $4 \mid 1 + f_U(2) + f_U(4)$ , it follows that  $f_U(4) = 2$  or  $2p$ . By Lemma 4.1,  $U$  is a dihedral or semi-dihedral group. If  $U$  is a dihedral one, then  $f_U(2) = 1 + 2^{n-t-1}$  and  $f_U(4) = 2$ . So  $n - t = 2^k + 1$ . Also  $U$  has an element of order  $2^{n-t-1}$ , so  $f_G(2^{n-t-1}p^m) = 2^{t+2^k+n-t-2}p^{m-1} = 2^{n+2^k-2}p^{m-1}$ . Then  $n + 2^k - 2 \leq n$ , that is  $k = 1$ . Thus  $p = 5$  and  $U \cong D_8$ . Since two elements of order 4 of  $U$  are conjugate, we have that  $f_{\bigcup_{g \in G} U^g}(4)$  is a 2-power, say  $2^i$ . Also clearly

$$f_G(4) = f_{\bigcup_{g \in G} U^g}(4) + f_{P_2 \setminus \bigcup_{g \in G} U^g}(4).$$

But  $5 \mid f_{P_2 \setminus \bigcup_{g \in G} U^g}(4)$  and  $f_G(4) \mid 2^n 5^m$ , so  $5 \nmid f_G(4)$ . Let  $f_G(4) = 2^h$  for  $h \geq 1$ . Set  $f_G(2) = 5^j$  for  $j \geq 1$ . By Frobenius's theorem we can obtain that  $4 \mid 1 + 2^i + 5^j$ , so  $h = 1$ , that is  $f_G(4) = 2$ . In view of Lemma 4.1, it is easy to see that  $P_2$  is also a dihedral group. Obviously, for all  $x \in P$ , the element  $x$  is an automorphism of  $P_2$ . On the other hand, the order of the automorphism group of a dihedral 2-group is still a 2-group, so  $P$  acts trivially on  $P_2$ . It leads to  $G = P \times P_2$ , i.e.,  $G \cong Z_{5^m} \times D_{2^n}$ . Moreover,  $f_G(2) = 1 + 2^{n-1}$ . So we get a Diophantine equation  $1 + 2^{n-1} = 5^j$ . By Lemma 3.4, the solution is  $n = 3$  and  $j = 1$ . Thus  $G \cong D_8 \times Z_{5^m}$ .

If  $U$  is a semi-dihedral one, then  $f_U(2) = 1 + 2^{n-t-2}$  and  $f_U(4) = 2 + 2^{n-t-2}$ . Since  $f_U(4) > 2$ ,  $f_U(4) = 2f_U(2)$ , which is impossible.

Next assume that  $f_U(4) = 0$ . Then  $U$  is an elementary abelian 2-group of order  $2^{n-t}$ . We may let  $U > 1$  (otherwise  $f_G(p^m) = 2^{n+2^k}p^{m-1} \mid 2^n p^m$ , a contradiction). By Lemma 2.3, we have  $2^n \mid f_G(p^m) + f_G(2p^m) = 2^{t+2^k}p^{m-1} + 2^{t+2^k}p^m$ , so  $n - t \leq 2^k + 1$ . In addition, let  $P$  act on the set of elements of order 2 in  $G$ , so we have

$$f_G(2) \equiv f_U(2) = 2^{n-t} - 1 \pmod{p}. \quad (4.8)$$

We make the equation (4.8) into two cases to consider.

Case I.  $f_G(2) = f_U(2) = 2^{n-t} - 1$ . Then  $2^{n-t} - 1 \mid p^m$ , and hence  $n - t = 2$  and  $p = 3$ .

Case II.  $f_G(2) > f_U(2)$ . Then  $n - t > 2^k$ , and thus  $n - t = 2^k + 1$ . Since  $f_G(2) \mid p^m$ , we have  $p \mid f_U(2) = 2^{2^k+1} - 1$ . So  $p = 3$  and  $n - t = 2$ .  $\square$

Using GAP software, we checked all small POS-groups  $G$  ( $|G| \leq 2000$ ),  $G$  has a cyclic Sylow  $p$ -subgroup or a generalized quaternion Sylow 2-group or a normal  $p$ -complement or a normal Sylow  $p$ -subgroup for every  $p \in \pi(G)$ . We put a conjecture to close this note.

**Conjecture 4.8.** *Let  $G$  be a POS-group and  $p \in \pi(G)$ . Then  $G$  satisfies one of following conditions:*

- (a)  $G$  has a cyclic Sylow  $p$ -subgroup or a generalized quaternion Sylow 2-group;
- (b)  $G$  has a normal  $p$ -complement;
- (c)  $G$  has a normal Sylow  $p$ -subgroup.

### Acknowledgments

Project supported by the NNSF of China (No.11171364, 11201133, 11201401), the China Postdoctoral Science Foundation (Grant No. 201104027) and the Innovation Foundation of Chongqing (KJTD201321).

### REFERENCES

- [1] Ya. Berkovich, On  $p$ -groups of finite order, *Siberian Math. J.* **9** (1968) 963–978.
- [2] A. K. Das, On finite groups having perfect order subsets, *Int. J. Algebra* **13** (2009), no. 3, 629–637.
- [3] W. Feit, On large Zsigmondy primes, *Proc. Amer. Math. Soc.* **102** (1988), no. 1, 29–36.
- [4] C. E. Finch and L. Jones, A curious connection between Fermat numbers and finite groups, *Amer. Math. Monthly* **109** (2002), no. 6, 517–524.
- [5] C. E. Finch and L. Jones, Nonabelian groups with perfect order subsets, *JP J. Algebra Number Theory Appl.* **3** (2003), no. 1, 13–26.
- [6] C. E. Finch and L. Jones, Corrigendum to: “Nonabelian groups with perfect order subsets”, *JP J. Algebra Number Theory Appl.* **4** (2004), no. 2, 413–416.
- [7] D. Gorenstein, *Finite Groups*, Chelsea Publishing Co., New York, 1980.
- [8] GAP software, <http://www.gap-system.org/>.
- [9] M. Hall, *The Theory of Groups*, The Macmillan Co., New York, 1959.
- [10] G. Higman, Finite groups in which every element has prime power order, *J. London Math. Soc.* **32** (1957) 335–342.
- [11] T. M. Keller, Solvable groups with a small number of prime divisors in the element orders, *J. Algebra* **170** (1994), no. 2, 625–648.
- [12] G. Malle, A. Moreto and G. Navarro, Element orders and Sylow structure, *Math. Z.* **252** (2006), no. 1, 223–230.
- [13] P. Ribenboim, *The Book of Prime Number Records*, Second Edition, Springer-Verlag, New York, 1989.



- [14] R. Shen, A note on finite groups having perfect order subsets, *Int. J. Algebra* **13** (2010), no. 4, 643–646.
- [15] J. S. Williams, Prime graph components of finite groups, *J. Algebra* **69** (1981), no. 2, 487–513.
- [16] L. Weisner, On the number of elements of a group which have a power in a given conjugate set, *Bull. Amer. Math. Soc.* **31** (1925), no. 9–10, 492–496.
- [17] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. und Phys.* **3** (1892), no. 1, 265–284.
- [18] H. Zassenhaus, Überendliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **11** (1936), no. 1, 187–220.

**Rulin Shen**

School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei, 430079, P.R. China

Email: [rulinshen@gmail.com](mailto:rulinshen@gmail.com)

**Wujie Shi**

Department of Mathematics, Chongqing University of Arts and Sciences, Yongchuan, Chongqing, 402160, P. R. China

Email: [shiwujie@gmail.com](mailto:shiwujie@gmail.com)

**Jiangtao Shi**

School of Mathematics and Information Science, Yantai University, Yantai 264005, P. R. China

Email: [shijt@pku.org.cn](mailto:shijt@pku.org.cn)