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DETERMINANTS AND PERMANENTS OF HESSENBERG MATRICES AND GENERALIZED LUCAS POLYNOMIALS

K. KAYGISIZ* AND A. ŞAHİN

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Abstract. In this paper, we give some determinantal and permanental representations of generalized Lucas polynomials, which are a general form of generalized bivariate Lucas p-polynomials, ordinary Lucas and Perrin sequences etc., by using various Hessenberg matrices. In addition, we show that determinant and permanent of these Hessenberg matrices can be obtained by using combinations. Then we show, the conditions under which the determinants of the Hessenberg matrix become its permanents.

1. Introduction

Fibonacci numbers f_n, Lucas numbers l_n and Perrin numbers r_n are defined by

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \text{ for } n > 2 \text{ and } f_1 = f_2 = 1, \\ l_n &= l_{n-1} + l_{n-2} \text{ for } n > 1 \text{ and } l_0 = 2,\ l_1 = 1, \\ r_n &= r_{n-2} + r_{n-3} \text{ for } n > 3 \text{ and } r_0 = 3,\ r_1 = 0,\ r_2 = 2, \end{aligned}$$

respectively.

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∗Corresponding author

Generalizations of these sequences have been studied by a number of researchers. For instance; Miles [16] defined generalized order-k Fibonacci numbers, Er [2] defined k sequences of generalized order-k Fibonacci numbers and Kaygısız and Bozkurt [4] defined k-generalized order-k Perrin numbers.

MacHenry [12] defined generalized Fibonacci polynomials $F_{k,n}(t)$ and Lucas polynomials $G_{k,n}(t)$ as follows;

$$\begin{array}{ll} F_{k,n}(t) = 0,\ n < 0, & G_{k,n}(t) = 0,\ n < 0, \\ F_{k,0}(t) = 1, & G_{k,0}(t) = \ k, \\ F_{k,n}(t) = \sum\limits_{j=1}^{k} t_j F_{k,n-j}(t), & G_{k,1}(t) = t_1, \\ & G_{k,n}(t) = G_{k-1,n}(t),\ 1 \le n < k, \\ & G_{k,n}(t) = \sum\limits_{j=1}^{k} t_j G_{k,n-j}(t),\ n \ge k \end{array}$$

where t_i $(1 \le i \le k)$ are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k.$$

In [13, 14], authors obtained some properties of this polynomials. In addition, in [15], authors obtained $(n, k \in \mathbb{N},\ n \ge 1)$

$$G_{k,n}(t) = \sum_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k} \tag{1.1}$$

where a_i are nonnegative integers for all i $(1 \le i \le k)$, with initial conditions given by

$$G_{k,0}(t) = k,\ G_{k,-1}(t) = 0,\ \cdots,\ G_{k,-k+1}(t) = 0.$$

Throughout this paper, the notations $a \vdash n$ and $|a|$ are used instead of $\sum\limits_{j=1}^{k} j a_j = n$ and $\sum\limits_{j=1}^{k} a_j$, respectively.

Kaygısız and Şahin [5] defined generalized Perrin polynomials $R_{k,n}(t)$ by using generalized Lucas polynomials.

The generalized bivariate Lucas p-polynomials [20] are defined by

$$L_{p,n}(x, y) = x L_{p,n-1}(x, y) + y L_{p,n-p-1}(x, y)$$

for $n > p$, with boundary conditions $L_{p,0}(x, y) = (p+1)$, $L_{p,n}(x, y) = x^n$, $n = 1, 2, ..., p$.

Table 1. Cognate polynomial sequences 1.

$\mathbf{k}$	$\mathbf{t_1}$	$\mathbf{t_i}(2 \le i \le (k-1))$	$\mathbf{t_k}$	$\mathbf{G_{k,n}(t)}$
k	0	t_i	t_k	$R_{k,n}(t)$
k	x	0	y	$L_{p,n}(x,y)$
3	0	1	1	r_n

Table 2. [20]Cognate polynomial sequences 2.

$\mathbf{x}$	$\mathbf{y}$	$\mathbf{p}$	$\mathbf{L_{p,n}(x,y)}$
x	y	1	bivariate Lucas polynomials $L_n(x,y)$
x	1	p	Lucas p-polynomials $L_{p,n}(x)$
x	1	1	Lucas polynomials $l_n(x)$
1	1	p	Lucas p-numbers $L_p(n)$
1	1	1	Lucas numbers L_n
$2x$	y	p	bivariate Pell-Lucas p-polynomials $L_{p,n}(2x,y)$
$2x$	y	1	bivariate Pell-Lucas polynomials $L_n(2x,y)$
$2x$	1	p	Pell-Lucas p-polynomials $Q_{p,n}(x)$
$2x$	1	1	Pell-Lucas polynomials $Q_n(x)$
2	1	1	Pell-Lucas numbers Q_n
$2x$	-1	1	Chebysev polynomials of the first kind $T_n(x)$
x	$2y$	p	bivariate Jacobsthal-Lucas p-polynomials $L_{p,n}(x,2y)$
x	$2y$	1	bivariate Jacobsthal-Lucas polynomials $L_n(x,2y)$
1	$2y$	1	Jacobsthal-Lucas polynomials $j_n(y)$
1	2	1	Jacobsthal-Lucas numbers j_n

Tables 1 and 2 show that $G_{k,n}(t)$ are general form of many sequences and polynomials. Therefore, any result obtained from the polynomials $G_{k,n}(t)$ is valid for all sequences and polynomials mentioned in these tables.

On the other hand, many researchers studied determinantal and permanental representations of k sequences of generalized order-k Fibonacci and Lucas numbers. For example, Minc [17] defined an $n \times n$ (0,1)-matrix $F(n,k)$ and showed that the permanents of $F(n,k)$ is equal to the generalized order-k Fibonacci numbers.

In [10, 11], authors defined two (0,1)-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. Öcal et al. [18] gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers and obtained Binet's formulas for these sequences. Kılıç and Stakhov [8] gave permanent representation of Fibonacci and Lucas p-numbers. Kılıç and Taşcı [9] studied permanents and determinants of Hessenberg matrices. Yılmaz and Bozkurt [22] derived some relationships between Pell and Perrin sequences, as well as permanents and determinants of a type of Hessenberg matrices. Kaygısız and Şahin [7] gave some determinantal and permanental representations of Fibonacci type numbers. Kaygısız and Şahin [6] gave some determinantal and permanental representations of generalized bivariate Lucas p-polynomials. In [3, 19, 21], authors gave some relations between determinants and permanents.

The main purpose of this paper is to give some determinantal and permanental representations of generalized Lucas polynomials by using various Hessenberg matrices. Then we provide some conditions under which the determinants of the Hessenberg matrix become its permanents.

2. The determinantal representations

An $n \times n$ matrix ${}_{+}A_n = (a_{ij})$ is called lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$ i.e.,

$$ {}_{+}A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix} \tag{2.1} $$

Lemma 2.1. [1] *Let ${}_{+}A_n$ be the $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define* $\det({}_{+}A_0) = 1$. *Then,* $\det({}_{+}A_1) = a_{11}$ *and for $n \geq 2$*

$$ \det({}_{+}A_n) = a_{n,n} \det({}_{+}A_{n-1}) + \sum_{r=1}^{n-1} [(-1)^{n-r} a_{n,r} (\prod_{j=r}^{n-1} a_{j,j+1}) \det({}_{+}A_{r-1})]. \tag{2.2} $$

Now, we define two Hessenberg matrices ${}_{+}C_{k,m}$ and ${}_{-}C_{k,m}$ whose determinants give the generalized Lucas polynomials.

Theorem 2.2. *Let $k \geq 2$ and $n \geq 1$ be integers, and let $G_{k,n}(t)$ be the generalized Lucas polynomials and ${}_{+}C_{k,n} = (c_{rs})$ an $n \times n$ Hessenberg* *matrix, given by*

$$c_{rs} = \begin{cases} i^{|r-s|}.\frac{t_{r-s+1}}{t_2^{(r-s)}}, & \text{if } s \neq 1 \text{ and } -1 \leq r-s < k, \\ i^{|r-s|}.\frac{t_{r-s+1}}{t_2^{(r-s)}}.(r-s+1), & \text{if } s = 1 \text{ and } -1 \leq r-s < k, \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$$_{+}C_{k,n} = \begin{bmatrix} t_1 & it_2 & 0 & 0 & \cdots & 0 \\ 2i & t_1 & it_2 & 0 & \cdots & 0 \\ 3i^2\frac{t_3}{t_2^2} & i & t_1 & it_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ ki^{k-1}\frac{t_k}{t_2^{k-1}} & i^{k-2}\frac{t_{k-1}}{t_2^{k-2}} & i^{k-3}\frac{t_{k-2}}{t_2^{k-3}} & i^{k-4}\frac{t_{k-3}}{t_2^{k-4}} & \cdots & 0 \\ 0 & i^{k-1}\frac{t_k}{t_2^{k-1}} & i^{k-2}\frac{t_{k-1}}{t_2^{k-2}} & i^{k-3}\frac{t_{k-2}}{t_2^{k-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & it_2 \\ 0 & 0 & 0 & \cdots & i & t_1 \end{bmatrix}$$

where $t_0 = 1$ *and* $i = \sqrt{-1}$. *Then,*

$$\det({}_{+}C_{k,n}) = G_{k,n}(t). \tag{2.3}$$

Proof. To prove (2.3), we use the mathematical induction on m. The result is true for $m = 1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to m, namely, $\det({}_{+}C_{k,m}) = G_{k,m}(t)$. Then, by using Lemma 2.1, we have

$$\begin{aligned} \det({}_{+}C_{k,m+1}) &= c_{m+1,m+1}\det({}_{+}C_{k,m}) \\ &\quad + \sum_{r=1}^{m}\left[(-1)^{m+1-r}c_{m+1,r}\prod_{j=r}^{m}c_{j,j+1}\det({}_{+}C_{k,r-1})\right] \\ &= t_1\det({}_{+}C_{k,m}) + \sum_{r=1}^{m-k+1}\left[(-1)^{m+1-r}c_{m+1,r}\prod_{j=r}^{m}c_{j,j+1}\det({}_{+}C_{k,r-1})\right] \\ &\quad + \sum_{r=m-k+2}^{m}\left[(-1)^{m+1-r}c_{m+1,r}\prod_{j=r}^{m}c_{j,j+1}\det({}_{+}C_{k,r-1})\right] \end{aligned}$$

$$\begin{aligned}
&= t_1 \det({}_+C_{k,m}) \\
&\quad + \sum_{r=m-k+2}^{m} \left[(-1)^{m+1-r} c_{m+1,r} \prod_{j=r}^{m} c_{j,j+1} \det({}_+C_{k,r-1}) \right] \\
&= t_1 \det({}_+C_{k,m}) \\
&\quad + \sum_{r=m-k+2}^{m} \left[(-1)^{m+1-r} . i^{m+1-r} \frac{t_{m-r+2}}{t_2^{(m-r+1)}} \prod_{j=r}^{m} it_2 \det({}_+C_{k,r-1}) \right] \\
&= t_1 \det({}_+C_{k,m}) \\
&\quad + \sum_{r=m-k+2}^{m} \left[(-i)^{m+1-r} \frac{t_{m-r+2}}{t_2^{(m-r+1)}} . i^{m+1-r} . t_2^{(m-r+1)} \det({}_+C_{k,r-1}) \right] \\
&= t_1 \det({}_+C_{k,m}) + \sum_{r=m-k+2}^{m} t_{m-r+2} \det({}_+C_{k,r-1}) \\
&= t_1 \det({}_+C_{k,m}) + t_2 \det({}_+C_{k,m-1}) + \cdots + t_k \det({}_+C_{k,m-(k-1)}).
\end{aligned}$$

From the hypothesis and definition of generalized Lucas polynomials we obtain

$$\det({}_+C_{k,m+1}) = t_1 G_{k,m}(t) + \cdots + t_k G_{k,m-(k-1)}(t) = G_{k,m+1}(t).$$

Therefore, (2.3) holds for all positive integers. □

Example 2.3. *We obtain* 6*-th generalized Lucas polynomial for* $k = 5$*, by using* (2.3)*.*

$$\begin{aligned}
\det({}_+C_{5,6}) &= \det \begin{bmatrix} t_1 & it_2 & 0 & 0 & 0 & 0 \\ 2i & t_1 & it_2 & 0 & 0 & 0 \\ 3\frac{-t_3}{t_2^2} & i & t_1 & it_2 & 0 & 0 \\ 4\frac{-it_4}{t_2^3} & \frac{-t_3}{t_2^2} & i & t_1 & it_2 & 0 \\ 5\frac{t_5}{t_2^4} & \frac{-it_4}{t_2^3} & \frac{-t_3}{t_2^2} & i & t_1 & it_2 \\ 0 & \frac{t_5}{t_2^4} & \frac{-it_4}{t_2^3} & \frac{-t_3}{t_2^2} & i & t_1 \end{bmatrix} \\
&= t_1^6 + 6t_1^4 t_2 + 9t_1^2 t_2^2 + 2t_2^3 + 6t_1^3 t_3 + 3t_3^2 + 12 t_1 t_2 t_3 \\
&\quad + 6t_1^2 t_4 + 6t_2 t_4 + 6t_1 t_5 \\
&= G_{5,6}(t).
\end{aligned}$$

Theorem 2.4. *Let* $k \geq 2$ *and* $n \geq 1$ *be integers,* $G_{k,n}$ *the generalized Lucas polynomial and* ${}_-C_{k,n} = (b_{ij})$ *an* $n \times n$ *lower Hessenberg matrix,* *given by*

$$b_{ij} = \begin{cases} -t_2, & \text{if } j = i+1, \\ \frac{t_{i-j+1}}{t_2^{(i-j)}}, & \text{if } j \neq 1 \text{ and } 0 \leq i-j < k, \\ \frac{t_{i-j+1}}{t_2^{(i-j)}}.(i-j+1), & \text{if } j = 1 \text{ and } 0 \leq i-j < k, \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$$_{-}C_{k,n} = \begin{bmatrix} t_1 & -t_2 & 0 & 0 & \cdots & 0 \\ 2 & t_1 & -t_2 & 0 & \cdots & 0 \\ 3\frac{t_3}{t_2^2} & 1 & t_1 & -t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ k\frac{t_k}{t_2^{k-1}} & \frac{t_{k-1}}{t_2^{k-2}} & \frac{t_{k-2}}{t_2^{k-3}} & \ddots & \cdots & 0 \\ 0 & \frac{t_k}{t_2^{k-1}} & \frac{t_{k-1}}{t_2^{k-2}} & \frac{t_{k-2}}{t_2^{k-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -t_2 \\ 0 & 0 & 0 & \cdots & \cdots & t_1 \end{bmatrix}$$

where $t_0 = 1$. *Then,*

$$\det({}_{-}C_{k,n}) = G_{k,n}(t). \tag{2.4}$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 2.1. □

Example 2.5. *We obtain* 5*-th generalized Lucas polynomial for* $k = 4$, *by using* (2.4).

$$\begin{aligned} \det({}_{-}C_{4,5}) &= \det \begin{bmatrix} t_1 & -t_2 & 0 & 0 & 0 \\ 2 & t_1 & -t_2 & 0 & 0 \\ 3\frac{t_3}{t_2^2} & 1 & t_1 & -t_2 & 0 \\ 4\frac{t_4}{t_2^3} & \frac{t_3}{t_2^2} & 1 & t_1 & -t_2 \\ 0 & \frac{t_4}{t_2^3} & \frac{t_3}{t_2^2} & 1 & t_1 \end{bmatrix} \\ &= t_1^5 + 5t_1^3t_2 + 5t_1t_2^2 + 5t_1^2t_3 + 5t_2t_3 + 5t_1t_4 \\ &= G_{4,5}(t). \end{aligned}$$

Corollary 2.6. *If we rewrite equalities* (2.3) *and* (2.4) *for* $t_i = 1$ *and* $k = 2$, *then we obtain*

$$\det({}_{+}C_{k,n}) = \det({}_{-}C_{k,n}) = l_n$$

where l_n *are the ordinary Lucas numbers.*

Corollary 2.7. *If we rewrite equalities* (2.3) *and* (2.4) *for* $t_1 = 0$*, then we obtain*

$$\det({}_{+}C_{k,n}) = \det({}_{-}C_{k,n}) = R_{k,n}(t)$$

where $R_{k,n}(t)$ *are the generalized Perrin polynomials.*

Corollary 2.8. *If we rewrite the right hand side of equalities* (2.3) *and* (2.4) *for* $t_1 = x$*,* $t_k = y$*,* $t_i = 0$ $(2 \leq i \leq k-1)$ *and* $k = (p+1)$*, then we obtain*

$$\det({}_{+}C_{k,n}) = \det({}_{-}C_{k,n}) = L_{p,n}(x,y)$$

where $L_{p,n}(x,y)$ *are the generalized bivariate Lucas p-polynomials.*

Corollary 2.9. *If we rewrite equalities* (2.3) *and* (2.4) *for* $t_1 = 0$ *and* $t_i = 1$ $(2 \leq i \leq k)$ *and* $k = 3$*, then we obtain*

$$\det({}_{+}C_{k,n}) = \det({}_{-}C_{k,n}) = r_n$$

where r_n *are the ordinary Perrin numbers.*

Now we show that determinants of Hessenberg matrices ${}_{-}C_{k,n}$ and ${}_{+}C_{k,n}$ can be obtained by using combinations.

Corollary 2.10.

$$\det({}_{+}C_{k,n}) = \det({}_{-}C_{k,n}) = \sum_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k}$$

Proof. It is obvious from Theorem 2.2, Theorem 2.4 and (1.1). □

3. The permanent representations

Let $A = (a_{i,j})$ be an $n \times n$ square matrix over a ring R. Then, it is well known that, the permanent of A is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where S_n denotes the symmetric group on n letters.

Lemma 3.1. [18] *Let* A_n *be an* $n \times n$ *lower Hessenberg matrix for all* $n \geq 1$ *and define per*$(A_0) = 1$*. Then, per*$(A_1) = a_{11}$ *and for* $n \geq 2$

$$per(A_n) = a_{n,n} per(A_{n-1}) + \sum_{r=1}^{n-1} (a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} per(A_{r-1})). \tag{3.1}$$

We define two Hessenberg matrices ${}_{-}H_{k,n}$ and ${}_{+}H_{k,n}$ whose permanents give the generalized Lucas polynomials.

Theorem 3.2. *Let $k \geq 2$ and $n \geq 1$ be integers, $G_{k,n}(t)$ the generalized Lucas polynomials and ${}_{-}H_{k,n} = (h_{rs})$ an $n \times n$ lower Hessenberg matrix, given by*

$$h_{rs} = \begin{cases} i^{(r-s)} . \frac{t_{r-s+1}}{t_2^{(r-s)}}, & \text{if } \; s \neq 1 \text{ and} -1 \leq r-s < k, \\ i^{(r-s)} . \frac{t_{r-s+1}}{t_2^{(r-s)}} .(r-s+1), & \text{if } \; s = 1 \text{ and } -1 \leq r-s < k, \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$${}_{-}H_{k,n} = \begin{bmatrix} t_1 & -it_2 & 0 & 0 & \cdots & 0 \\ 2i & t_1 & -it_2 & 0 & \cdots & 0 \\ 3i^2 \frac{t_3}{t_2^2} & i & t_1 & -it_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ ki^{k-1} \frac{t_k}{t_2^{k-1}} & i^{k-2} \frac{t_{k-1}}{t_2^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_2^{k-3}} & i^{k-4} \frac{t_{k-3}}{t_2^{k-4}} & \cdots & 0 \\ 0 & i^{k-1} \frac{t_k}{t_2^{k-1}} & i^{k-2} \frac{t_{k-1}}{t_2^{k-2}} & i^{k-3} \frac{t_{k-2}}{t_2^{k-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \cdots & t_1 \end{bmatrix}$$

where $t_0 = 1$ and $i = \sqrt{-1}$. Then,

$$per({}_{-}H_{k,n}) = G_{k,n}(t). \tag{3.2}$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 3.1. □

Example 3.3. *We obtain the 3-rd generalized Lucas polynomial for $k = 4$, by using* (3.2)

$$\begin{aligned} per({}_{-}H_{4,3}) &= per \begin{bmatrix} t_1 & -it_2 & 0 \\ 2i & t_1 & -it_2 \\ 3\frac{-t_3}{t_2} & i & t_1 \end{bmatrix} \\ &= t_1^3 + 3t_1t_2 + 3t_3. \end{aligned}$$

Theorem 3.4. *Let $k \geq 2$ and $n \geq 1$ be integers, $G_{k,n}(t)$ the generalized Lucas polynomials and ${}_{+}H_{k,n} = (p_{ij})$ an $n \times n$ lower Hessenberg matrix* *given by*

$$p_{ij} = \begin{cases} \frac{t_{i-j+1}}{t_2^{(i-j)}}, & if \;\; j \neq 1 \;\; and \; -1 \leq i-j < k, \\ \frac{t_{i-j+1}}{t_2^{(i-j)}}.(i-j+1), & if \;\; j = 1 \;\; and \; 0 \leq i-j < k, \\ 0, & otherwise \end{cases}$$

i.e.,

$$_{+}H_{k,n} = \begin{bmatrix} t_1 & t_2 & 0 & 0 & \cdots & 0 \\ 2 & t_1 & t_2 & 0 & \cdots & 0 \\ 3\frac{t_3}{t_2^2} & 1 & t_1 & t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ k\frac{t_k}{t_2^{k-1}} & \frac{t_{k-1}}{t_2^{k-2}} & \frac{t_{k-2}}{t_2^{k-3}} & \frac{t_{k-3}}{t_2^{k-4}} & \cdots & 0 \\ 0 & \frac{t_k}{t_2^{k-1}} & \frac{t_{k-1}}{t_2^{k-2}} & \frac{t_{k-2}}{t_2^{k-3}} & \cdots & 0 \\ & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \cdots & t_1 \end{bmatrix}$$

where $t_0 = 1$*. Then,*

$$per(_{+}H_{k,n}) = G_{k,n}(t). \tag{3.3}$$

Proof. The proof is similar to the proof of Theorem 2.2, by using Lemma 3.1. □

Corollary 3.5. *If we rewrite equalities* (3.2) *and* (3.3) *for* $t_i = 1$ $(1 \leq i \leq k)$ *and* $k = 2$*, then we obtain*

$$per(_{-}H_{k,n}) = per(_{+}H_{k,n}) = l_n$$

where l_n *are the ordinary Lucas numbers.*

Corollary 3.6. *If we rewrite equalities* (3.2) *and* (3.3) *for* $t_1 = 0$ *and* $t_i = 1$ $(2 \leq i \leq k)$*, then we obtain*

$$per(_{-}H_{k,n}) = per(_{+}H_{k,n}) = R_{k,n}(t)$$

where $R_{k,n}(t)$ *are the generalized Perrin polynomials.*

Corollary 3.7. *If we rewrite the right hand side of equalities* (3.2) *and* (3.3) *for* $t_1 = x$*,* $t_k = y$*,* $t_i = 0$ $(2 \leq i \leq k-1)$ *and* $k = (p+1)$*, then we obtain*

$$per(_{-}H_{k,n}) = per(_{+}H_{k,n}) = L_{p,n}(x, y)$$

where $L_{p,n}(x, y)$ *are the generalized bivariate Lucas p-polynomials.*

Corollary 3.8. *If we rewrite equalities* (3.2) *and* (3.3) *for* $t_1 = 0$, $t_i = 1$ $(2 \le i \le k)$ *and* $k = 3$, *then we obtain*

$$per({}_{-}H_{k,n}) = per({}_{+}H_{k,n}) = r_n$$

where r_n *are the ordinary Perrin numbers.*

Now we show that permanent of Hessenberg matrices ${}_{-}H_{k,n}$ and ${}_{+}H_{k,n}$ can be obtained by using combinations.

Corollary 3.9.

$$per({}_{-}H_{k,n}) = per({}_{+}H_{k,n}) = \sum_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_{1,\dots,}a_k} t_1^{a_1} \dots t_k^{a_k}$$

Proof. It is obvious from Theorem 3.2, Theorem 3.4 and (1.1). □

4. Determinant and permanent of a Hessenberg matrix

Gibson [3] gave an identity between permanent and determinant of a semitriangular matrix. We give a different proof of this identity for Hessenberg matrices.

Theorem 4.1. *Let* ${}_{+}A_n$ *be the Hessenberg matrix in* (2.1) *and*

${}_{-}A_n = (b_{ij})$ *an* $n \times n$ *Hessenberg matrix, given by*

$$b_{ij} = \begin{cases} 0, & \text{if } j - i > 1, \\ -a_{ij}, & \text{if } j - i = 1, \\ a_{ij}, & \text{otherwise} \end{cases}$$

i.e.,

$${}_{-}A_n = \begin{bmatrix} a_{11} & -a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & -a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & -a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}.$$

Then,

$$\det({}_{-}A_n) = per({}_{+}A_n) \text{ and } \det({}_{+}A_n) = per({}_{-}A_n). \tag{4.1}$$

Proof. To prove (4.1), we use the mathematical induction on m. The result is true for $m = 1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to m, namely $\det({}_{-}A_m)$ =$\mathrm{per}({}_{+}A_m)$. Then, by using (2.2) and (3.1), we have

$$\begin{aligned}
\det({}_{-}A_{m+1}) &= a_{m+1,m+1}\det({}_{-}A_m) \\
&\quad + \sum_{r=1}^{m}[(-1)^{m+1-r}a_{m+1,r}\prod_{j=r}^{m}b_{j,j+1}\det({}_{-}A_{r-1})] \\
&= a_{m+1,m+1}\mathrm{per}({}_{+}A_m) \\
&\quad + \sum_{r=1}^{m}[(-1)^{m+1-r}a_{m+1,r}\prod_{j=r}^{m}(-a_{j,j+1})\mathrm{per}({}_{+}A_{r-1})]
\end{aligned}$$

$$\begin{aligned}
&= a_{m+1,m+1}\mathrm{per}({}_{+}A_m) \\
&\quad + \sum_{r=1}^{m}[(-1)^{m+1-r}a_{m+1,r}(-1)^{m+1-r}\prod_{j=r}^{m}a_{j,j+1}\mathrm{per}({}_{+}A_{r-1})] \\
&= a_{m+1,m+1}\mathrm{per}({}_{+}A_m) + \sum_{r=1}^{m}[a_{m+1,r}\prod_{j=r}^{m+1}a_{j,j+1}\mathrm{per}({}_{+}A_{r-1})] \\
&= \mathrm{per}({}_{+}A_{m+1}).
\end{aligned}$$

Therefore, the result is true for all positive integers. □

Conclusion

Generalized Lucas polynomials are a general form of several polynomials and number sequences. Therefore any result obtained from these polynomials is applicable to the others. In addition, the relation between determinant and permanent of a Hessenberg matrix make it possible to transfer any result between them.

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K. Kaygısız
Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpaşa University, 60250 Tokat, Turkey
Email: kenan.kaygisiz@gop.edu.tr

A. Şahin
Department of Mathematics, Faculty of Education, Gaziosmanpaşa University, 60250 Tokat, Turkey
Email: adem.sahin@gop.edu.tr