

SOME CLASSES OF STRONGLY CLEAN RINGS

H. CHEN

Communicated by Fariborz Azarpanah

ABSTRACT. A ring R is a strongly clean ring if every element in R is the sum of an idempotent and a unit that commute. We construct some classes of strongly clean rings which have stable range one. It is shown that such cleanness of 2×2 matrices over commutative local rings is completely determined in terms of solvability of quadratic equations.

1. Introduction

Throughout, all rings are associative rings with identity. A ring R is strongly clean provided that for any $x \in R$ there exists an idempotent $e \in R$ such that $x - e \in U(R)$ and $ex = xe$. Many authors investigated strong cleanness of 2×2 matrices over local rings [1–5, 7] and [10]. Recall that a ring R has stable range one provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. A long standing question asks whether strongly clean rings have stable range one. So to check that strongly clean rings have stable range one, some subclasses of such rings are introduced. A ring R is strongly rad clean in case for any $x \in R$, there is an idempotent $e \in R$ such that $x - e \in U(R)$, $ex = xe$ and $ex \in J(R)$, where $J(R)$ is the Jacobson radical of R . A ring R is strongly π -rad clean in case for any $x \in R$, there exist an idempotent $e \in R$ and an $n \in \mathbb{N}$ such that $x - e \in U(R)$, $ex = xe$ and $(ex)^n \in J(R)$. For a fixed natural number n , we say that $x \in R$ is strongly J_n -clean provided

MSC(2010): Primary: 16E50; Secondary: 16U99.

Keywords: Strongly J_n -clean ring, 2×2 matrix, local ring.

Received: 20 September 2011, Accepted: 24 October 2012.

© 2013 Iranian Mathematical Society.

that there exists an idempotent $e \in R$ such that $x - e \in U(R)$, $ex = xe$ and $(ex)^n \in J(R)$. A ring R is strongly J_n -clean in case each element in R is strongly J_n -clean. Clearly, a ring R is strongly rad clean if and only if R is strongly J_1 -clean. For elementary properties of these rings, we refer the reader to [6] and [12].

The motivation of this article is to give a characterization of strongly J_n -clean rings by virtue of strongly π -regularity, and then construct many of such rings. Finally, we determine strong J_n -cleanness for 2×2 matrices over commutative local rings in terms of solvability of quadratic equations. We will show that strong J_n -cleanness ($n \geq 2$), strong J_2 -cleanness and strongly π -rad cleanness coincide for 2×2 matrices over local rings.

Throughout, $J(R)$ and $U(R)$ will denote, respectively, the Jacobson radical and the group of units in R . Furthermore $M_n(R)$ stands for the ring of all $n \times n$ matrices over a ring R . $tr(A)$ denotes the trace of $A \in M_n(R)$, $t_A = tr^2(A) - 4det(A)$, and \mathbb{N} is the set of all natural numbers.

2. Strong π -regularity

A ring R is strongly π -regular in case for any $x \in R$ there exists some $n \in \mathbb{N}$ such that $x^n \in x^{n+1}R$. The aim of this section is to characterize the strongly J_n -cleanness by means of the strong π -regularity.

Theorem 2.1. *Let R be a ring, and let n be a fixed natural number. Then the following are equivalent:*

- (1) R is strongly J_n -clean.
- (2) For any $x \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x^n \equiv eu \equiv ue \pmod{J(R)}$, $ex = xe$.

Proof. (1) \Rightarrow (2) For any $x \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$, $ex = xe$ and $ex^n \in J(R)$. Clearly, $x = ex + (1 - e)x$. This implies that $x^n = ex^n + (1 - e)x^n$. It is easy to verify that $(1 - e)x = (1 - e)u$, and so $(1 - e)x^n = (1 - e)u^n$. Let $f = 1 - e$ and $v = u^n$. Then $x^n \equiv fv \equiv vf \pmod{J(R)}$ and $fx = xf$, as required.

(2) \Rightarrow (1) Let $x \in R$. Then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x^n \equiv eu \equiv ue \pmod{J(R)}$, $ex = xe$. Let $f = 1 - e$

and $w = x - f$. Choose $v = x^{n-1}u^{-1}e - (1 + x + \cdots + x^{n-1})f$. Then

$$\begin{aligned} wv &= (x - f)(x^{n-1}u^{-1}e - (1 + x + \cdots + x^{n-1})f) \\ &= x^n u^{-1}e + (1 - x)(1 + x + \cdots + x^{n-1})f - f x^{n-1} u^{-1}e \\ &\equiv e + (1 - x^n)f + f x^{n-1} u^{-1} x^n u^{-1} \pmod{J(R)} \\ &\equiv e + (1 - x^n)f + f x^{n-1} e u^{-1} \pmod{J(R)} \\ &= e + f \\ &= 1, \end{aligned}$$

and so $w \in R$ is right invertible. Likewise, we get $vw = 1$, and so $w \in R$ is left invertible. Hence, $w \in U(R)$. Therefore, $x = f + w$ is a sum of an idempotent $f \in R$ and a unit $w \in R$, and that $fw = wf$. One easily checks that $fx^n \in J(R)$, as required. \square

For a fixed natural number n , a ring R is said to be n -regular provided that for any $x \in R$ there exists a $y \in R$ such that $x^n = x^n y x^n$. Clearly, $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ is 2-regular, while it is not regular. Let $S = T_n(\mathbb{Z}_2)$, the ring of all $n \times n$ upper triangular matrix over \mathbb{Z}_2 . Then S is n -regular.

Let $R = \left\{ \begin{pmatrix} A_i & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \mid A_i \text{ is an } i \times i \text{ upper triangular matrix over } \mathbb{Z}_2, i = 1, 2, 3, \dots \right\}$. Then R is π -regular, while it is not n -regular for all $n \in \mathbb{N}$. Thus,

$$\{ \text{regular rings} \} \subsetneq \{ n\text{-regular rings} \} \subsetneq \{ \pi\text{-regular rings} \}$$

for all $n \geq 2$.

Corollary 2.2. *Let R be commutative, and let n be a fixed natural number. Then the following are equivalent:*

- (1) R is strongly J_n -clean.
- (2) $R/J(R)$ is n -regular and every idempotent lifts modulo $J(R)$.

Proof. (1) \Rightarrow (2) Since R is strongly J_n -clean, it is clean. In view of [11, Theorem 30.2], R is an exchange ring. Hence, every idempotent lifts modulo $J(R)$, by [11, Theorem 29.2]. Let $x \in R$. According to Theorem 2.1, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x^n \equiv eu \equiv ue \pmod{J(R)}$, $ex = xe$. This implies that $x^n \equiv ex^n \equiv u^{-1}x^{2n} \pmod{J(R)}$. Therefore, $x^n u^{-1} x^n \equiv x^n e \equiv (ue)e = ue \equiv x^n \pmod{J(R)}$, and so $R/J(R)$ is n -regular, as required.

(2) \Rightarrow (1) As every idempotent lifts modulo $J(R)$, $R/J(R)$ is commutative. For any $x \in R$, there exists a $y \in R$ such that $x^n \equiv x^n y x^n \pmod{J(R)}$. Set $z = y x^n y$. Then $x^n \equiv x^n z x^n \pmod{J(R)}$ and $z \equiv z x^n z \pmod{J(R)}$. Set $u = z + 1 - x^n z$. Since R is commutative, it is easy to verify that

$$(x^n + 1 - x^n z)u \equiv u(x^n + 1 - x^n z) \equiv 1 \pmod{J(R)}.$$

Hence, $u \in U(R)$. Let $e = x^n z$. Then we can find an idempotent $f \in R$ such that $e \equiv f \pmod{J(R)}$. In addition, $x^n \equiv f u \equiv u f \pmod{J(R)}$. Further, $f x = x f$. According to Theorem 2.1, R is strongly J_n -clean. \square

In view of [11, Remark 29.7], all commutative n -regular rings are exchange rings; hence, it follows from [11, Theorem 29.2] and Corollary 2.2 that every commutative n -regular ring is strongly J_n -clean. Obviously, \mathbb{Z}_4 is 2-regular, while it is not regular. Also we see that \mathbb{Z}_8 is 3-regular, while it is not 2-regular. In light of Proposition 2.4, \mathbb{Z}_4 is strongly J_2 -clean and \mathbb{Z}_8 is strongly J_3 -clean. Let $R = \mathbb{F}[x]/(x^2) = \{a + bt \mid a, b \in \mathbb{F}, t^2 = 0\}$, where \mathbb{F} is a field of characteristic 2. Then for any $a \in R$, there exists a $b \in R$ such that $a^2 = a^3 b$. Hence, $a^2 = a^2 b^2 a^2$. That is, R is commutative 2-regular. Thus, R is strongly J_2 -clean. In this case, R is not commutative regular.

Recall that a ring R is of bounded index n provided that $x^n = 0$ for any nilpotent $x \in R$.

Lemma 2.3. *Let R be a strongly π -regular ring of bounded index n . Then for any $x \in R$, there exists \mathbb{N} such that $x^n R = x^{n+1} R$.*

Proof. Let $x \in R$. In light of [4, Proposition 13.1.18], there exist $e = e^2 \in R$, $u \in U(R)$ and a nilpotent $w \in R$ such that $x = eu + w$ and e, u, w commute. It is easy to verify that $x - x^2 u^{-1} = (w + eu) - (w + eu)^2 u^{-1} = w - w(2e + wu^{-1})$. By hypothesis, $w^n = 0$, and so $(x - x^2 u^{-1})^n = 0$. This implies that $x^n \in x^{n+1} R$, as required. \square

As is well known, a ring R is strongly π -regular if and only if for any $x \in R$ there exist $y \in R$ and $n \in \mathbb{N}$ such that $x^n = x^{2n} y$, $y = y^2 x$ and $xy = yx$. By a similar method of the proof of this fact, we now derive the following.

Theorem 2.4. *Every strongly π -regular ring of bounded index n is strongly J_n -clean.*

Proof. Let R be a strongly π -regular ring of bounded index n , and let $x \in R$. According to Lemma 2.3, we have some $y \in R$ such that $x^n = x^{n+1}y$, and then $x^n = x^{n+i}y^i$ for any $i \in \mathbb{N}$. Also we have some $z \in R$ such that $y^n = y^{n+1}z$; hence, $y^n = y^{n+j}z^j$ for any $j \in \mathbb{N}$. Furthermore, we get $x^n = x^{2n}y^n$. As in the proof of [4, Proposition 13.15], we get $x^n \in Rx^{n+1}$.

Write $x^n = x^{n+1}y = rx^{n+1}$. Then $x^ny = rx^{n+1}y = rx^n$. By induction, we get $x^ny^k = r^kx^n$ for any $k \in \mathbb{N}$. Let $t = x^ny^{n+1}$. Then $t = r^{n+1}x^n$. It is easy to verify that $xt = x^{n+1}y^{n+1} = (x^{n+1}y)y^n = x^ny^n = r^n x^n = r^n(rx^{n+1}) = (r^{n+1}x^n)x = tx$; $x^{n+1}t = x^{2n+1}y^{n+1} = x^n(x^{n+1}y)y^n = x^{2n}y^n = \dots = x^{n+1}y = x^n$; $t^2x = t(xt) = t(x^{n+1}y^{n+1}) = x^{n+1}ty^{n+1} = x^ny^{n+1} = t$. Let $s = t^n$. Then $x^n = x^{2n}s$, $xs = sx$, $s = s^2x^n$. Set $u = s + 1 - x^n s$. Then $x^nu = x^ns + x^n(1 - x^n s) = x^ns$ is an idempotent. In addition, $u^{-1} = x^n + 1 - x^n s$. Let $e = x^ns$. Then $x^n = eu^{-1} = u^{-1}e$ and $ex = xe$. Accordingly, the result follows by Theorem 2.1. \square

As a consequence, we deduce that every finite ring R is strongly J_n -clean for all $n \geq |R|$. In view of [8, Theorem 7.15] and Theorem 2.4, every regular ring of bounded index n is strongly J_n -clean. Further, we show that every right P -exchange ring of bounded index n is strongly J_n -clean [4, Corollary 15.3.8].

By the decompositions of 2×2 matrices, Ying considered the strong J_2 -cleanness of 2×2 matrix ring over a division ring [12, Theorem 3.3.10]. As a consequence of Theorem 2.4, we extend Ying's result to the general case.

Example 2.5. *Let D be a division ring. Then $M_n(D)$ is strongly J_n -clean, while it is not strongly J_{n-1} -clean.*

Proof. In view of [8, Theorem 7.2], $M_n(D)$ is a strongly π -regular ring of bounded index n . Therefore, $M_n(D)$ is strongly J_n -clean from Theorem 2.4. By a similar method [12, Proposition 3.3.9], one can check the

next assertion. Choose $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(D)$. Then A

is not strongly J_{n-1} -clean. Otherwise, we have some $E = E^2 \in M_n(D)$ and $U \in GL_n(D)$ such that $EA = AE$ and $(EA)^{n-1} \in J(M_n(R))$. Write $E = (e_{ij})$. As $EA^{n-1} = A^{n-1}E$, we get $e_{ni} = 0$ ($i = 1, \dots, n-1$).

As $(EA)^{n-1} \in J(M_n(R))$, we get $e_{nn} = 0$. So every entry at the last row of U are zero, a contradiction. Accordingly, $M_n(D)$ is not strongly J_{n-1} -clean. \square

Example 2.6. Let D be a division ring, and let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$. Then R is strongly J_n -clean for all $n \in \mathbb{N}$.

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$ is invertible. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$. Accordingly, R is strongly J_n -clean for all $n \in \mathbb{N}$. \square

3. Examples

Let R be a ring, and let $a \in R$. Let $\ell(a) = \{r \in R \mid ra = 0\}$ and $r(a) = \{r \in R \mid ar = 0\}$. We begin this section with considering the strong J_n -cleanness in corner rings.

Lemma 3.1. Let R be a ring, and let $a = e + w$ be a strongly J_n -clean decomposition of a in R . Then $\ell(a) \subseteq \ell(e)$ and $r(a) \subseteq r(e)$.

Proof. Let $r \in \ell(a)$. Then $ra = 0$. Write $a = e + w, e = e^2, w \in U(R), ew = we$ and $ew^n \in J(R)$. Hence, $ew^{2n+1} \in J(R)$. We observe that $1 + ew \in U(R)$. Thus, $re = -rw$, and so $re = -rwe = -rew$. It follows that $re(1 + ew) = 0$, and then $re = 0$. That is, $r \in \ell(e)$. Therefore, $\ell(a) \subseteq \ell(e)$. A similar argument shows that $r(a) \subseteq r(e)$. \square

Theorem 3.2. Let R be a ring, let n be a fixed natural number, and let $f \in R$ be an idempotent. Then $a \in fRf$ is strongly J_n -clean in R if and only if a is strongly J_n -clean in fRf .

Proof. Suppose $a = e - w, e = e^2 \in fRf, w \in U(fRf), ew = we$ and $ea^n \in J(fRf)$. Hence, $a = (e+1-f) - (1-f+w) \in fRf$ is strongly clean in R . In addition, $(e+1-f)a^n = ea^n \in J(R)$. Clearly, $(e+1-f)a = ea = ae = a(e+1-f)$, and thus, $a \in fRf$ is strongly J_n -clean in R .

Conversely, suppose that $a = e + w, e = e^2 \in R, w \in U(R), ew = we$ and $ew^n \in J(R)$. As $a \in fRf$, it follows from Lemma 3.1 that

$$1-f \in \ell(a) \cap r(a) \subseteq \ell(e) \cap r(e) = R(1-e) \cap (1-e)R = (1-e)R(1-e).$$

Hence, $ef = e = fe$. We observe that $a = fef + fwf, (fef)^2 = fef, fwf = a - fef = a - e = w \in U(fRf)$. Furthermore, $fef \cdot$

$fwf = fewf = fewf = fwf \cdot fef$. In addition, $fef(fwf)^n = few^n f \in fJ(R)f \subseteq J(fRf)$, as required. \square

As a consequence, every corner of a strongly J_n -clean ring is strongly J_n -clean.

Example 3.3. Let D be a division ring, and let $R = \bigoplus_{i=1}^{\infty} M_i(D)$. Then R is strongly π -rad clean, while R is not strongly J_n -clean for all $n \in \mathbb{N}$.

Proof. Let $x \in R$. Write $x = (x_1, x_2, \dots, x_m, 0, 0, \dots) \in R$. In view of Corollary 2.2, all $x_i \in M_i(D)$ ($i = 1, \dots, m$) are strongly J_m -clean. Therefore we conclude that $x \in R$ is strongly J_m -clean; hence, it is strongly π -rad clean. If R is strongly J_n -clean, then so is $M_{n+1}(D)$ by Theorem 3.2. This contradicts the result of Corollary 2.2, and we are done. \square

For a commutative ring R , R is strongly J_n -clean if and only if R is strongly π -rad clean. But such rings are very different in the noncommutative case. From Example 3.3, we see that

$$\begin{aligned} \{\text{strongly rad clean}\} &= \{\text{strongly } J_1\text{-clean rings}\} \\ &\subsetneq \{\text{strongly } J_2\text{-clean rings}\} \subsetneq \{\text{strongly } J_3\text{-clean rings}\} \\ &\subsetneq \dots \subsetneq \{\text{strongly } \pi\text{-rad clean rings}\}. \end{aligned}$$

We say that B is a subring of a ring A provided that B is a non-empty subset of A such that for any $x, y \in B$, $x - y, xy \in B$ and that $1_A \in B$. Let B be a subring of A . Set $R[A, B] = \{(a_1, a_2, \dots, a_n, b, b, \dots) \mid a_i \in A, b \in B, n \geq 1\}$. Then $R[A, B]$ is a ring under the general addition and multiplication.

Theorem 3.4. Let B be a subring of a ring A , and let n be a fixed natural number. Then $R[A, B]$ is strongly J_n -clean if and only if

- (1) A is strongly J_n -clean.
- (2) For any $x \in B$, there exists an idempotent $e \in B$ such that $x - e \in U(B)$, $ex = xe$ and $(ex)^n \in J(A) \cap J(B)$.

Proof. Suppose $R[A, B]$ is strongly J_n -clean. Then A is strongly J_n -clean by Theorem 3.2. For any $x \in B$, we see that $(0, x, x, \dots) \in R[A, B]$. By hypothesis, there exist an idempotent $(f_1, \dots, f_s, f, f, \dots) \in R[A, B]$ and a unit $(v_1, \dots, v_t, v, v, \dots) \in R[A, B]$ such that

$$(0, x, x, \dots) = (f_1, \dots, f_s, f, f, \dots) + (v_1, \dots, v_t, v, v, \dots)$$

is a strongly clean expression and $(f_1, \dots, f_s, f, f, \dots)(0, x, x, \dots)^n \in J(R[A, B])$. This implies that $x = f + v$ is a strongly clean expression in B . One easily checks that $J(R[A, B]) = [J(A), J(A) \cap J(B)]$. As a result, we see that $fx^n \in J(A) \cap J(B)$, as required.

Conversely, assume that (1) and (2) hold. Let $(a_1, \dots, a_m, a, a, \dots) \in R[A, B]$. Then we have strongly clean expressions $a_1 = e_1 + u_1, \dots, a_m = e_m + u_m, a = e + u$, and that $e_1 a_1^n, \dots, e_m a_m^n \in J(A), ea^n \in J(A) \cap J(B)$. Thus, we have a strongly clean decomposition $(a_1, \dots, a_m, a, a, \dots) = (e_1, \dots, e_m, e, e, \dots) + (u_1, \dots, u_m, u, u, \dots)$. In addition, $(e_1, \dots, e_m, e, e, \dots)(a_1, \dots, a_m, a, a, \dots)^n \in R[J(A), J(A) \cap J(B)]$. Therefore $R[A, B]$ is strongly J_n -clean. \square

Example 3.5. Let D be a division ring, and let $A = M_2(D)$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$. Then $R[A, B]$ is strongly J_2 -clean.

Proof. In view of Example 2.5, A is strongly J_2 -clean. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in J(A) \cap J(B)$. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ is a strongly clean expression. In addition, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 \in J(A) \cap J(B)$. According to Theorem 3.4, $R[A, B]$ is strongly J_2 -clean. In this case, $R[A, B]$ is not strongly J_1 -clean. \square

Example 3.6. Let D be a division ring, and let $A = M_3(D)$, $B = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in D \right\}$. Then $R[A, B]$ is strongly J_3 -clean.

Proof. In view of Example 2.5, A is strongly J_3 -clean. Let $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is

a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in$

$J(A) \cap J(B)$. If $a = 0$,

then $\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b & c \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a strongly

clean expression. In addition, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^3 \in J(A) \cap J(B)$.

By virtue of Theorem 3.4, $R[A, B]$ is strongly J_3 -clean. In this case, $R[A, B]$ is not strongly J_2 -clean. \square

4. Matrices over local rings

Let $a \in R$. Let $l_a : R \rightarrow R$ and $r_a : R \rightarrow R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. Following Diesl, a local ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b, l_b - r_a$ are both surjective. Let $T_2(R)$ denote the ring of all upper triangular 2×2 matrices over R . For J_n -cleanness of $T_2(R)$, we can derive the following.

Proposition 4.1. *Let R be a local ring, and let n be a fixed natural number. Then the following are equivalent:*

- (1) $T_2(R)$ is strongly J_n -clean.
- (2) $T_2(R)$ is strongly J_1 -clean.
- (3) $T_2(R)$ is strongly π -rad clean.
- (4) R is bleached.

Proof. (1) \Rightarrow (3) is trivial.

(3) \Rightarrow (4) Since $T_2(R)$ is strongly π -rad clean, it follows from [6, Theorem 4.2.4] that R is bleached, as desired.

(4) \Rightarrow (2) As R is bleached, it follows from [6, Theorem 4.2.4] that $T_2(R)$ is strongly rad clean, and so R is strongly J_1 -clean.

(2) \Rightarrow (1) is trivial. \square

Let R be a commutative local ring. Then $T_2(R)$ is strongly J_n -clean for all $n \in \mathbb{N}$. But it is hard to determine the strongly J_n -cleanness for

general case. In the next, we discuss the strongly J_n -cleanness for full matrices.

Lemma 4.2. *Let R be a local ring, and let $E \in M_2(R)$ be idempotent. Then $E \in M_2(R)$ is similar to an idempotent matrix.*

Proof. In view of [4, Lemma 16.4.11], $E \in GL_2(R)$ or $I_2 - E \in GL_2(R)$ or there exists some $P \in GL_2(R)$ such that $P^{-1}EP = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. If $E \in GL_2(R)$, then $E = I_2$. If $I_2 - E \in GL_2(R)$, then $E = 0$. If there exists some $P \in GL_2(R)$ such that $P^{-1}EP = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, then $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \in M_2(R)$ is idempotent; hence, $\lambda = 0$ and $\mu = 1$. One easily checks that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, E is similar to an idempotent matrix. \square

Lemma 4.3. *Let $a \in R, u \in U(R)$, and let n be a fixed natural number. Then the following are equivalent:*

- (1) $a \in R$ is strongly J_n -clean.
- (2) $uau^{-1} \in R$ is strongly J_n -clean.

Proof. (1) \Rightarrow (2) Write $a = e + w, e = e^2, w \in U(R), ew = we$ and $ea^n \in J(R)$. Then $uau^{-1} = ueu^{-1} + uwu^{-1}$. Clearly, $ueu^{-1} \in R$ is an idempotent and $uwu^{-1} \in U(R)$. In addition, $(ueu^{-1})(uwu^{-1}) = (uwu^{-1})(ueu^{-1})$. Further, $(ueu^{-1})(uau^{-1})^n = uea^n u^{-1} \in J(R)$. Therefore, $uau^{-1} \in R$ is strongly J_n -clean.

(2) \Rightarrow (1) is proved in the same manner. \square

Theorem 4.4. *Let R be a local ring, and let n be a fixed natural number. Then $A \in M_2(R)$ is strongly J_n -clean if and only if $A \in GL_2(R)$ or $A^n \in J(M_2(R))$ or A admits a diagonal reduction.*

Proof. If $A \in GL_2(R)$, then $A = 0 + A$ is a strongly J_n -clean decomposition. If $A^n \in J(M_2(R))$, then $A = I_2 + (A - I_2)$ is a strongly J_n -clean decomposition. Suppose A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If

$\alpha \in J(R), \beta \in U(R)$, then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}$$

is a strongly J_1 -clean decomposition. In view of Lemma 4.3, A is strongly J_1 -clean. If $\alpha \in U(R), \beta \in J(R)$, similarly, A is strongly J_1 -clean. If $\alpha, \beta \in U(R)$, then $A \in GL_2(R)$. If $\alpha, \beta \in J(R)$, then $A \in J(M_2(R))$, and so A is strongly J_1 -clean. Thus one direction is proved.

Conversely, assume that $A \in M_2(R)$ is strongly J_n -clean. Then there exists an idempotent $E \in M_2(R)$ and a $W \in GL_2(R)$ such that $A = E + W, EW = WE$ and $EA^n \in J(M_2(R))$. According to Lemma 4.2, there exists $K \in GL_2(R)$ such that $KEK^{-1} = \text{diag}(f_1, f_2)$, where $f_1, f_2 \in R$ are idempotents. As R is local, every idempotent in R is 0 or 1. If $f_1 = f_2 = 0$, then $A \in GL_2(R)$. If $f_1 = f_2 = 1$, then $E = I_2$; hence, $A^n \in J(M_2(R))$. So we may assume that $f_1 = 1, f_2 = 0$ or $f_1 = 0, f_2 = 1$. Thus, there exists some $H \in GL_2(R)$ such that $HEH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Hence

$$HAH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + HWH^{-1}.$$

Set $V = (v_{ij}) := HWH^{-1}$. It follows from $EW = WE$ that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in U(R)$. We conclude that $HAH^{-1} = \begin{pmatrix} 1 + v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, and therefore A admits a diagonal reduction. \square

Corollary 4.5. *Let R be a local ring, and let $n \geq 2$ be a fixed natural number. Then the following are equivalent:*

- (1) $A \in M_2(R)$ is strongly J_n -clean.
- (2) $A \in M_2(R)$ is strongly J_2 -clean.
- (3) $A \in M_2(R)$ is strongly π -rad clean.

Proof. (1) \Rightarrow (3) is trivial.

(3) \Rightarrow (2) Suppose $A \in M_2(R)$ is strongly π -rad clean. Then there exists some $m \in \mathbb{N}$ such that $A \in M_2(R)$ is strongly J_m -clean. By virtue of Theorem 4.4, $A \in GL_2(R)$ or $A^n \in J(M_2(R))$ or A admits a diagonal reduction. If $A^n \in J(M_2(R))$, then $\bar{A}^n = \bar{0}$ in $M_2(R/J(R))$. As $R/J(R)$ is a division ring, it follows by [8, Theorem 7.2] that $M_2(R/J(R))$ is a

regular ring of bounded index 2. Hence, $\overline{A}^2 = \overline{0}$, and so $A^2 \in J(M_2(R))$. By using Theorem 4.4 again, $A \in M_2(R)$ is strongly J_2 -clean.

(2) \Rightarrow (1) is obvious. \square

Corollary 4.6. *Let R be a commutative local ring. Then the following are equivalent:*

- (1) $M_2(R)$ is strongly clean.
- (2) $M_2(R)$ is strongly J_2 -clean.
- (3) $M_2(R)$ is strongly π -rad clean.

Proof. (1) \Rightarrow (3) is clear by [6, Corollary 4.3.7].

(3) \Rightarrow (2) is obvious from Corollary 4.5.

(2) \Rightarrow (1) is trivial. \square

We naturally ask whether $M_2(R)$ is strongly clean if and only if $M_2(R)$ is strongly J_1 -clean? The answer is negative. For instance, letting $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$, one easily checks that A is not strongly J_1 -clean, and so $M_2(\mathbb{Z}_2)$ is not strongly J_1 -clean, but $M_2(\mathbb{Z}_2)$ is strongly clean.

Corollary 4.7. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly clean if and only if $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$ or $A \in M_2(R)$ is strongly J_1 -clean.*

Proof. One direction is obvious. Conversely, assume that $A \in M_2(R)$ is strongly clean. If $A \notin GL_2(R)$ or $I_2 - A \notin GL_2(R)$, it follows by [4, Corollary 16.4.7] that there exists some $U \in GL_2(R)$ such that $UAU^{-1} = \text{diag}(1 + \alpha, \beta)$, where $\alpha, \beta \in U(R)$. Thus, $A \in M_2(R)$ is strongly J_1 -clean by Theorem 4.4. \square

Recall that $A \in M_2(R)$ is purely singular provided that $A, I_2 - A \notin GL_2(R)$.

Corollary 4.8. *Let R be a commutative local ring, and let $A \in M_2(R)$ be purely singular. Then the following are equivalent:*

- (1) $A \in M_2(R)$ is strongly J_1 -clean.
- (2) $A \in M_2(R)$ is strongly J_2 -clean.
- (3) $A \in M_2(R)$ is strongly clean.

Proof. (1) \Rightarrow (2) \Rightarrow (3) are trivial.

(3) \Rightarrow (2) Since $A \in M_2(R)$ is purely singular, we complete the proof by Corollary 4.7. \square

Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{N}, 2 \nmid n\}$. Then $\begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is strongly J_1 -clean for all $n \in \mathbb{N}$, by Corollary 4.8.

Let $\chi(A)$ denote the characteristic polynomial of a matrix $A \in M_2(R)$, i.e., $\chi(A) = x^2 - \text{tr}(A)x + \det(A)$. Furthermore, we can derive the following.

Corollary 4.9. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly J_2 -clean if and only if $\det(A) \in U(R)$, or $\chi(A) \equiv t^2 \pmod{J(R)}$, or A admits a diagonal reduction.*

Proof. Suppose $A \in M_2(R)$ is strongly J_2 -clean. In light of Theorem 4.4, $A \in GL_2(R)$ or $A^2 \in J(M_2(R))$ or A admits a diagonal reduction. If $A^2 \in J(M_2(R))$, then $\bar{A} \in M_2(R/J(R))$ is nilpotent. Therefore, $\chi(\bar{A}) \equiv t^2 \pmod{N(R/J(R))}$. Write $\chi(A) = t^2 + at + b$. Then there exists some $m \in \mathbb{N}$ such that $a^m \in J(R)$. As $J(R)$ is maximal, it is prime. Hence, $a \in J(R)$. Likewise, $b \in J(R)$. This shows that $\chi(A) \equiv t^2 \pmod{J(R)}$, as required.

Conversely, assume that $\det(A) \in U(R)$, or $\chi(A) \equiv t^2 \pmod{J(R)}$, or A admits a diagonal reduction. If $\chi(A) \equiv t^2 \pmod{J(R)}$, then $\chi(\bar{A}) \equiv t^2 \pmod{R/J(R)}$. This implies that \bar{A} is nilpotent in $M_2(R/J(R))$. Therefore, $A^m \in J(M_2(R))$. In light of Theorem 4.4, $A \in M_2(R)$ is strongly J_m -clean, and so it is strongly π -rad clean. Accordingly, $A \in M_2(R)$ is strongly J_2 -clean, by Corollary 4.6. \square

5. Criteria via quadratic equations

Many authors characterized strong cleanness by means of quadratic equations. The purpose of this section is to study the strongly J_n -cleanness of a single 2×2 matrix over commutative local rings. It is shown that such property can be characterized by a kind of quadratic equations. We begin this section with two elementary lemmas.

Lemma 5.1. *Let R be a commutative local ring. Then $A \in M_2(R)$ is an idempotent if and only if $A = 0$ or $A = I_2$ or $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$ in R .*

Proof. See [4, Lemma 16.4.10]. \square

Lemma 5.2. *Let R be a commutative ring, and let $A = (a_{ij}) \in M_2(R)$. If $a_{21} \in U(R)$ and $\chi(A)$ has two roots $x_1, x_2 \in R$ such that $x_1 - x_2 \in U(R)$, then A is similar to $\text{diag}(x_1, x_2)$.*

Proof. See [4, Lemma 16.4.30]. \square

Lemma 5.3. *Let R be a commutative local ring, let n be a fixed natural number, and let $A = (a_{ij}) \in M_2(R)$ be strongly J_n -clean. Then $A \in GL_2(R)$ or $A^n \in J(M_2(R))$, or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$.*

Proof. If $A \notin GL_2(R)$ and $A^n \notin J(M_2(R))$. It follows from Theorem 4.4 that there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$. Thus, $B_{21}(1)P^{-1}APB_{21}(-1) = \begin{pmatrix} \alpha & 0 \\ \alpha - \beta & \beta \end{pmatrix}$. As $\alpha - \beta \in U(R)$, it is easy to check that

$$\begin{aligned} & B_{12}(-\alpha(\alpha - \beta)^{-1})B_{21}(1)P^{-1}APB_{21}(-1)B_{12}(\alpha(\alpha - \beta)^{-1}) \\ &= \begin{pmatrix} 0 & -\alpha(\alpha - \beta)^{-1}\beta \\ \alpha - \beta & (\alpha - \beta)\alpha(\alpha - \beta)^{-1} + \beta \end{pmatrix}. \end{aligned}$$

This implies that

$$\begin{aligned} & [\alpha - \beta, 1]B_{12}(-\alpha(\alpha - \beta)^{-1})B_{21}(1)P^{-1}APB_{21}(-1) \\ & B_{12}(\alpha(\alpha - \beta)^{-1})[(\alpha - \beta)^{-1}, 1] \\ &= \begin{pmatrix} 0 & -(\alpha - \beta)\alpha(\alpha - \beta)^{-1}\beta \\ 1 & (\alpha - \beta)\alpha(\alpha - \beta)^{-1} + \beta \end{pmatrix}. \end{aligned}$$

Let $\lambda = -(\alpha - \beta)\alpha(\alpha - \beta)^{-1}\beta$ and $\mu = (\alpha - \beta)\alpha(\alpha - \beta)^{-1} + \beta$.

As $\alpha \in J(R)$, we see that $\beta - (\beta - \alpha)\alpha(\alpha - \beta)^{-1} \in U(R)$, i.e., $\mu \in U(R)$.

Therefore A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. \square

Theorem 5.4. *Let R be a commutative local ring, and let $n \in \mathbb{N}$ be a fixed natural number. Then the following are equivalent:*

- (1) $A \in M_2(R)$ is strongly J_n -clean.
- (2) $A \in GL_2(R)$ or $A^n \in J(M_2(R))$, or $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$.

Proof. (1) \Rightarrow (2) Let $A \in M_2(R)$ be strongly J_n -clean. Assume that $A \notin GL_2(R)$ and $A^n \notin J(M_2(R))$. In view of Theorem 4.4, A is similar

to the matrix $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_2(R)$, where $\lambda^n \in J(R), \mu \in U(R)$.

Thus,

$$\chi(A) = \det(xI_2 - A) = \det(xI_2 - B) = (x - \alpha)(x - \beta),$$

and so $\chi(A) = 0$ has a root α and a root β . As $J(R)$ is maximal, it is prime. Hence, $\alpha \in J(R)$, as required.

(2) \Rightarrow (1) Let $A \in M_2(R)$. If $A \in GL_2(R)$ or $A^n \in J(M_2(R))$, then $A \in M_2(R)$ is strongly J_n -clean. Otherwise, it follows by the hypothesis that the equation $\chi(A) = 0$ has a root $x_1 \in J(R)$ and a root $x_2 \in U(R)$. Clearly, $x_1 - x_2 \in U(R)$.

Case I. $a_{21} \in U(R)$. It follows from Lemma 5.2 that there exists a $P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ for some $\alpha_1 \in J(R), \alpha_2 \in U(R)$. Thus,

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 - 1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

is a strongly J_n -clean expression. Therefore, $A \in M_2(R)$ is strongly J_n -clean by Lemma 4.3.

Case II. $a_{12} \in U(R)$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} \in M_2(R).$$

Applying Case I and Lemma 4.3, we see that $A \in M_2(R)$ is strongly J_1 -clean.

Case III. $a_{12}, a_{21}, a_{11} \in J(R)$. Clearly, $\text{tr}(A) = x_1 + x_2$ and $\det(A) = x_1x_2$. As $\text{tr}(A) \in U(R), a_{22} \in U(R)$. Obviously,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} - a_{11} - a_{12} & a_{22} - a_{12} \end{pmatrix} \in M_2(R),$$

where $a_{21} + a_{22} - a_{11} - a_{12} \in U(R)$. Applying Case I and Lemma 4.3, $A \in M_2(R)$ is strongly J_1 -clean.

Case IV. $a_{12}, a_{21} \in J(R), a_{11} \in U(R)$. Clearly, $\det(A) = x_1x_2 \in J(R)$. This implies that $a_{22} \notin U(R)$, and so $a_{22} \in J(R)$. Obviously,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} & a_{12} \\ a_{21} + a_{22} - a_{11} - a_{12} & a_{22} - a_{12} \end{pmatrix} \in M_2(R),$$

where $a_{21} + a_{22} - a_{11} - a_{12} \in U(R)$. Applying Case I and Lemma 4.3, $A \in M_2(R)$ is strongly J_1 -clean.

In any case, $A \in M_2(R)$ is strongly J_n -clean, as asserted. \square

Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. In light of Theorem 5.4, we claim that $A \in M_2(\mathbb{Z}_{(2)})$ is strongly J_2 -clean, but it is not strongly J_1 -clean. Clearly, $A^2 \in J(M_2(\mathbb{Z}_{(2)}))$, and so $A \in M_2(\mathbb{Z}_{(2)})$ is strongly J_2 -clean. But $A \notin GL_2(\mathbb{Z}_{(2)})$ and $A \notin J(M_2(\mathbb{Z}_{(2)}))$. On the other hand, χ_A has no root in $J(\mathbb{Z}_{(2)})$. Accordingly, $A \in M_2(\mathbb{Z}_{(2)})$ is not strongly J_1 -clean, by Theorem 5.4. Furthermore, we can derive the following.

Corollary 5.5. *Let R be a commutative local ring, let $n \in \mathbb{N}$ be a fixed natural number, and let $A \in M_2(R)$. Then the following are equivalent:*

- (1) $A \in M_2(R)$ is strongly J_n -clean.
- (2) $A \in GL_2(R)$ or $A^n \in J(M_2(R))$, or the equation $x^2 - x = t_A^{-1} \det(A)$ is a root in $J(R)$.

Corollary 5.6. *Let R be a commutative local ring, let $n \in \mathbb{N}$ be a fixed natural number, and let $A \in M_2(R)$. If $\frac{1}{2} \in R$, then the following are equivalent:*

- (1) $A \in M_2(R)$ is strongly J_n -clean.
- (2) $A \in GL_2(R)$ or $A^n \in J(M_2(R))$, or $\text{tr}^2(A) - 4\det(A) = u^2$ for some $u \in U(R)$.

Acknowledgments

The research of the author was supported by the Natural Science Foundation of Zhejiang Province (LY13A010019).

REFERENCES

- [1] D. D. Anderson and V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra* **30** (2002), no. 7, 3327–3336.
- [2] G. Borooh, A. J. Diesl and T. J. Dorsey, Strongly clean matrix rings over commutative local rings, *J. Pure Appl. Algebra* **212** (2008), no. 1, 281–296.
- [3] H. Chen, On strongly J -clean rings, *Comm. Algebra* **38** (2010), no. 10, 3790–3804.
- [4] H. Chen, Rings Related Stable Range Conditions, Series in Algebra 11. Hackensack, NJ: World Scientific, 2011.
- [5] H. Chen, On uniquely clean rings, *Comm. Algebra* **39** (2011), no. 1, 189–198.
- [6] A. J. Diesl, Classes of Strongly Clean Rings, Ph.D. Thesis, University of California, Berkeley, University of California, Berkeley, 2006.
- [7] L. Fan and X. Yang, A note on strongly clean matrix rings, *Comm. Algebra* **38** (2010), no. 3, 799–806.

- [8] K. R. Goodearl, Von Neumann Regular Rings, Monographs and Studies in Mathematics, 4, Pitman (Advanced Publishing Program), London, 1979.
- [9] W. K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra* **27** (1999), no. 8, 3583–3592.
- [10] W. K. Nicholson, Clean rings: A Survey, *Advances in Ring Theory*, 181–198, World Sci. Publ., Hackensack, 2005.
- [11] A. A. Tuganbaev, Rings Close to Regular, Kluwer Academic Publishers, Dordrecht, 2002.
- [12] Z. L. Ying, Clean Rings and Regular Local Rings, Ph.D. Thesis, Southeast University, Nanjing, 2008.

Huanyin Chen

Department of Mathematics, Hangzhou Normal University, 310036, Hangzhou, China

Email: huanyinchen@yahoo.cn

Archive of SID