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ANNIHILATOR-SMALL SUBMODULES

T. AMOUZEGAR-KALATI AND D. KESKIN-TÜTÜNCÜ*

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Abstract. Let M_R be a module with $S = \mathrm{End}(M_R)$. We call a submodule K of M_R annihilator-small if $K + T = M$, T a submodule of M_R, implies that $\ell_S(T) = 0$, where ℓ_S indicates the left annihilator of T over S. The sum $A_R(M)$ of all such submodules of M_R contains the Jacobson radical $\mathrm{Rad}(M)$ and the left singular submodule $Z_S(M)$. If M_R is cyclic, then $A_R(M)$ is the unique largest annihilator-small submodule of M_R. We study $A_R(M)$ and $K_S(M)$ in this paper. Conditions when $A_R(M)$ is annihilator-small and $K_S(M) = J(S) = \mathrm{Tot}(M, M)$ are given.

1. Introduction

Throughout this paper all rings are associative with identity and modules are unitary right modules. Let M_R be any module. The endomorphism ring $\mathrm{End}(M)$ of the right R-module M will be denoted by S. We abbreviate the Jacobson radical as $\mathrm{Rad}(M)$ for any right R-module M. The notations $N \subseteq^{ess} M$ and $N \subseteq^{max} M$ mean respectively that a submodule N of M is essential and maximal in the module M_R. The left annihilator of any submodule X of M is denoted by $\ell_S(X)$ while the right annihilator of any endomorphism f of M, namely the kernel of f, is denoted by $r_M(f)$.

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*Corresponding author

In [3], Nicholson and Zhou defined annihilator-small right (left) ideals. In this work, inspired by this nice work we introduce annihilator-small submodules of any right R-module M. Let M_R be a module and $K \subseteq M_R$ a submodule of M_R. We say that K is an *annihilator-small* submodule of M_R if $K+X=M$, X a submodule of M_R, implies that $\ell_S(X)=0$. Clearly every small submodule is annihilator-small. In Proposition 2.2, we prove that the converse is true if M_R is a coretractable module. Let M_R be a semi-projective module and $k \in S$. Then we prove the following which generalizes [3, Lemma 4]:

The submodule $k(M)$ is annihilator-small in M_R if and only if $bk(M) \subsetneq b(M)$ for all $0 \neq b \in S$ if and only if $\ell_S(1_S - ks) = 0$ for all $s \in S$ if and only if $\ell_S(1_S - sk) = 0$ for all $s \in S$ if and only if $\ell_S(k - ksk) = \ell_S(k)$ for all $s \in S$ (see Lemma 2.7).

In this note our aim is to generalize the other results of [3] from the ring case to the module case in light of Lemma 2.7. For example, we examine when the equalities $J(S) = K_S(M) = \mathrm{Tot}(M, M)$ are satisfied. As we mentioned in the abstract we study $A_R(M)$ which is the sum of all annihilator-small submodules of M_R. Relevant with it we prove Proposition 3.5 as a generalization of [3, Theorem 11].

2. Annihilator-small submodules

Definition 2.1. We say that a submodule K of a module M_R is *annihilator-small (a-small)* if $K + X = M$, X a submodule of M_R, implies that $\ell_S(X) = 0$ where $S = \mathrm{End}(M)$. In this case, we write $K \ll_a M$.

It is clear that every small submodule is a-small, but the converse is not true in general (consider the submodule $n\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$).

An R-module M_R is called *coretractable* if, for any proper submodule K of M, there exists a nonzero homomorphism $f : M \to M$ with $f(K) = 0$, that is, $\mathrm{Hom}\,(M/K, M) \neq 0$.

Proposition 2.2. *Let M_R be a coretractable module. If $K \ll_a M$, then $K \ll M$.*

Proof. Let $K + X = M$ for any submodule X of M. By hypothesis, $\ell_S(X) = 0$. But M_R is coretractable, thus $X = M$, and so $K \ll M$. □

Lemma 2.3. *Let M_R be a module. If $N \subseteq K \ll_a M$, where N is a submodule of M, then $N \ll_a M$.*

Proof. Clear. □

Let M_R be any module. We set $Z_S(M) = \{m \in M \mid \ell_S(m) = \ell_S(mR) \subseteq^{ess} {}_SS\}$.

Proposition 2.4. *If K is an a-small submodule of a finitely generated module M_R, then so is $K + Rad(M) + Z_S(M)$.*

Proof. Let $(K + \mathrm{Rad}(M) + Z_S(M)) + X = M$ where X is a submodule of M_R. Since $\mathrm{Rad}(M) \ll M$, $K + Z_S(M) + X = M$. Assume that $M_R = \sum_{i=1}^n a_iR$. Now, $k_i + z_i + x_i = a_i$ where $k_i \in K$, $z_i \in Z_S(M)$, $x_i \in X$. Hence $K + \sum_{i=1}^n z_iR + X = M$. Thus $0 = \ell_S(\sum_{i=1}^n z_iR + X) = (\cap_{i=1}^n \ell_S(z_iR)) \cap \ell_S(X)$ since $K \ll_a M$. As $\ell_S(z_i) \subseteq^{ess} {}_SS$, we have $\ell_S(X) = 0$. □

Lemma 2.5. *If T is a submodule of M_R and $\ell_S(T) \subseteq^{ess} {}_SS$, then $r_M\ell_S(T) \ll_a M_R$. In particular, $T \ll_a M_R$.*

Proof. Let $r_M\ell_S(T)+X = M$. Then $0 = \ell_S(M) = \ell_Sr_M\ell_S(T)\cap\ell_S(X) = \ell_S(T)\cap\ell_S(X)$, so $\ell_S(X) = 0$ since $\ell_S(T) \subseteq^{ess} {}_SS$. The last observation is by Lemma 2.3 since $T \subseteq r_M\ell_S(T)$ always holds. □

Note that the converse of Lemma 2.5 is true if $r_M[\ell_S(T) \cap Sb] = r_M\ell_S(T)+r_M(b)$ holds for all submodules T of M_R and all $b \in S$. To see this, let $\ell_S(T)\cap Sb = 0$ for an element b of S. Then $r_M\ell_S(T)+r_M(b) = M$, so $\ell_Sr_M(b) = 0$ since $r_M\ell_S(T) \ll_a M_R$. Hence $b = 0$ because $Sb \subseteq \ell_Sr_M(b)$, proving that $\ell_S(T) \subseteq^{ess} {}_SS$.

Following Wisbauer [5, p. 261], an R-module M_R is called *semi-injective* if for any $f \in S$,

$$Sf = \ell_S(ker(f)) = \ell_S(r_M(f))$$

(equivalently, for any monomorphism $f : N \to M$, where N is a factor module of M_R, and for any homomorphism $g : N \to M$, there exists $h : M \to M$ such that $hf = g$).

Proposition 2.6. *Let M_R be a coretractable semi-injective module and T a submodule of M_R. Then $\ell_S(T) \subseteq^{ess} {}_SS$ if $T \ll_a M$.*

Proof. This follows by Proposition 2.2 and [1, Proposition 4.5]. □

Recall that a module M_R is called *semi-projective* if for any epimorphism $f : M \to N$, where N is a submodule of M_R, and for any homomorphism $g : M \to N$, there exists $h : M \to M$ such that $fh = g$.

Lemma 2.7. *Consider the following conditions for a right R-module M and $k \in S$:*

(1) $k(M) \ll_a M_R$.
(2) $bk(M) \subsetneqq b(M)$ *for all* $0 \neq b \in S$.
(3) $\ell_S(1_S - ks) = 0$ *for all* $s \in S$.
(4) $\ell_S(1_S - sk) = 0$ *for all* $s \in S$.
(5) $\ell_S(k - ksk) = \ell_S(k)$ *for all* $s \in S$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *If* M_R *is semi-projective, then* (5) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Assume that $b \in S$ and $bk(M) = b(M)$. Let $m \in M$. Then $b(m) = bk(m')$ for some $m' \in M$. Hence $m - k(m') \in r_M(b)$. Therefore $m \in r_M(b) + k(M)$. Namely, $M = r_M(b) + k(M)$. Since $k(M) \ll_a M_R$, $\ell_S r_M(b) = 0$. As $Sb \subseteq \ell_S r_M(b)$, $b = 0$.

(2) $\Rightarrow$ (3) Let $s \in S$ and $b \in \ell_S(1_S - ks)$. Then $b = bks$ implies that $b(M) = bks(M) \subseteq bk(M)$. By (2), $b = 0$.

(3) $\Rightarrow$ (4) Let $s \in S$ and $b \in \ell_S(1_S - sk)$. Then $b(1_S - sk) = 0$ implies that $bs(1_S - ks) = b(s - sks) = b(1_S - sk)s = 0$. Hence $bs = 0$ by (3), and so $b = bsk = 0$.

(4) $\Rightarrow$ (5) Let $s \in S$ and $b \in \ell_S(k - ksk)$. By (4), $bk = 0$. Hence $b \in \ell_S(k)$. The other inclusion always holds.

(5) $\Rightarrow$ (1) Assume that M_R is semi-projective. Let $M = k(M) + X$ for a submodule X of M_R. Let $b \in \ell_S(X)$ and $m \in M$. Then there exist $m' \in M$ and $x \in X$ such that $m = k(m') + x$. Now $b(m) = bk(m')$, and so $b(M) = bk(M)$. Since M_R is semi-projective, there exists a homomorphism $s \in S$ such that $bks = b$. Note that $b(k - ksk) = 0$. Hence $b \in \ell_S(k - ksk) = \ell_S(k)$. Therefore $bk = 0$, and hence $b = 0$. □

Note that condition 2 in Lemma 2.7 implies that if $k(M) \ll_a M_R$ and $k \in S$ is not nilpotent, then $k(M) \supsetneqq k^2(M) \supsetneqq k^3(M) \supsetneqq \cdots$ is strictly decreasing.

Corollary 2.8. (See [3, Lemma 4]) *If* R *is a ring, then the following are equivalent for* $k \in R$:

(1) $kR \ll_a R_R$, *namely if* $R = kR + X$, X *a right ideal of* R, *then* $\ell_R(X) = 0$.
(2) $bR \supsetneqq bkR$ *for all* $0 \neq b \in R$.
(3) $\ell_R(1 - kr) = 0$ *for all* $r \in R$.
(4) $\ell_R(1 - rk) = 0$ *for all* $r \in R$.
(5) $\ell_R(k - krk) = \ell_R(k)$ *for all* $r \in R$.

Let us define $K_S(M) = \{s \in S \mid s(M) \ll_a M_R\}$ for any module M_R.

Corollary 2.9. *Let M_R be a module and $k \in K_S(M)$. Then $kS \subseteq K_S(M)$. If M_R is semi-projective, then $Sk \subseteq K_S(M)$.*

Proof. By Lemma 2.3, $kS \subseteq K_S(M)$. Now assume that M_R is semi-projective. Let $s \in S$. We show that $sk(M) \ll_a M_R$. Let $g \in S$. Then $\ell_S(1_S - gsk) = 0$ since $k(M) \ll_a M_R$, by Lemma 2.7(4). Again by Lemma 2.7(4), $sk(M) \ll_a M_R$. Hence $Sk \subseteq K_S(M)$. □

Corollary 2.10. *We have $K_S(M) \subseteq r_S(Soc(S_S))$. Moreover, $J(S) \subseteq K_S(M)$ provided that M_R is semi-projective.*

Proof. Let $s \in K_S(M)$. We need to show that $\mathrm{Soc}(S_S)s = 0$. Let $0 \neq t \in \mathrm{Soc}(S_S)$. Then $t \in S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where $S_1, \cdots, S_n$ are the simple right ideals of S. Assume $ts \neq 0$ and $t = t_1 + t_2 + \cdots + t_n$ where $t_i \in S_i$. Then $t_i s \neq 0$ for some $i \in \{1, \cdots, n\}$. Since S_i is simple, $t_i sS = S_i$. Now, $t_i = t_i s\alpha$ for some $\alpha \in S$. Then $t_i(1_S - s\alpha) = 0$, namely $t_i \in \ell_S(1_S - s\alpha)$. Since $s(M) \ll_a M$, $\ell_S(1_S - s\alpha) = 0$ by Lemma 2.7, hence $t_i = 0$, a contradiction. Thus $ts = 0$. So we proved that $\mathrm{Soc}(S_S)K_S(M) = 0$, hence $K_S(M) \subseteq r_S(\mathrm{Soc}(S_S))$.

Now let $k \in J(S)$. We show that $k \in K_S(M)$. Let $s \in S$. Take $\alpha \in \ell_S(1_S - ks)$. Then $\alpha(1_S - ks) = 0$. Since $1_S - ks$ is invertible, $\alpha = 0$. Thus $\ell_S(1_S - ks) = 0$ for all $s \in S$. By Lemma 2.7, $k \in K_S(M)$. □

Corollary 2.11. *Let M_R be a quasi-projective module. Then $K_S(M) = J(S) = \nabla(M)$, where $\nabla(M) = \{\phi \in S \mid Im\phi \ll M\}$.*

Proof. Let $f \in K_S(M)$. We show that $fS \ll S_S$. Let $I + fS = S$ for a right ideal $I \subseteq S$. Then $1 = fs + g$ for some $s \in S$, $g \in I$ and $M = fs(M) + g(M) \subseteq f(M) + g(M)$. Then the composition $M \overset{f}{\to} M \overset{\rho}{\to} M/g(M)$ is an epimorphism and there exists $\lambda \in S$ with $\rho = \rho f\lambda$. This means that $\rho(1 - f\lambda) = 0$. Since $f(M) \ll_a M$, by Lemma 2.7, $\ell_S(1 - f\lambda) = 0$. Thus $\rho = 0$, namely $g(M) = M$. As M_R is quasi-projective, there exists $h \in S$ with $1 = gh$ which means $I = S$. Now we have the equalities by using Corollary 2.10 and [2, 4.25]. □

Corollary 2.12. *Let M_R be a module and $f \in S$. If $f(M) \ll_a M_R$, then $fS \ll_a S_S$. The converse is true if M_R is semi-projective.*

Proof. First, assume that $f(M) \ll_a M$. Let $S = fS + I$ where I is a right ideal of S. Then $1_S = fs + x$, $s \in S$, $x \in I$. Hence $M = fs(M) + x(M) = f(M) + x(M)$. Since $f(M) \ll_a M$, $\ell_S(x(M)) = 0$. Thus $\ell_S(IM) = 0$, and so $\ell_S(I) = 0$. Therefore $fS \ll_a S_S$. Conversely, let $fS \ll_a S_S$. By Corollary 2.8, $\ell_S(f - fsf) = \ell_S(f)$ for all $s \in S$. By Lemma 2.7, $f(M) \ll_a M_R$. □

Corollary 2.13. *Let M_R be any module. If $f^2 = f \in K_S(M)$, then $f = 0$.*

Proof. Observe that by Lemma 2.7 (4) $f(M) \ll_a M_R$ implies $\ell_S(1_S - f) = 0$. Since $f \in \ell_S(1_S - f)$, $f = 0$. □

Corollary 2.14. *Let M_R be any module. The following are equivalent for a maximal left ideal I of $S = End(M)$:*

(1) $r_M(I) \ll_a M_R$.
(2) $I \subseteq^{ess} {}_SS$.

Proof. (1) ⇒ (2) Let $r_M(I) \ll_a M_R$. Assume that I is not essential in ${}_SS$. Then there exists a nonzero left ideal J of S such that $I \cap J = 0$. Since I is a maximal left ideal of S, then I is a direct summand of ${}_SS$. So, there exists an idempotent $e \in S$ such that $I = Se$. Hence $r_M(I) = (1 - e)(M) \ll_a M$. Then $1 - e \in K_S(M)$. By Corollary 2.13, $e = 1$, a contradiction.

(2) ⇒ (1) Let $I \subseteq^{ess} {}_SS$. Let $M = r_M(I) + X$ for a submodule X of M_R. Then $\ell_S(M) = 0 = \ell_S r_M(I) \cap \ell_S(X)$ implies that $I \cap \ell_S(X) = 0$. Since I is essential in ${}_SS$, $\ell_S(X) = 0$. □

Let f be an element in S. Then f is said to be *partially invertible* if, fS (equivalently, Sf) contains a nonzero idempotent.

For an R-module M_R, the total of M_R is defined as

$$\mathrm{Tot}(S) = \mathrm{Tot}(M, M) = \{f \in S \mid f \text{ is not partially invertible}\}.$$

The total may not be closed under addition. In fact, if 0 and 1 are the only idempotents in S, then total of M_R is the set of non-isomorphisms.

Proposition 2.15. *If M_R is a module, then $K_S(M) \subseteq Tot(M, M)$.*

Proof. If $f \in K_S(M)$ but $f \notin \mathrm{Tot}(M, M)$, then f is partially invertible. So, there exists $0 \neq e^2 = e \in fS$. By Corollary 2.9, $e \in K_S(M)$, which contradicts Corollary 2.13. □

If I is a subset of a ring R, then R is said to be *I-semipotent* if every right (equivalently, left) ideal not contained in I contains a nonzero idempotent, equivalently if every element $a \notin I$ has a partial inverse. A ring R is called *semipotent* if R is $J(R)$-semipotent.

Lemma 2.16. *Let I be a subset of $S = End(M_R)$. Then the following are equivalent:*

(1) *S is I-semipotent.*
(2) *$Tot(M, M) \subseteq I$.*

Proof. See [3, Lemma 20]. □

Let U be a submodule of an R-module M_R. The module M_R is called *U-semipotent* if, for every submodule A of M such that $A \nsubseteq U$, there exists a nonzero idempotent $e : M \to M$ such that $e(M) \subseteq A$ and $e(M) \nsubseteq U$. Clearly R is a semipotent ring if and only if R_R is $J(R)$-semipotent (see [4, Definition 2.5]).

Lemma 2.17. *Let U be a submodule of a semi-projective module M_R. If M is U-semipotent, then $Tot(M, M)M \subseteq U$.*

Proof. Let $a \in \mathrm{Tot}(M, M)$. If $a(M) \nsubseteq U$, then by hypothesis, there exists a nonzero idempotent $e : M \to M$ such that $e(M) \subseteq a(M)$ and $e(M) \nsubseteq U$. Since M_R is semi-projective, there exists $f : M \to M$ such that $af = e$, it is a contradiction. Therefore $a(M) \subseteq U$, hence $\mathrm{Tot}(M, M)M \subseteq U$. □

Proposition 2.18. *Let $S = End(M_R)$ for any module M_R. Then S is semipotent if and only if $J(S) = Tot(M, M)$.*

Proof. See [3, Theorem 21]. □

Proposition 2.19. *Let $S = End(M_R)$ for any semi-projective module M_R. Then $J(S) = K_S(M) = Tot(M, M)$ if S is semipotent.*

Proof. By Corollary 2.10, $J(S) \subseteq K_S(M)$. Let $s \in K_S(M)$. If $s \notin J(S)$, then since S is $J(S)$-semipotent, $K_S(M)$ have a nonzero idempotent, which is a contradiction (see Corollary 2.13). Thus $J(S) = K_S(M)$. By Proposition 2.15, $K_S(M) \subseteq \mathrm{Tot}(M, M)$. On the other hand, S is $K_S(M)$-semipotent since $J(S) = K_S(M)$. So by Lemma 2.16, $\mathrm{Tot}(M, M) \subseteq K_S(M)$ (also see Proposition 2.18). □

Proposition 2.20. *Let $S = End(M_R)$ for any semi-projective module M_R in which $\ell_S(a) = 0$, $a \in S$, implies $aS = S$. Then $K_S(M) = J(S)$.*

Proof. Observe that $J(S) \subseteq K_S(M)$ by Corollary 2.10. Let $k \in K_S(M)$. Then $k(M) \ll_a M$, so $\ell_S(1_S - ks) = 0$ for all $s \in S$ by Lemma 2.7. Hence $(1_S - ks)S = S$ by hypothesis. Thus $k \in J(S)$. □

A ring R is called *right Kasch* if each simple right R-module embeds in R; equivalently, if $\ell_R(T) \neq 0$ for every (maximal) right ideal T of R. Call R left *principally injective* if every R-linear map $Ra \to R$, $a \in R$, extends to $R \to R$; equivalently if aR is a right annihilator in R for each $a \in R$. Finally, call R a left C_2 ring if every left ideal that is isomorphic to a direct summand of ${}_RR$ is itself a direct summand of ${}_RR$.

Example 2.21. *In each of the following cases we have $J(S) = K_S(M)$ for a semi-projective module M_R:*

(1) *S is semipotent.*
(2) *S is right Kasch.*
(3) *S is left principally injective.*
(4) *S is a left C_2 ring.*

Proof. (1) Follows by Proposition 2.19.

(2) Let $a \in S$ and $\ell_S(a) = 0$. If $aS \neq S$, then $\ell_S(aS) \neq 0$ by (2); that is, $\ell_S(a) \neq 0$, a contradiction. Thus by Proposition 2.20, $J(S) = K_S(M)$.

(3) Let $a \in S$ and $\ell_S(a) = 0$. By (3), $aS = r_S(X)$. Then $X \subseteq \ell_S(a) = 0$, so $aS = r_S(X) = S$. Thus by Proposition 2.20, $J(S) = K_S(M)$.

(4) Let $a \in S$ and $\ell_S(a) = 0$. Then $Sa \cong S$. By (4), Sa is a direct summand of S. Then $a = aba$ for some element b of S. But then $0 = \ell_S(a) = \ell_S(ab) = S(1_S - ab)$. Now $ab = 1_S$ and $S = (ab)S \oplus (1_S - ab)S$ imply that $S = aS$. By Proposition 2.20, $K_S(M) = J(S)$. □

3. The submodule $A_R(M)$

Lemma 3.1. *Let $M = mR$, where $m \in M$, be a cyclic R-module. Then the following are equivalent for $k \in M$:*

(1) *$kR \ll_a M$.*
(2) *$f(kR) \subsetneq f(M)$ for all $0 \neq f \in S$.*
(3) *$\ell_S(m - kr) = 0$ for all $r \in R$.*

Proof. (1) ⇒ (2) If $f(kR) = f(M)$, then $f(m) = f(kr)$ for some $r \in R$. Thus $f \in \ell_S(m - kr)$. But $kR + (m - kr)R = mR = M$. So, by (1), $\ell_S(m - kr) = 0$. Thus $f = 0$.

(2) ⇒ (3) If $f \in \ell_S(m - kr)$ and $r \in R$, then $f(m) = f(kr) \subseteq f(kR)$. By (2), $f = 0$.

(3) ⇒ (1) If $kR + X = M$, where X is a submodule of M_R, then $m = kr + x$, $r \in R$, $x \in X$. If $f \in \ell_S(X)$, then $f(m) = f(kr)$. So $f \in \ell_S(m - kr)$. Hence $f = 0$ by (3). □

Let M_R be a module. An element $k \in M$ is called *a-small* if $kR \ll_a M$. For convenience, define

$$K_R(M) = \{k \in M \mid k \text{ is a} - \text{small in } M\} = \{k \in M \mid kR \ll_a M\}.$$

Note that $K_R(M)$ may not be closed under addition: for example, consider -2 and 3 in the $\mathbb{Z}$-module $\mathbb{Z}$.

Proposition 3.2. *Let $M = mR$ be a cyclic R-module and K any submodule of M_R. Then the following are equivalent:*

(1) *K is a-small in M.*
(2) *$K \subseteq K_R(M)$.*
(3) *$\ell_S(m - k) = 0$ for every $k \in K$.*

Proof. (1) $\Rightarrow$ (2) By Lemma 2.3.

(2) $\Rightarrow$ (3) Lemma 3.1.

(3) $\Rightarrow$ (1) Let $K + X = M$, where X is a submodule of M_R. If $m = k + x$, $k \in K$, $x \in X$, then $\ell_S(X) \subseteq \ell_S(m - k) = 0$ by (3). Hence $K \ll_a M$. □

The sum of a-small submodules need not be a-small: for example, consider $3\mathbb{Z} + (-2)\mathbb{Z}$ in the $\mathbb{Z}$-module $\mathbb{Z}$.

Let M_R be a module. We define

$$A_R(M) = \sum\{K \leq M_R \mid K \ll_a M\}.$$

Clearly, $K_R(M) \subseteq A_R(M)$ in every right R-module M_R, but this may not be equality (consider the $\mathbb{Z}$-module $\mathbb{Z}$).

Proposition 3.3. *Let M_R be a module. Then:*

(1) $A_R(M) = \{x_1 + x_2 + \cdots + x_n \mid x_i \in K_R(M) \ for \ each \ i, \ n \geq 1\}$.
(2) $A_R(M) = K_R(M)R$.
(3) $Rad(M) \subseteq K_R(M)$ *and* $Z_S(M) \subseteq K_R(M)$.

Proof. (1) Set $X = \{x_1 + x_2 + \cdots + x_n \mid x_i \in K_R(M) \text{ for each i}, \ n \geq 1\}$. If $x \in A_R(M)$, then $x \in X_1 + X_2 + \cdots + X_n$ where $X_i \ll_a M_R$ for each i. If $x = x_1 + x_2 + \cdots + x_n$, $x_i \in X_i$, then $x_iR \ll_a M_R$ by Lemma 2.3. Hence $x_i \in K_R(M)$ for each i. Thus $A_R(M) \subseteq X$. It is easy to see that $X \subseteq A_R(M)$.

(2) Follows by (1) and the fact that $K_R(M) \subseteq A_R(M)$.

(3) Let $x \in \text{Rad}(M)$. Then $xR \ll M$ and hence $xR \ll_a M$. So $x \in K_R(M)$. Therefore $\text{Rad}(M) \subseteq K_R(M)$. Now let $y \in Z_S(M)$. Then $\ell_S(y) = \ell_S(yR) \subseteq^{ess} {}_SS$. By Lemma 2.5, $yR \ll_a M$. So $y \in K_R(M)$. Therefore $Z_S(M) \subseteq K_R(M)$. □

Proposition 3.4. *Let M_R be a coretractable module. Then $Rad(M) = A_R(M) = K_R(M)$. Moreover, if M_R is semi-injective, then $Rad(M) = A_R(M) = K_R(M) = r_M(Soc(_SS)) = Z_S(M)$.*

Proof. By Proposition 2.2, $\mathrm{Rad}(M) = A_R(M) = K_R(M)$. Now suppose that M_R is semi-injective. Then by [1, Corollary 4.7], $\mathrm{Rad}(M) = A_R(M) = K_R(M) = r_M(\mathrm{Soc}(_SS))$. Now, let $x \in K_R(M)$. Then $xR \ll_a M$. By Proposition 2.6, $\ell_S(xR) \subseteq^{ess} {}_SS$. Thus $x \in Z_S(M)$. Hence $Z_S(M) = K_R(M)$ by Proposition 3.3(3). □

Proposition 3.5. *Let M_R be a module. Consider the following conditions:*

(1) *If $K \ll_a M$ and $L \ll_a M$, then $K + L \ll_a M$.*
(2) *$K_R(M)$ is closed under addition.*
(3) *$A_R(M) = K_R(M)$.*
(4) *$A_R(M) \ll_a M$.*

Then (1) ⇒ (2) ⇒ (3) and (4) ⇒ (1) hold. If M is cyclic, then (3) ⇒ (4) holds.

Moreover, if $M = mR$, where $m \in M$, and one of the above conditions holds, then we have:

(a) *$A_R(M)$ is the unique largest a-small submodule of M.*
(b) *$A_R(M) = \{k \in M \mid \ell_S(m - kr) = 0 \ for \ all \ r \in R\}$.*
(c) *$A_R(M) = \bigcap\{U \subseteq^{max} M \mid A_R(M) \subseteq U\}$.*

Proof. (1) ⇒ (2) Since $(k + l)R \subseteq kR + lR$, $K_R(M)$ is closed under addition by Lemma 2.3.

(2) ⇒ (3) It is clear that $K_R(M) \subseteq A_R(M)$. By (2) and Proposition 3.3(1), $A_R(M) \subseteq K_R(M)$.

(4) ⇒ (1) Let $K \ll_a M$ and $L \ll_a M$. Then $K \subseteq A_R(M)$ and $L \subseteq A_R(M)$, so $K + L \subseteq A_R(M)$. Thus, by (4) and Lemma 2.3, $K + L \ll_a M$.

(3) ⇒ (4) Let $M = mR$ for some $m \in M$ and $A_R(M) + X = M$ for a submodule X of M_R. So $K_R(M) + X = M$ by (3). If $m = k + x$ with $k \in K_R(M)$ and $x \in X$, then $M = kR + X$ and $kR \ll_a M$. Hence $\ell_S(X) = 0$, so $A_R(M) \ll_a M$.

Finally, (a) is clear by (4), and (b) follows from (3) and Lemma 3.1. As to (c): If $a \notin A_R(M)$, then aR is not a-small by (3), so $aR + X = M$ for some submodule X of M_R with $\ell_S(X) \neq 0$. As $A_R(M) \ll_a M$ by (4), we have $A_R(M) + X \neq M$. If $A_R(M) + X \subseteq U \subseteq^{max} M$, then $a \notin U$, this proves (c). □

Corollary 3.6. *Let M_R be a cyclic module. If $K_R(M)$ is closed under addition, then $Rad(M/A_R(M)) = Rad(M/K_R(M)) = 0$.*

Proof. This follows by part (c) of Proposition 3.5. □

Proposition 3.7. *Let M_R be a finitely generated module. If $A_R(M) \subseteq Rad(M) + Z_S(M)$, then the sum of any two a-small submodules is a-small.*

Proof. Let $K \ll_a M_R$ and $L \ll_a M_R$. Then $K + L \subseteq A_R(M)$. By Proposition 2.4 and Lemma 2.3, $K + L \ll_a M_R$. □

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Tayyebeh Amouzegar-Kalati

Department of Mathematics, Quchan University of Advanced Technologies Engineering, Quchan, Iran

Email: `t.amoozegar@yahoo.com`

Derya Keskin-Tütüncü

Department of Mathematics, University of Hacettepe, P.O. Box 06800, Beytepe, Ankara, Turkey

Email: `keskin@hacettepe.edu.tr`