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GEODESIC METRIC SPACES AND GENERALIZED NONEXPANSIVE MULTIVALUED MAPPINGS

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ABSTRACT. In this paper, we present some common fixed point theorems for two generalized nonexpansive multivalued mappings in CAT(0) spaces as well as in UCED Banach spaces. Moreover, we prove the existence of fixed points for generalized nonexpansive multivalued mappings in complete geodesic metric spaces with convex metric for which the asymptotic center of a bounded sequence in a bounded closed convex subset is nonempty and singleton. The results obtained in this paper extend and improve some recent results.

1. Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [18], [19]). He showed that every nonexpansive single valued mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in R-trees) can be applied to graph theory, biology, and computer science. (see e.g., [2, 10]).

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In [5], Dhompongsa et al. presented a fixed point theorem for a commuting pair consisting of a single valued nonexpansive, and a multivalued nonexpansive mapping in a CAT(0) space. In the recent years, the existence of common fixed points for a commuting pair of mappings, including a single valued and a multivalued nonexpansive (generalized nonexpansive) mapping in geodesic metric spaces, have been studied extensively by many authors (see, e.g, [1,8,9,11,17,21,23–25]). But to our best knowledge, the existence of common fixed points for two multivalued mappings, even in a Hilbert space case, has not yet been studied. In this paper we intend to present a common fixed point theorem for two generalized nonexpansive multivalued mappings in CAT(0) spaces, as well as in UCED Banach spaces. Moreover we prove the existence of fixed points for generalized nonexpansive multivalued mappings in a complete geodesic metric space with convex metric for which the asymptotic center of a bounded sequence in a bounded closed convex subset is nonempty and singleton. The results we obtain in the paper extend and improve some recent known results.

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, the mapping c is an isometry and $d(x, y) = l$. The image of c is called a geodesic segment joining x and y which when it is unique it is denoted by $[x, y]$. We write $c(\alpha 0 + (1-\alpha)l) = \alpha x + (1-\alpha)y$ for $0 \leq \alpha \leq 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset K of X is called convex if K contains every geodesic segment joining any two points of it.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$

and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x,y) \leq d_{R^2}(\overline{x}, \overline{y})$.
It is known (see [3]) that if (X,d) is a $CAT(0)$ space and $K \subset X$ is a complete and convex subset of X, then for any $x \in X$, there exists a unique closest point $y \in K$ to x.

Definition 2.1. *In a geodesic space (X,d), the metric $d : X \times X \to \mathbb{R}$ is convex if for any $x, y, z \in X$ we have*

$$d(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)d(x,y) + \alpha d(x,z) \qquad \text{for all} \quad \alpha \in [0,1].$$

It is well-known that CAT(0) spaces are unique geodesic spaces with convex metric, (see [7]).

Definition 2.2. *([16]) A geodesic space (X,d) is uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that if $w, x, y \in X$ with $d(x,w) \leq r$, $d(y,w) \leq r$ and $d(x,y) \geq \varepsilon r$, then*

$$d(\frac{x}{2} \oplus \frac{y}{2}, w) \leq (1-\delta)r.$$

Definition 2.3. *([20]) For given $r > 0$ and $\varepsilon \in (0,2]$, the mapping $\delta : (0,\infty) \times (0,2] \to (0,1]$ is called a modulus of uniform convexity. The mapping δ is monotone (resp. lower semi-continuous from the right) if for every fixed ε it decreases (resp. is lower semi-continuous from the right) with respect to r.*

A UC space is by definition, a uniformly convex metric space with monotone (or lower-semicontinuous from the right) modulus of uniform convexity.

Definition 2.4. *([12]) A geodesic metric space (X,d) which satisfies the inequality*

$$d(x,y)d(z,p) \leq d(x,z)d(y,p) + d(x,p)d(y,z) \qquad \text{for every} \quad x, y, z, p \in X,$$

is called a geodesic Ptolemy space.

Definition 2.5. *([12]) Let X be a geodesic space. We say that X admits a uniformly continuous midpoint map if there exists a map $m : X \times X \to X$ such that*

$$d(x, m(x,y)) = d(y, m(x,y)) = \frac{d(x,y)}{2} \quad \text{for all} \quad x, y \in X,$$

and that for $n \in \mathbb{N}$ and $x_n, x'_n, y_n, y'_n \in X$ with $\lim_{n\to\infty} d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} d(y_n, y'_n) = 0$, we have

$$\lim_{n\to\infty} d(m(x_n, x'_n), m(y_n, y'_n)) = 0.$$

Every geodesic Ptolemy space with a uniformly continuous midpoint map is uniquely geodesic (see [12]).

The following lemma is a consequence of Proposition 2 proved by Goebel and Kirk [15].

Lemma 2.6. *Let X be a geodesic metric space with convex metric, $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in X, and let $0 < \lambda < 1$. If for every natural number n we have $z_{n+1} = \lambda w_n \oplus (1-\lambda)z_n$ and $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$, then $\lim_{n\to\infty} d(w_n, z_n) = 0$.*

Lemma 2.7. *([7]) Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)(d(x, y))^2.$$

Let (X, d) be a geodesic metric space. A subset $K \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x, y) = dist(x, K) = inf\{d(x, z) : z \in K\}.$$

We denote by $CB(K), P(K), KC(K)$ and $CBP(K)$ the collection of all nonempty closed bounded subsets, nonempty proximal subsets, nonempty compact convex subsets, and nonempty closed bounded proximal subsets of K, respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(X)$.
Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 2.8. *A multivalued mapping $T : X \to CB(X)$ is called*

(i) *nonexpansive if*

$$H(T(x), T(y)) \leq d(x, y), \quad x, y \in X.$$

(ii) *quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq d(x, p)$ for all $x \in H$ and all $p \in F(T)$.*

In [13], J. Garcia-Falset, E. Liorens -Fuster and T. Suzuki introduced two types of generalization for nonexpansive mappings. In the following, we modify their conditions for multivalued mappings in the framework of a geodesic metric space, (see also [1]).

Definition 2.9. *A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (C_λ) for some $\lambda \in (0,1)$ provided that*

$$\lambda\, dist(x, T(x)) \le d(x,y) \implies H(T(x),T(y)) \le d(x,y), \quad x,y \in X.$$

Definition 2.10. *A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition $(E_{\mu,\eta})$ provided that*

$$dist(x, T(y)) \le \mu\, dist(x, T(x)) + \eta\, d(x,y), \quad x,y \in X.$$

We say that T satisfies condition (E) whenever T satisfies $(E_{\mu,\eta})$ for some $\mu, \eta \ge 1$.

Lemma 2.11. *([1]) Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping, then T satisfies condition (E).*

The following theorem was proved in the setting of CAT(0) spaces [1]. It is easy to see that this result stands true in a more general context. We formulate this result in the framework of a geodesic metric space, but omit the details of its proof.

Theorem 2.12. *Let X be a geodesic metric space with convex metric and K be a nonempty bounded convex subset of X. Let $T : K \to CBP(K)$ satisfy the condition (C_λ) on K for some $\lambda \in (0,1)$. Then there exists a sequence $\{x_n\}$ in K such that*

$$\lim_{n\to\infty} dist(x_n, T(x_n)) = 0.$$

Let $\{x_n\}$ be a bounded sequence in X and K be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\},$$

where

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x),$$

and the set

$$A = A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to K. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to K. In a complete CAT(0) space, the asymptotic center $A = A(K, \{x_n\})$ of (x_n) consists of exactly one point whenever K is closed and convex (see for example [6]). Likewise, the same holds for complete uniformly convex metric spaces with a monotone (or lower semi-continuous from the right) modulus of uniform convexity (see [8] for details) and complete

Ptolemy spaces with a uniformly continuous midpoint map (see [12] for details).

Definition 2.13. *A bounded sequence $\{x_n\}$ is said to be regular with respect to K if for every subsequence $\{x'_n\}$ we have*

$$r(K,\{x_n\}) = r(K,\{x'_n\}).$$

The following lemma was proved by Goebel and Lim (see [14] and [22]) in the framework of Banach spaces. Since the proof has a metric nature the lemma holds true in complete geodesic metric spaces with convex metric.

Lemma 2.14. *Let X be a complete geodesic metric space with convex metric, $\{x_n\}$ be a bounded sequence in X and let K be a nonempty closed convex subset of X. Then there exists a subsequence of $\{x_n\}$ which is regular relative to K.*

3. Geodesic metric spaces

Theorem 3.1. *Let X be a complete geodesic metric space with convex metric, and K be a nonempty closed convex bounded subset of X. Let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If the asymptotic center relative to K of each sequence in K is nonempty and singleton, then T has a fixed point.*

Proof. By Theorem 2.12 there exists an approximate fixed point sequence $\{x_n\}$ for T in K. We can choose a sequence $y_n \in T(x_n)$ such that

$$\lim_{n\to\infty} dist(x_n, T(x_n)) = \lim_{n\to\infty} d(x_n, y_n) = 0.$$

From Lemma 2.14, by passing to a subsequence, we may assume that $\{x_n\}$ is regular. By assumption $A(K,\{x_n\}) = \{z\}$ is singleton. If $r = r(K,\{x_n\}) = 0$, then we have $x_n \to z$. Since T satisfies condition (E), there exists $\eta, \mu \geq 1$ such that

$$\begin{aligned} dist(z, T(z)) &\leq d(z, x_n) + dist(x_n, T(z)) \\ &\leq (\eta + 1)d(z, x_n) + \mu\, dist(x_n, T(x_n)) \to 0 \quad n \to \infty, \end{aligned}$$

which implies that $z \in T(z)$. In the other case if $r > 0$ there exists a natural number n_0 such that for every $n \geq n_0$,

$$\lambda\, dist(x_n, T(x_n)) \leq d(x_n, z)$$

and hence from our assumption we have

$$H(T(x_n), T(z)) \le d(x_n, z), \qquad \forall n \ge n_0.$$

The compactness of $T(z)$ implies that for each $n \ge 1$ we can take $z_n \in T(z)$ such that

$$d(y_n, z_n) = dist(y_n, T(z)).$$

Also we have

$$d(y_n, z_n) = dist(y_n, T(z)) \le H(T(x_n), T(z)) \le d(x_n, z), \qquad \forall n \ge n_0.$$

Since $T(z)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in T(z)$. Note that

$$\begin{aligned} d(x_{n_k}, w) &\le d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ &\le d(x_{n_k}, y_{n_k}) + d(x_{n_k}, z) + d(z_{n_k}, w), \end{aligned}$$

for $n_k \ge n_0$. This implies that

$$\limsup_{k\to\infty} d(x_{n_k}, w) \le \limsup_{k\to\infty} d(x_{n_k}, z) \le r.$$

This implies by the regularity of $\{x_n\}$ and by uniqueness of asymptotic center that $z = w \in T(z)$. This completes the proof. □

Corollary 3.2. *Let X be a complete UC space, and K be a nonempty closed convex bounded subset of X. Let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Then T has a fixed point.*

Corollary 3.3. *Let X be a complete Ptolemy space with a uniformly continuous midpoint map and K be a nonempty closed convex bounded subset of X. Let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Then T has a fixed point.*

4. CAT(0) spaces

We need the following lemma for the proof of our main result.

Lemma 4.1. *Let K be a closed convex subset of a CAT(0) space X. Let $T : K \to P(K)$ be a multivalued mapping such that P_T is quasi-nonexpansive, where $P_T(x) = \{y \in T(x) : d(x,y) = dist(x, T(x))\}$. Then $F(T)$ is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to z$ as $n \to \infty$. Then $P_T(p_n) = \{p_n\}$. By quasi-nonexpansivness of P_T we have

$$\begin{aligned} dist(z, T(z)) &\leq dist(z, P_T(z)) \leq d(z, p_n) + dist(p_n, P_T(z)) \\ &\leq d(z, p_n) + H(P_T(p_n), P_T(z)) \\ &\leq 2\, d(z, p_n) \to 0 \quad n \to \infty. \end{aligned}$$

This implies that $z \in T(z)$, hence $z \in F(T)$. We now show that $F(T)$ is convex. For $x, y \in F(T)$ we have $P_T(x) = \{x\}$ and $P_T(y) = \{y\}$. For $\alpha \in [0, 1]$, put $z = \alpha x \oplus (1-\alpha)y$. Let $w \in P_T(z)$, then by Lemma 2.7 we have

$$\begin{aligned} d(w, z)^2 &= d(\alpha x \oplus (1-\alpha)y, w)^2 \\ &\leq \alpha d(w, x)^2 + (1-\alpha)d(w, y)^2 - \alpha(1-\alpha)d(x, y)^2 \\ &= \alpha dist(w, P_T(x))^2 + (1-\alpha)dist(w, P_T(y))^2 \\ &\quad -\alpha(1-\alpha)d(x, y)^2 \\ &\leq \alpha H(P_T(z), P_T(x))^2 + (1-\alpha)H(P_T(z), P_T(y))^2 \\ &\quad -\alpha(1-\alpha)d(x, y)^2 \\ &\leq \alpha d(x, z)^2 + (1-\alpha)d(z, y)^2 - \alpha(1-\alpha)d(x, y)^2 \\ &\leq \alpha(1-\alpha)^2 d(x, y)^2 + (1-\alpha)\alpha^2 d(x, y)^2 - \alpha(1-\alpha)d(x, y)^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)d(x, y)^2 = 0 \end{aligned}$$

so that $z = w \in P_T(z) \subset T(z)$ and finally $z \in F(T)$. □

We remark that there exist examples of multivalued mappings for which P_T is nonexpansive (see [26] for details), so that the assumption on T is not artificial.

Lemma 4.2. *Let K be a closed convex subset of a CAT(0) space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multivalued mapping. Assume that $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to z$ as $n \to \infty$. Then by assumption $T(p_n) = \{p_n\}$. It follows that

$$\begin{aligned} dist(z, T(z)) &\leq d(z, p_n) + dist(p_n, T(z)) \\ &\leq d(z, p_n) + H(T(p_n), T(z)) \\ &\leq 2\, d(z, p_n) \to 0 \quad n \to \infty. \end{aligned}$$

This implies that $z \in T(z)$, hence $z \in F(T)$. We now show that $F(T)$ is convex. For $x, y \in F(T)$ we have $T(x) = \{x\}$ and $T(y) = \{y\}$. For

$\alpha \in [0,1]$, put $z = \alpha x \oplus (1-\alpha)y$. Let $w \in T(z)$, then we have

$$\begin{aligned} d(w,z)^2 &= d(\alpha x \oplus (1-\alpha)y, w)^2 \\ &\leq \alpha d(w,x)^2 + (1-\alpha)d(w,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha dist(w,T(x))^2 + (1-\alpha)dist(w,T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\ &\leq \alpha H(T(z),T(x))^2 + (1-\alpha)H(T(z),T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\ &\leq \alpha d(x,z)^2 + (1-\alpha)d(z,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &\leq \alpha(1-\alpha)^2 d(x,y)^2 + (1-\alpha)\alpha^2 d(x,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)d(x,y)^2 = 0 \end{aligned}$$

so that $z = w \in T(z)$ and finally $z \in F(T)$. □

As a corollary, we obtain the following result of Chaoha and Phon-On [4].

Corollary 4.3. *Let K be a closed convex subset of a CAT(0) space X. Let $T : K \to K$ be a quasi-nonexpansive single valued mapping. Then $F(T)$ is closed and convex.*

Definition 4.4. *Let K be a nonempty subset of a CAT(0) space X, and $T_1, T_2 : X \to 2^X$ be two multivalued mappings. We say that a pair (T_1, T_2) commutes in K if for all $x \in K$*

$$T_1(T_2(x)) = T_2(T_1(x),$$

where $T_1(T_2(x)) = \bigcup_{y \in T_2(x)} T_1(y)$.

Now we state the main result of this section.

Theorem 4.5. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $S : K \to CB(K)$ be a quasi-nonexpansive multivalued mapping, and let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If the pair (S,T) commutes in $F(S)$ and $S(p) = \{p\}$ for all $p \in F(S)$, then S and T have a common fixed point.*

Proof. By Lemma 4.2, it follows that $F(S)$ is a nonempty closed convex subset of X. We show that for $x \in F(S)$, $T(x) \cap F(S) \neq \emptyset$. To see this, let $x \in F(S)$, and let $y \in T(x)$ be the unique closest point to x. Since S and T commute in x and $S(x) = \{x\}$ we have

$$S(y) \subset ST(x) \subset TS(x) = T(x),$$

and since S is quasi-nonexpansive, we have

$$dist(S(y), x) \leq H(S(y), S(x)) \leq d(y, x),$$

hence there exists $z \in S(y) \subset T(x)$ such that $d(z, x) = dist(S(y), x) \leq d(y, x)$. Now by the uniqueness of y as the closest point to x, we get $y = z \in S(y)$ and therefore $T(x) \cap F(S) \neq \emptyset$, for $x \in F(S)$.

Now we find an approximate fixed point sequence in $F(S)$ for T. Take $x_0 \in F(S)$, since $T(x_0) \cap F(S) \neq \emptyset$, we can find $y_0 \in T(x_0) \cap F(S)$. Define

$$x_1 = (1 - \lambda)x_0 \oplus \lambda y_0.$$

Since $F(S)$ is convex, we have $x_1 \in F(S)$. Let $y_1 \in T(x_1)$ be taken in such a way that

$$d(y_0, y_1) = dist(y_0, T(x_1)).$$

By the method described above we can prove that $y_1 \in F(S)$. Similarly, we put $x_2 = (1 - \lambda)x_1 \oplus \lambda y_1$, and again we choose $y_2 \in T(x_2)$ in such a way that

$$d(y_1, y_2) = dist(y_1, T(x_2)).$$

By the same argument, we get $y_2 \in F(S)$. In this way we will find a sequence $\{x_n\}$ in $F(S)$ such that $x_{n+1} = (1 - \lambda)x_n \oplus \lambda y_n$ where $y_n \in T(x_n) \cap F(S)$ and

$$d(y_{n-1}, y_n) = dist(y_{n-1}, T(x_n)).$$

Therefore for every natural number $n \geq 1$ we have

$$\lambda\, d(x_n, y_n) = d(x_n, x_{n+1})$$

from which it follows that

$$\lambda\, dist(x_n, T(x_n)) \leq \lambda d(x_n, y_n) = d(x_n, x_{n+1}), \quad n \geq 1.$$

Since T satisfies the condition (C_λ) we have

$$H(T(x_n), T(x_{n+1})) \leq d(x_n, x_{n+1}), \quad n \geq 1,$$

hence for each $n \geq 1$

$$d(y_n, y_{n+1}) = dist(y_n, T(x_{n+1})) \leq H(T(x_n), T(x_{n+1})) \leq d(x_n, x_{n+1}).$$

We now apply Lemma 2.6 to conclude that $\lim_{n\to\infty} d(x_n, y_n) = 0$ where $y_n \in T(x_n)$. From Lemma 2.14 by passing to a subsequence we may assume that $\{x_n\}$ is regular. Put $A(F(S), \{x_n\}) = \{z\}$. If

$$r = r(F(S), \{x_n\}) = 0,$$

then $x_n \to z$ as $n \to \infty$. Since T satisfies condition (E), there exist $\eta, \mu \geq 1$ such that

$$\begin{aligned} dist(z, T(z)) &\leq d(z, x_n) + dist(x_n, T(z)) \\ &\leq (\eta + 1)d(z, x_n) + \mu\, dist(x_n, T(x_n)) \to 0 \quad n \to \infty, \end{aligned}$$

which implies that $z \in T(z)$. In the other case, if $r > 0$, there exists a natural number n_0 such that for every $n \geq n_0$,

$$\lambda\, dist(x_n, T(x_n)) \leq d(x_n, z)$$

and hence from our assumption we have

$$H(T(x_n), T(z)) \leq d(x_n, z), \qquad \forall n \geq n_0.$$

The compactness of $T(z)$ implies that for each $n \geq 1$ we can take $z_n \in T(z)$ such that

$$d(y_n, z_n) = dist(y_n, T(z)).$$

Since $y_n \in F(S)$, by a similar argument, we obtain $z_n \in F(S)$. We also have

$$d(y_n, z_n) = dist(y_n, T(z)) \leq H(T(x_n), T(z)) \leq d(x_n, z), \qquad \forall n \geq n_0.$$

Since $T(z)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in T(z)$. Now, the closedness of $F(S)$ implies that $w \in F(S)$. Note that

$$\begin{aligned} d(x_{n_k}, w) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ \leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, z) + d(z_{n_k}, w), \end{aligned}$$

for $n_k \geq n_0$. This implies that

$$\limsup_{k\to\infty} d(x_{n_k}, w) \leq \limsup_{k\to\infty} d(x_{n_k}, z) \leq r.$$

This implies, by the regularity of $\{x_n\}$ and by the uniqueness of asymptotic center, that $z = w \in T(z)$. Hence $z \in F(S) \bigcap F(T)$. □

Theorem 4.6. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $S : K \to P(K)$ be a multivalued mapping such that P_S is quasi-nonexpansive, and let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If the pair (P_S, T) commutes in $F(S)$, then they have a common fixed point.*

Proof. According to Lemma 4.1, it follows that $F(S)$ is a nonempty closed convex subset of X. We show that for $x \in F(S)$, $T(x) \cap F(S) \neq \emptyset$. To see this, let $x \in F(S)$ then we have $P_S(x) = \{x\}$. Let $y \in T(x)$ be the unique closest point to x, since the pair (P_S, T) commutes in x and we have

$$P_S(y) \subset P_S T(x) \subset T P_S(x) = T(x)$$

and since P_S is quasi-nonexpansive, we have

$$dist(P_S(y), x) \leq H(P_S(y), P_S(x)) \leq d(y, x).$$

Thus there exists $z \in P_S(y) \subset T(x)$ such that $d(z, x) = dist(P_S(y), x) \leq d(y, x)$. Now by the uniqueness of y as the closest point to x, we get $y = z \in P_S(y) \subset S(y)$ and therefore $T(x) \cap F(S) \neq \emptyset$, for $x \in F(S)$. The rest of proof is exactly similar to the proof of Theorem 4.5. □

Corollary 4.7. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $t : K \to K$ be a quasi nonexpansive single valued mapping, and let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and C_λ for some $\lambda \in (0, 1)$. If t and T commute, then they have a common fixed point, i.e. there exists a point $z \in K$ such that $z = t(z) \in T(z)$.*

We now give an example to illustrate Theorem 4.5.

Example. Define two mappings T and S on the closed interval $[0, 5]$ by

$$S(x) = [\frac{x}{6}, \frac{x}{2}], \qquad T(x) = \begin{cases} [0, \frac{x}{5}], & x \neq 5 \\ \{1\} & x = 5. \end{cases}$$

Then S is quasi-nonexpansive, and 0 is the only fixed point of S. Also T satisfies conditions (C_λ) and E (for details, see [1]). It is easy to see that S and T commute on $F(S) = \{0\}$ and that 0 is a common fixed point of T and S.

5. Banach spaces

Let X be a Banach space. X is said to be strictly convex if $\|x+y\| < 2$ for all $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space X is said to be *uniformly convex in every direction* (UCED, for short) provided that for every $\epsilon \in (0, 2]$ and $z \in X$ with $\|z\| = 1$, there exists a positive number δ (depending on ϵ and z) such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $x - y \in \{tz : t \in [-2, -\epsilon] \cup [\epsilon, 2]\}$ we have $\|x + y\| \leq 2(1 - \delta)$. X is said to be *uniformly convex* if X is UCED and

$inf\{\delta(\epsilon, z) : \|z\| = 1\} > 0$ for all $\epsilon \in (0, 2]$. It is rather obvious that uniform convexity implies UCED, and UCED implies strict convexity.

It is also known that in a UCED Banach space X, the asymptotic center of a bounded sequence with respect to a weakly compact convex set is a singleton;

Lemma 5.1. *Let K be a nonempty closed convex subset of a strictly convex Banach space X. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is quasi-nonexpansive, where $P_T(x) = \{y \in T(x) : \|x - y\| = dist(x, T(x))\}$. Then $F(T)$ is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to z$ as $n \to \infty$. Then $P_T(p_n) = \{p_n\}$. Since P_T is quasi-nonexpansive we have

$$\begin{aligned} dist(z, P_T(z)) &\leq d(z, p_n) + dist(p_n, P_T(z)) \\ &\leq d(z, p_n) + H(P_T(p_n), P_T(z)) \\ &\leq 2\, d(z, p_n) \to 0 \quad n \to \infty. \end{aligned}$$

This implies that $z \in Tz$, hence $z \in F(T)$. We now show that $F(T)$ is convex. For $x, y \in F(T)$ we have $P_T(x) = \{x\}$ and $P_T(y) = \{y\}$. For $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Let $w \in P_T(z)$, then we have

$$\begin{aligned} \|x - y\| &\leq \|x - w\| + \|w - y\| \\ &= dist(w, P_T(x)) + dist(w, P_T(y)) \\ &\leq H(P_T(z), P_T(x)) + H(P_T(z), P_T(y)) \\ &\leq \|x - z\| + \|z - y\| \\ &= \|x - y\| \end{aligned}$$

By the strict convexity of X, there exist $\lambda \in [0, 1]$ such that $w = \lambda x + (1 - \lambda)y$. Since

$$\begin{aligned} \lambda\|x - y\| = \|w - y\| &= dist(w, P_T(y)) \\ &\leq H(P_T(z), P_T(y)) \\ &\leq \|y - z\| \\ &= \alpha\|x - y\| \end{aligned}$$

and

$$\begin{aligned} (1 - \lambda)\|x - y\| &= \|w - x\| \\ &= dist(w, P_T(x)) \\ &\leq H(P_T(z), P_T(x)) \\ &\leq \|x - z\| = (1 - \alpha)\|x - y\|, \end{aligned}$$

we have $1 - \lambda \leq 1 - \alpha$ and $\lambda \leq \alpha$. These imply that $\lambda = \alpha$. Hence $z = w \in P_T(z) \subset T(z)$ and finally $z \in F(T)$. □

By the uniqueness of asymptotic center and using Lemma 5.1 and a similar argument as in the proof of Theorem 4.6 we obtain the following theorem.

Theorem 5.2. *Let K be a nonempty weakly compact convex subset of a UCED Banach space X. Let $S : K \to P(K)$ be a multivalued mapping such that P_S is quasi-nonexpansive, and let $T : K \to KC(K)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If the pair (P_S, T) commutes in $F(S)$ then they have a common fixed point.*

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References

[1] A. Abkar and M. Eslamian, Common fixed point results in CAT(0) spaces, *Nonlinear Anal.* **74** (2011), no. 5, 1835–1840.

[2] M. Bestvina, R-Trees in Topology, Geometry, and Group Theory, Handbook of Geometric Topology, 55–91. North- Holland, Amsterdam, 2002.

[3] M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin, 1999.

[4] P. Chaoha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, *J. Math. Anal. Appl.* **320** (2006), no. 2, 983–987.

[5] S. Dhompongsa, A. Kaewkhao and B. Panyanak, Lim's theorem for multivalued mappings in CAT(0) spaces, *J. Math. Anal. Appl.* **312** (2005), no. 2, 478–487.

[6] S. Dhompongsa, W. A. Kirk and B. Sims, Fixed points of uniformly lipschitzian mappings, *Nonlinear Anal.* **65** (2006), no. 4, 762–772.

[7] S. Dhompongsa and B. Panyanak, On △-convergence theorems in CAT(0) spaces, *Comput. Math. Appl.* **56** (2008), no. 10, 2572–2579.

[8] R. Espinola, A. Fernandez-Leon and B. Piatek, Fixed points of single- and set-valued mappings in uniformly convex metric spaces with no metric convexity, *Fixed Point Theory Appl.* (2010), Article ID 169837, 16 pages.

[9] R. Espinola and N. Hussain, Common fixed points for multimaps in metric spaces, *Fixed Point Theory Appl.* (2010), Article ID 204981, 14 pages.

[10] R. Espinola and W. A. Kirk, Fixed point theorems in R-trees with applications to graph theory, *Topology Appl.* **153** (2006), no. 7, 1046–1055.

[11] R. Espinola, P. Lorenzo and A. Nicolae, Fixed points, selections and common fixed points for nonexpansive-type mappings, *J. Math. Anal. Appl.* **382** (2011), no. 2, 503–515.

[12] R. Espinola and A. Nicolae, Geodesic Ptolemy spaces and fixed points, *Nonlinear Anal.* **74** (2011), no. 1, 27–34.

[13] J. Garcia-Falset, E. Llorens-Fuster and T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *J. Math. Anal. Appl.* **375** (2011), no. 1, 185–195.

[14] K. Goebel, On a fixed point theorem for multivalued nonexpansive mappings, *Ann. Univ. Mariae Curie-Skodowska Sect. A* **29** (1975) 69–72.

[15] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mapping, 115–123, Contemp. Math., 21, Amer. Math. Soc., Providence, 1983.

[16] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, Inc., New York, 1984.

[17] N. Hussain and M. A. Khamsi, On asymptotic pointwise contractions in metric spaces, *Nonlinear Anal.* **71** (2009), no. 10, 4423–4429.

[18] W. A. Kirk, Geodesic geometry and fixed point theory, 195–225, Seminar of Mathematical Analysis (Malaga/Seville, 2002-2003), Colec. Abierta, 64, Univ. Sevilla Secr. Publ. Seville, 2003.

[19] W. A. Kirk, Geodesic geometry and fixed point theory, 113–142, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004.

[20] U. Kohlenbach and L. Leustean, Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 1, 71–92.

[21] W. Laowang and B. Panyanak, Common fixed points for some generalized multivalued nonexpansive mappings in uniformly convex metric spaces, *Fixed Point Theory Appl.* (2011) 9 pages.

[22] T. C. Lim, A fixed point theorem for multivalued nonexpansive mapping in a uniformly convex Banach space, *Bull. Amer. Math. Soc.* **80** (1974) 1123–1126.

[23] N. Shahzad, Fixed point results for multimaps in CAT(0) spaces, *Topology Appl.* **156** (2009), no. 5, 997–1001.

[24] N. Shahzad, Invariant approximations in CAT(0) spaces, *Nonlinear Anal.* **70** (2009), no. 12, 4338–4340.

[25] N. Shahzad and J. Markin, Invariant approximations for commuting mappings in CAT(0) and hyperconvex spaces, *J. Math. Anal. Appl.* **337** (2008), no. 2, 1457–1464.

[26] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Anal.* **71** (2009), no 3-4, 838–844.

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