

RICCI TENSOR FOR GCR -LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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ABSTRACT. We obtain the expression of Ricci tensor for a GCR -lightlike submanifold of indefinite complex space form and discuss its properties on a totally geodesic GCR -lightlike submanifold of an indefinite complex space form. Moreover, we prove that every proper totally umbilical GCR -lightlike submanifold of an indefinite Kaehler manifold is a totally geodesic GCR -lightlike submanifold.

1. Introduction

The geometry of CR -submanifolds of Kaehler manifolds was initiated by Bejancu [2], as a generalization of complex and totally real submanifolds of Kaehler manifolds and has been further developed by many others [3–6, 12, 13]. Due to growing importance of lightlike geometry in mathematical physics and relativity, Duggal and Bejancu [7], introduced the notion of CR -lightlike submanifolds of indefinite Kaehler manifolds, which does not include complex and totally real subcases. Then Duggal and Sahin [9], obtained a new class of submanifolds, namely, SCR -lightlike submanifolds of indefinite Kaehler manifolds but there was no inclusion relation between CR and SCR -cases. Later on, Duggal and Sahin [10], introduced GCR -lightlike submanifolds of indefinite

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Kaehler manifolds. This class of submanifolds is an umbrella over real hypersurfaces, invariant, screen real and CR -lightlike submanifolds. In this paper, we obtain the expression of Ricci tensor for a GCR -lightlike submanifold of indefinite complex space form and discuss its properties for a totally geodesic GCR -lightlike submanifold of an indefinite complex space form. We also prove that every proper totally umbilical GCR -lightlike submanifold of an indefinite Kaehler manifold is a totally geodesic GCR -lightlike submanifold.

2. Lightlike submanifolds

Let (\bar{M}, \bar{g}) be a real $(m+n)$ -dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m -dimensional submanifold of \bar{M} and g the induced metric of \bar{g} on M . If g is degenerate on the tangent bundle TM of M , then M is called a lightlike submanifold of \bar{M} , (for detail see [7]). For a degenerate metric g on M

$$(2.1) \quad T_x M^\perp = \cup \{u \in T_x \bar{M} : \bar{g}(u, v) = 0, \forall v \in T_x M, x \in M\},$$

is a degenerate n -dimensional subspace of $T_x \bar{M}$. Thus both $T_x M$ and $T_x M^\perp$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad T_x M = T_x M \cap T_x M^\perp$, which is known as radical (null) subspace. If the mapping

$$(2.2) \quad Rad TM : x \in M \longrightarrow Rad T_x M,$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of \bar{M} is called an r -lightlike submanifold and $Rad TM$ is called the radical distribution on M .

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM , that is

$$(2.3) \quad TM = Rad TM \oplus S(TM),$$

and $S(TM^\perp)$ is a complementary vector subbundle to $Rad TM$ in TM^\perp . Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and to $Rad TM$ in $S(TM^\perp)^\perp$ respectively. Then we have

$$(2.4) \quad tr(TM) = ltr(TM) \perp S(TM^\perp),$$

$$(2.5) \quad T\bar{M}|_M = TM \oplus tr(TM) = (Rad TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp).$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of \bar{M} along M , on u as $\{\xi_1, \dots, \xi_r, W_{r+1}, \dots, W_n, N_1, \dots, N_r, X_{r+1}, \dots, X_m\}$, where $\{\xi_1, \dots, \xi_r\}$, $\{N_1, \dots, N_r\}$ are local lightlike bases of $\Gamma(\text{Rad}TM|_u)$, $\Gamma(\text{ltr}(TM)|_u)$, respectively and $\{W_{r+1}, \dots, W_n\}$, $\{X_{r+1}, \dots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^\perp)|_u)$, $\Gamma(S(TM)|_u)$, respectively. For this quasi-orthonormal fields of frames, one has

Theorem 2.1. ([7]). *Let $(M, g, S(TM), S(TM^\perp))$ be an r -lightlike submanifold of a semi-Riemannian manifold (\bar{M}, \bar{g}) . Then, there exists a complementary vector bundle $\text{ltr}(TM)$ of $\text{Rad}TM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(\text{ltr}(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp|_u$, where u is a coordinate neighborhood of M such that*

$$(2.6) \quad \bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for any } i, j \in \{1, 2, \dots, r\},$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on \bar{M} then, using the decomposition of $T\bar{M}|_M$ in (2.5), the Gauss and Weingarten formulas are given by

$$(2.7) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(2.8) \quad \bar{\nabla}_X U = -A_U X + \nabla_X^\perp U, \quad \forall X \in \Gamma(TM), U \in \Gamma(\text{tr}(TM)),$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^\perp U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M , h is a symmetric bilinear form on $\Gamma(TM)$, which is called second fundamental form, A_U is a linear operator on M , known as shape operator.

Considering the projection morphisms L and S of $\text{tr}(TM)$ on $\text{ltr}(TM)$ and $S(TM^\perp)$, respectively, then (2.7) and (2.8) become

$$(2.9) \quad \bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y),$$

$$(2.10) \quad \bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U,$$

where we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D_X^l U = L(\nabla_X^\perp U)$, $D_X^s U = S(\nabla_X^\perp U)$.

As h^l and h^s are $\Gamma(\text{ltr}(TM))$ -valued and $\Gamma(S(TM^\perp))$ -valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M . In particular

$$(2.11) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N),$$

$$(2.12) \quad \bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W),$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (2.9)-(2.12) we obtain

$$(2.13) \quad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y),$$

$$(2.14) \quad \bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0,$$

$$(2.15) \quad \bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X),$$

$$(2.16) \quad \bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0,$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^\perp))$ and $N, N' \in \Gamma(ltr(TM))$.

Let \bar{P} be the projection morphism of TM on $S(TM)$, then using (2.3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$(2.17) \quad \nabla_X \bar{P}Y = \nabla_X^* \bar{P}Y + h^*(X, \bar{P}Y),$$

$$(2.18) \quad \nabla_X \xi = -A_\xi^* X + \nabla_X^{*t} \xi,$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla_X^* \bar{P}Y, A_\xi^* X\}$ and $\{h^*(X, \bar{P}Y), \nabla_X^{*t} \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$ -valued and $\Gamma(S(TM))$ -valued bilinear forms, known as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

Using (2.9), (2.10), (2.17) and (2.18), the following relations hold good.

$$(2.19) \quad \bar{g}(h^l(X, \bar{P}Y), \xi) = g(A_\xi^* X, \bar{P}Y),$$

$$(2.20) \quad \bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y),$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Denote by \bar{R} and R the curvature tensors of $\bar{\nabla}$ and ∇ , respectively, then by straightforward calculations ([7]), we have

$$(2.21) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h^l(X, Z)} Y - A_{h^l(Y, Z)} X + A_{h^s(X, Z)} Y \\ &\quad - A_{h^s(Y, Z)} X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) \\ &\quad + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ &\quad - (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)), \end{aligned}$$

where

$$(2.22) \quad (\nabla_X h^s)(Y, Z) = \nabla_X^s h^s(Y, Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z),$$

$$(2.23) \quad (\nabla_X h^l)(Y, Z) = \nabla_X^l h^l(Y, Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z).$$

Let $\bar{M}(c)$ be a complex space form of constant holomorphic curvature c , then the curvature tensor \bar{R} is given by

$$(2.24) \quad \bar{R}(X, Y)Z = \frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY + 2\bar{g}(X, JY)JZ\},$$

for X, Y, Z vector fields on \bar{M} .

Using (2.24) and (2.21), we obtain

$$(2.25) \quad \begin{aligned} g(R(X, Y)Z, W) = & \frac{c}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & + g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W) \\ & + 2g(X, JY)g(JZ, W)\} - g(A_{h^l(X, Z)}Y, W) \\ & + g(A_{h^l(Y, Z)}X, W) - g(A_{h^s(X, Z)}Y, W) \\ & + g(A_{h^s(Y, Z)}X, W) - \bar{g}((\nabla_X h^l)(Y, Z), W) \\ & + \bar{g}((\nabla_Y h^l)(X, Z), W) - \bar{g}(D^l(X, h^s(Y, Z)), W) \\ & + \bar{g}(D^l(Y, h^s(X, Z)), W). \end{aligned}$$

Definition 2.2. ([1]): Let (\bar{M}, J, \bar{g}) be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on \bar{M} with respect to \bar{g} . Then \bar{M} is called an indefinite Kaehler manifold if J is parallel with respect to $\bar{\nabla}$, that is

$$(2.26) \quad (\bar{\nabla}_X J)Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}).$$

3. Generalized Cauchy-Riemann lightlike submanifolds

Definition 3.1. ([10]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold (\bar{M}, \bar{g}, J) then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$(3.1) \quad Rad(TM) = D_1 \oplus D_2, \quad J(D_1) = D_1, \quad J(D_2) \subset S(TM).$$

(B) *There exist two subbundles D_0 and D' of $S(TM)$ such that*

$$(3.2) \quad S(TM) = \{JD_2 \oplus D'\} \perp D_0, \quad J(D_0) = D_0, \quad J(D') = L_1 \perp L_2,$$

where D_0 is a non-degenerate distribution on M , L_1 and L_2 are vector subbundle of $\text{ltr}(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as

$$(3.3) \quad TM = D \oplus D', \quad \text{where } D = \text{Rad}(TM) \oplus D_0 \oplus JD_2.$$

M is called a proper *GCR-lightlike* submanifold if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let Q, P_1 and P_2 be the projections on D , $J(L_1) = M_1 \subset D'$ and $J(L_2) = M_2 \subset D'$, respectively. Then for any $X \in \Gamma(TM)$, we have

$$(3.4) \quad X = QX + P_1X + P_2X = QX + PX,$$

applying J to (3.4), we obtain

$$(3.5) \quad JX = TX + wP_1X + wP_2X = TX + wX,$$

where TX and wX are the tangential and transversal components of JX , respectively.

Similarly

$$(3.6) \quad JV = BV + CV,$$

for any $V \in \Gamma(\text{tr}(TM))$, where BV and CV are the sections of TM and $\text{tr}(TM)$, respectively. Differentiating (3.5) and using (2.9)-(2.12) and (3.6), we have

$$(3.7) \quad D^s(X, wP_1Y) = -\nabla_X^s wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y),$$

$$(3.8) \quad D^l(X, wP_2Y) = -\nabla_X^l wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y).$$

Using Kaehlerian property of $\bar{\nabla}$ with (2.11) and (2.12), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then we have*

$$(3.9) \quad (\nabla_X T)Y = A_{wY}X + Bh(X, Y)$$

and

$$(3.10) \quad (\nabla_X^t w)Y = Ch(X, Y) - h(X, TY),$$

where $X, Y \in \Gamma(TM)$ and

$$(3.11) \quad (\nabla_X T)Y = \nabla_X TY - T\nabla_X Y,$$

$$(3.12) \quad (\nabla_X^t w)Y = \nabla_X^t wY - w\nabla_X Y.$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then we have*

$$(3.13) \quad (\nabla_X B)V = A_{CV}X - TA_V X$$

and

$$(3.14) \quad (\nabla_X^t C)V = -wA_V X - h(X, BV),$$

where $X \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$ and

$$(3.15) \quad (\nabla_X B)V = \nabla_X BV - B\nabla_X^t V,$$

$$(3.16) \quad (\nabla_X^t C)V = \nabla_X^t CV - C\nabla_X^t V.$$

4. Ricci tensor of GCR-lightlike Submanifold of an indefinite complex space form

Let $\{E_1, E_2, \dots, E_m\}$ be a local orthonormal frame field on M such that $\{E_1, E_2, \dots, E_p, E_{p+1} = JE_1, E_{p+2} = JE_2, \dots, E_{2p} = JE_p\}$, $\{\xi_1, \xi_2, \dots, \xi_s, \xi_{s+1} = J\xi_1, \xi_{s+2} = J\xi_2, \dots, \xi_{2s} = J\xi_s\}$, $\{\xi_{2s+1}, \xi_{2s+2}, \dots, \xi_r\}$ and $\{J\xi_{2s+1}, J\xi_{2s+2}, \dots, J\xi_r\}$ be local frame fields on D_0 , D_1 , D_2 and JD_2 , respectively and $\{F_1, F_2, \dots, F_q\}$ be a local frame field on D' , then by direct computation, we have

$$(4.1) \quad \sum_{i=1}^m g(U, E_i)g(E_i, V) = g(U, V),$$

$$(4.2) \quad \sum_{i=r+1}^m g(U, E_i)g(E_i, V) = g(\bar{P}U, \bar{P}V),$$

$$(4.3) \quad \sum_{i=1}^{m-q} g(U, E_i)g(E_i, V) = g(QU, QV)$$

and

$$(4.4) \quad \sum_{i=1}^q g(U, E_i)g(E_i, V) = g(PU, PV),$$

for any $U, V \in \Gamma(TM)$ and the Ricci tensor is given by

$$(4.5) \quad Ric(U, V) = \sum_{a=1}^r g(R(U, \xi_a)V, N_a) + \sum_{b=r+1}^m g(R(U, U_b)V, U_b).$$

Using (2.25), we obtain

$$\begin{aligned}
 \sum_{a=1}^r g(R(U, \xi_a)V, N_a) &= -\frac{c}{4} \sum_{a=1}^r g(JV, \xi_a)g(JU, N_a) - \frac{cT}{4}g(U, V) \\
 &\quad - \frac{c}{4} \sum_{a=1}^r g(JU, V)g(J\xi_a, N_a) \\
 &\quad - \frac{c}{2} \sum_{a=1}^r g(JU, \xi_a)g(JV, N_a) \\
 &\quad + \sum_{a=1}^r g(A_{h^t(\xi_a, V)}U, N_a) - \sum_{a=1}^r g(A_{h^t(U, V)}\xi_a, N_a) \\
 (4.6) \quad &\quad + \sum_{a=1}^r g(A_{h^s(\xi_a, V)}U, N_a) - \sum_{a=1}^r g(A_{h^s(U, V)}\xi_a, N_a).
 \end{aligned}$$

Now, using equation (2.30) of [7] at page 158, for any $U \in \Gamma(T(M))$, define a differential 1-form as

$$\eta_a(U) = \bar{g}(U, N_a), \forall a \in \{1, 2, \dots, r\},$$

then any vector field U on M can be expressed as

$$(4.7) \quad U = \bar{P}U + \sum_{a=1}^r \eta_a(U)\xi_a,$$

where \bar{P} is the projection morphism of TM on $S(TM)$. Therefore, we have

$$(4.8) \quad g(U, JV) = g(\bar{P}U, JV) + \sum_{a=1}^r \bar{g}(U, N_a)g(\xi_a, JV).$$

Also, using (2.15), (2.16) and (4.8) in (4.6), we obtain

$$\begin{aligned}
 \sum_{a=1}^r g(R(U, \xi_a)V, N_a) &= -\frac{c(r+3)}{4}g(U, V) + \frac{3c}{4}g(TU, TV) \\
 &\quad -\frac{c}{4}\sum_{a=1}^r g(JU, V)g(J\xi_a, N_a) \\
 &\quad -\sum_{a=1}^r g(A_{N_a}U, h^l(\xi_a, V)) \\
 &\quad +\sum_{a=1}^r g(A_{N_a}\xi_a, h^l(U, V)) \\
 &\quad +\sum_{a=1}^r g(D^s(U, N_a), h^s(\xi_a, V)) \\
 &\quad -\sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(U, V)).
 \end{aligned}
 \tag{4.9}$$

Using (2.13), (2.20), (2.25) and (4.2), we obtain

$$\begin{aligned}
 \sum_{b=r+1}^m g(R(U, U_b)V, U_b) &= -\frac{c}{2}g(\bar{P}U, \bar{P}V) - \frac{(m-r)c}{4}g(U, V) \\
 &\quad +\sum_{b=r+1}^m \{g(h^l(U_b, V), h^*(U, U_b)) \\
 &\quad -g(h^l(U, V), h^*(U_b, U_b)) \\
 &\quad +g(h^s(U_b, V), h^s(U, U_b)) \\
 &\quad -g(h^s(U, V), h^s(U_b, U_b))\}.
 \end{aligned}
 \tag{4.10}$$

Thus substituting (4.9) and (4.10) in (4.5), we obtain the expression of Ricci tensor of a *GCR*-lightlike submanifold as

$$\begin{aligned}
 Ric(U, V) &= -\frac{(m+3)c}{4}g(U, V) + \frac{3c}{4}g(TU, TV) - \frac{c}{2}g(\bar{P}U, \bar{P}V) \\
 &\quad - \frac{c}{4} \sum_{a=1}^r g(JU, V)g(J\xi_a, N_a) + \sum_{a=1}^r g(A_{N_a}\xi_a, h^l(U, V)) \\
 &\quad - \sum_{a=1}^r g(A_{N_a}U, h^l(\xi_a, V)) + \sum_{a=1}^r g(D^s(U, N_a), h^s(\xi_a, V)) \\
 &\quad - \sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(U, V)) - \sum_{b=r+1}^m g(h^l(U, V), h^*(U_b, U_b)) \\
 &\quad + \sum_{b=r+1}^m g(h^l(U_b, V), h^*(U, U_b)) - \sum_{b=r+1}^m g(h^s(U, V), h^s(U_b, U_b)) \\
 (4.11) \quad &\quad + \sum_{b=r+1}^m g(h^s(U_b, V), h^s(U, U_b)).
 \end{aligned}$$

Next, by using an orthonormal frame fields on D' , D_0 , JD_2 and $Rad(TM)$, we can also express the Ricci tensors as

$$\begin{aligned}
 Ric(U, V) &= \sum_{i=1}^q g(R(U, F_i)V, F_i) + \sum_{k=1}^{2p} g(R(U, E_k)V, E_k) \\
 (4.12) \quad &\quad + \sum_{l=2s+1}^r g(R(U, J\xi_l)V, JN_l) + \sum_{a=1}^r g(R(U, \xi_a)V, N_a),
 \end{aligned}$$

therefore, we have

$$\begin{aligned}
 Ric_D(U, V) &= \sum_{k=1}^{2p} g(R(U, E_k)V, E_k) + \sum_{l=2s+1}^r g(R(U, J\xi_l)V, JN_l) \\
 (4.13) \quad &\quad + \sum_{a=1}^r g(R(U, \xi_a)V, N_a),
 \end{aligned}$$

$$(4.14) \quad Ric_{D'}(U, V) = \sum_{i=1}^q g(R(U, F_i)V, F_i).$$

Using (2.15), (2.20), (2.25) and (4.3), we obtain

$$\begin{aligned}
 \sum_{k=1}^{2p} g(R(U, E_k)V, E_k) &= -\frac{c}{2}g(QU, QV) - \frac{pc}{2}g(U, V) \\
 &+ \sum_{k=1}^{2p} g(h^l(E_k, V), h^*(U, E_k)) \\
 &- \sum_{k=1}^{2p} g(h^l(U, V), h^*(E_k, E_k)) \\
 &+ \sum_{k=1}^{2p} g(h^s(E_k, V), h^s(U, E_k)) \\
 &- \sum_{k=1}^{2p} g(h^s(U, V), h^s(E_k, E_k))
 \end{aligned}
 \tag{4.15}$$

and

$$\begin{aligned}
 \sum_{l=2s+1}^r g(R(U, J\xi_l)V, JN_l) &= -\frac{c(r-2s)}{4}g(U, V) \\
 &+ \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, V)g(U, JN_l) \\
 &- \sum_{l=2s+1}^r g(A_{h^l(U, V)}J\xi_l, JN_l) \\
 &+ \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, V)}U, JN_l) \\
 &- \sum_{l=2s+1}^r g(A_{h^s(U, V)}J\xi_l, JN_l) \\
 &+ \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, V)}U, JN_l).
 \end{aligned}
 \tag{4.16}$$

Using (4.9), (4.15) and (4.16) in (4.13), we obtain

$$\begin{aligned}
 Ric_D(U, V) = & \frac{3c}{4}g(TU, TV) - \frac{c(2p+2r-2s+3)}{4}g(U, V) - \frac{c}{2}g(QU, QV) \\
 & - \frac{c}{4} \sum_{a=1}^r g(JU, V)g(J\xi_a, N_a) + \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, V)g(U, JN_l) \\
 & + \sum_{a=1}^r g(A_{N_a}\xi_a, h^l(U, V)) - \sum_{a=1}^r g(A_{N_a}U, h^l(\xi_a, V)) \\
 & - \sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(U, V)) + \sum_{a=1}^r g(D^s(U, N_a), h^s(\xi_a, V)) \\
 & + \sum_{k=1}^{2p} g(h^l(E_k, V), h^*(U, E_k)) - \sum_{k=1}^{2p} g(h^l(U, V), h^*(E_k, E_k)) \\
 & + \sum_{k=1}^{2p} g(h^s(E_k, V), h^s(U, E_k)) - \sum_{k=1}^{2p} g(h^s(U, V), h^s(E_k, E_k)) \\
 & - \sum_{l=2s+1}^r g(A_{h^l(U, V)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, V)}U, JN_l) \\
 (4.17) \quad & - \sum_{l=2s+1}^r g(A_{h^s(U, V)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, V)}U, JN_l).
 \end{aligned}$$

Also using (2.13), (2.20), (2.25) and (4.4), we obtain

$$\begin{aligned}
 Ric_{D'}(U, V) = & -\frac{c}{2}g(PU, PV) - \frac{qc}{4}g(U, V) - \sum_{i=1}^q g(h^l(U, V), h^*(F_i, F_i)) \\
 & + \sum_{i=1}^q g(h^l(F_i, V), h^*(U, F_i)) - \sum_{i=1}^q g(h^s(F_i, F_i), h^s(U, V)) \\
 (4.18) \quad & + \sum_{i=1}^q g(h^s(F_i, V), h^s(U, F_i)).
 \end{aligned}$$

Let $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D')$, then particularly, we have

$$\begin{aligned}
 Ric_{D'}(X, Y) = & -\frac{qc}{4}g(X, Y) - \sum_{i=1}^q g(h^l(X, Y), h^*(F_i, F_i)) \\
 & + \sum_{i=1}^q g(h^l(F_i, Y), h^*(X, F_i)) - \sum_{i=1}^q g(h^s(F_i, F_i), h^s(X, Y)) \\
 (4.19) \quad & + \sum_{i=1}^q g(h^s(F_i, Y), h^s(X, F_i)),
 \end{aligned}$$

$$\begin{aligned}
 Ric_D(X, Y) = & \frac{c(s-r-p-1)}{2}g(X, Y) - \frac{c}{4} \sum_{a=1}^r g(JX, Y)g(J\xi_a, N_a) \\
 & + \sum_{a=1}^r g(A_{N_a}\xi_a, h^l(X, Y)) - \sum_{a=1}^r g(A_{N_a}X, h^l(\xi_a, Y)) \\
 & - \sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(X, Y)) + \sum_{a=1}^r g(D^s(X, N_a), h^s(\xi_a, Y)) \\
 & + \sum_{k=1}^{2p} g(h^l(E_k, Y), h^*(X, E_k)) - \sum_{k=1}^{2p} g(h^l(X, Y), h^*(E_k, E_k)) \\
 & + \sum_{k=1}^{2p} g(h^s(E_k, Y), h^s(X, E_k)) - \sum_{k=1}^{2p} g(h^s(X, Y), h^s(E_k, E_k)) \\
 & - \sum_{l=2s+1}^r g(A_{h^l(X, Y)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, Y)}X, JN_l) \\
 (4.20) \quad & - \sum_{l=2s+1}^r g(A_{h^s(X, Y)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, Y)}X, JN_l),
 \end{aligned}$$

$$\begin{aligned}
 Ric_{D'}(X, Z) = & \sum_{i=1}^q g(h^l(X, Z), h^*(F_i, F_i)) + \sum_{i=1}^q g(h^l(F_i, Z), h^*(X, F_i)) \\
 (4.21) \quad & - \sum_{i=1}^q g(h^s(F_i, F_i), h^s(X, Z)) + \sum_{i=1}^q g(h^s(F_i, Z), h^s(X, F_i)),
 \end{aligned}$$

$$\begin{aligned}
Ric_D(X, Z) = & \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, Z)g(X, JN_l) + \sum_{a=1}^r g(A_{N_a}\xi_a, h^l(X, Z)) \\
& - \sum_{a=1}^r g(A_{N_a}X, h^l(\xi_a, Z)) - \sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(X, Z)) \\
& + \sum_{a=1}^r g(D^s(X, N_a), h^s(\xi_a, Z)) + \sum_{k=1}^{2p} g(h^l(E_k, Z), h^*(X, E_k)) \\
& - \sum_{k=1}^{2p} g(h^l(X, Z), h^*(E_k, E_k)) + \sum_{k=1}^{2p} g(h^s(E_k, Z), h^s(X, E_k)) \\
& - \sum_{k=1}^{2p} g(h^s(X, Z), h^s(E_k, E_k)) - \sum_{l=2s+1}^r g(A_{h^l(X, Z)}J\xi_l, JN_l) \\
& + \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, Z)}X, JN_l) - \sum_{l=2s+1}^r g(A_{h^s(X, Z)}J\xi_l, JN_l) \\
(4.22) \quad & + \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, Z)}X, JN_l)
\end{aligned}$$

and

$$\begin{aligned}
Ric_{D'}(Z, W) = & -\frac{(q+2)c}{4}g(Z, W) - \sum_{i=1}^q g(h^l(Z, W), h^*(F_i, F_i)) \\
& + \sum_{i=1}^q g(h^l(F_i, W), h^*(Z, F_i)) - \sum_{i=1}^q g(h^s(F_i, F_i), h^s(Z, W)) \\
(4.23) \quad & + \sum_{i=1}^q g(h^s(F_i, W), h^s(Z, F_i)).
\end{aligned}$$

$$\begin{aligned}
Ric_D(Z, W) = & -\frac{c(2p + 2r - 2s + 3)}{4}g(Z, W) \\
& + \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, W)g(Z, JN_l) + \sum_{a=1}^r g(A_{N_a}\xi_a, h^l(Z, W)) \\
& - \sum_{a=1}^r g(A_{N_a}Z, h^l(\xi_a, W)) - \sum_{a=1}^r g(D^s(\xi_a, N_a), h^s(Z, W)) \\
& + \sum_{a=1}^r g(D^s(Z, N_a), h^s(\xi_a, W)) + \sum_{k=1}^{2p} g(h^l(E_k, W), h^*(Z, E_k)) \\
& - \sum_{k=1}^{2p} g(h^l(Z, W), h^*(E_k, E_k)) \\
& + \sum_{k=1}^{2p} g(h^s(E_k, W), h^s(Z, E_k)) - \sum_{k=1}^{2p} g(h^s(Z, W), h^s(E_k, E_k)) \\
& - \sum_{l=2s+1}^r g(A_{h^l(Z, W)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, W)}Z, JN_l) \\
(4.24) \quad & - \sum_{l=2s+1}^r g(A_{h^s(Z, W)}J\xi_l, JN_l) + \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, W)}Z, JN_l),
\end{aligned}$$

Definition 4.1. A GCR-lightlike submanifold of an indefinite Kaehler manifold is called:

- (i) *Totally geodesic GCR-lightlike submanifold if its second fundamental form h vanishes, that is, $h(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.*
- (ii) *D-geodesic GCR-lightlike submanifold if $h(X, Y) = 0$, for any $X, Y \in \Gamma(D)$.*
- (iii) *D' -geodesic GCR-lightlike submanifold $h(X, Y) = 0$, for any $X, Y \in \Gamma(D')$.*
- (iv) *Mixed-geodesic GCR-lightlike submanifold if $h(X, Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.*

Thus from (4.19) to (4.24), we have the following results.

Theorem 4.2. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X, Y \in \Gamma(D)$*

and $Z, W \in \Gamma(D')$

$$Ric_D(X, Y) = \frac{c(s-r-p-1)}{2}g(X, Y) - \frac{c}{4} \sum_{a=1}^r g(JX, Y)g(J\xi_a, N_a),$$

$$Ric_D(X, Z) = \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, Z)g(X, JN_l),$$

$$Ric_D(Z, W) = -\frac{c(2p+2r-2s+3)}{4}g(Z, W) + \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, W)g(Z, JN_l)$$

and

$$Ric_{D'}(X, Y) = -\frac{qc}{4}g(X, Y),$$

$$Ric_{D'}(X, Z) = 0,$$

$$Ric_{D'}(Z, W) = -\frac{(q+2)c}{4}g(Z, W).$$

Theorem 4.3. Let M be a D -geodesic GCR -lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X, Y \in \Gamma(D)$

$$Ric_D(X, Y) = \frac{c(s-r-p-1)}{2}g(X, Y) - \frac{c}{4} \sum_{a=1}^r g(JX, Y)g(J\xi_a, N_a)$$

and

$$Ric_{D'}(X, Y) = -\frac{qc}{4}g(X, Y) + \sum_{i=1}^q g(h^l(F_i, Y), h^*(X, F_i)) + \sum_{i=1}^q g(h^s(F_i, Y), h^s(X, F_i)).$$

Theorem 4.4. Let M be a D' -geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $Z, W \in \Gamma(D')$

$$\begin{aligned} Ric_D(Z, W) = & -\frac{c(2p+2r-2s+3)}{4}g(Z, W) + \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, W)g(Z, JN_l) \\ & - \sum_{a=1}^r g(A_{N_a}Z, h^l(\xi_a, W)) + \sum_{a=1}^r g(D^s(Z, N_a), h^s(\xi_a, W)) \\ & + \sum_{k=1}^{2p} g(h^l(E_k, W), h^*(Z, E_k)) + \sum_{k=1}^{2p} g(h^s(E_k, W), h^s(Z, E_k)) \\ & + \sum_{l=2s+1}^r g(A_{h^l(J\xi_l, W)}Z, JN_l) + \sum_{l=2s+1}^r g(A_{h^s(J\xi_l, W)}Z, JN_l) \end{aligned}$$

and

$$\begin{aligned} Ric_{D'}(Z, W) = & -\frac{(q+2)c}{4}g(Z, W) + \sum_{i=1}^q g(h^l(F_i, W), h^*(Z, F_i)) \\ & + \sum_{i=1}^q g(h^s(F_i, W), h^s(Z, F_i)). \end{aligned}$$

Theorem 4.5. Let M be a mixed-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X \in \Gamma(D)$ and $Z \in \Gamma(D')$

$$Ric_D(X, Z) = \frac{c}{4} \sum_{l=2s+1}^r g(J\xi_l, Z)g(X, JN_l)$$

and

$$Ric_{D'}(X, Z) = \sum_{i=1}^q g(h^l(F_i, Z), h^*(X, F_i)) + \sum_{i=1}^q g(h^s(F_i, Z), h^s(X, F_i)).$$

Definition 4.6. ([8]). A lightlike submanifold (M, g) of a semi-Riemannian manifold (\bar{M}, \bar{g}) is said to be a totally umbilical in \bar{M} , if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M , called the transversal curvature vector field of M , such that, for $X, Y \in \Gamma(TM)$

$$(4.25) \quad h(X, Y) = H\bar{g}(X, Y).$$

Using (2.9), it is clear that M is totally umbilic, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(\text{ltr}(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$(4.26) \quad \begin{aligned} h^l(X, Y) &= H^l g(X, Y), & h^s(X, Y) &= H^s g(X, Y), \\ D^l(X, W) &= 0, \end{aligned}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$.

Theorem 4.7. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then $H^l = 0$.*

Proof. Using (2.9), (2.10) and (2.26) with the hypothesis that M is a totally umbilic GCR-lightlike submanifold and then taking tangential parts of the resulting equation, we obtain

$$(4.27) \quad A_{wZ}Z + T\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) = 0,$$

where $Z \in \Gamma(JL_2)$. Taking inner product with $\xi \in \Gamma(D_2)$, we obtain $g(A_{wZ}Z, J\xi) + g(h^l(Z, Z), \xi) = 0$ and further using (2.13), we have $g(h^s(Z, J\xi), wZ) + g(h^l(Z, Z), \xi) = 0$. Hence, using (4.26), we get $g(Z, Z)g(H^l, \xi) = 0$, then the non-degeneracy of JL_2 implies that $H^l = 0$, which completes the proof. \square

Lemma 4.8. *Let M be a totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then $\nabla_X X \in \Gamma(D)$, for any $X \in \Gamma(D)$.*

Proof. Since $D' = J(L_1 \perp L_2)$, therefore $\nabla_X X \in \Gamma(D)$, if and only if, $g(\nabla_X X, J\xi) = 0$ and $g(\nabla_X X, JW) = 0$, for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Since M is a totally umbilic GCR-lightlike submanifold, we obtain

$$(4.28) \quad \begin{aligned} g(\nabla_X X, J\xi) &= -\bar{g}(\bar{\nabla}_X JX, \xi) = -\bar{g}(h^l(X, JX), \xi) \\ &= -\bar{g}(H^l, \xi)g(X, JX) = 0 \end{aligned}$$

and

$$(4.29) \quad \begin{aligned} g(\nabla_X X, JW) &= -\bar{g}(\bar{\nabla}_X JX, W) = -\bar{g}(h^s(X, JX), W) \\ &= -\bar{g}(H^s, W)g(X, JX) = 0. \end{aligned}$$

Thus from (4.28) and (4.29), the result follows. \square

Theorem 4.9. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then $H^s \in \Gamma(L_2)$.*

Proof. Since M is a totally umbilic GCR-lightlike submanifold, therefore using (3.7), we get $g(X, JX)H^s = wP_2\nabla_X X + g(X, X)CH^s$, for any $X \in \Gamma(D_0)$. Then, using the Lemma (4.8), we have $g(X, X)CH^s = 0$. As D_0 is non-degenerate, we get $CH^s = 0$. Hence $H^s \in \Gamma(L_2)$. \square

Theorem 4.10. *Let M be a totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then $\nabla_X JX = J\nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof. Using (3.10) and (3.12), we have $w\nabla_X X = h(X, JX) - Ch(X, X)$, for any $X \in \Gamma(D_0)$. Since M is totally umbilic, using (4.25), we have $w\nabla_X X = Hg(X, JX) - CH^l g(X, X) - CH^s g(X, X)$. Consequently, using the Theorems (4.7) and (4.9), we get $w\nabla_X X = 0$. Hence

$$(4.30) \quad \nabla_X X \in \Gamma(D).$$

Moreover, using (2.26) and the fact that D_0 is non-degenerate, we obtain that $\nabla_X JX = J\nabla_X X$, for any $X \in \Gamma(D_0)$. \square

Theorem 4.11. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then $H^s = 0$.*

Proof. For any $W \in \Gamma(S(TM^\perp))$ and $X \in \Gamma(D_0)$, using (2.26), (4.25) and the Theorem (4.10), we have

$$\begin{aligned} g(J\bar{\nabla}_X X, JW) &= g(\bar{\nabla}_X JX, JW) = g(\nabla_X JX, JW) + g(h(X, JX), JW) \\ &= g(\nabla_X JX, JW) + g(X, JX)g(H, JW) \\ (4.31) \quad &= g(J\nabla_X X, JW) = g(\nabla_X X, W) = 0. \end{aligned}$$

Also using (4.26), we have

$$\begin{aligned} g(J\bar{\nabla}_X X, JW) &= g(\bar{\nabla}_X X, W) = g(h^s(X, X), W) \\ (4.32) \quad &= g(X, X)g(H^s, W). \end{aligned}$$

Comparing (4.31) and (4.32), we have $g(X, X)g(H^s, W) = 0$. Thus, the non-degeneracy of D_0 and $S(TM^\perp)$ implies that $H^s = 0$. Hence the result follows. \square

As a consequence of Theorems (4.7) and (4.11), we obtain the following result.

Theorem 4.12. *Let M be a proper totally umbilic GCR-lightlike submanifold of an indefinite Kaehler manifold \bar{M} , then M is a totally geodesic GCR-lightlike submanifold.*

In [10], Duggal and Sahin proved a theorem for non-existence of totally umbilic proper *GCR*-lightlike submanifolds in a complex space form as

Theorem 4.13. (*[10]*). *There exist no totally umbilic proper GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, such that $c \neq 0$.*

Finally, using the theorems (4.12) and (4.13) in (4.11), we obtain

Theorem 4.14. *Let M be a totally umbilic GCR-lightlike submanifold of an complex space form $\bar{M}(c)$, then $Ric(U, V) = 0$, for any $U, V \in \Gamma(TM)$.*

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