

## CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

J. WU AND Z. WU \*

Communicated by Javad Mashreghi

ABSTRACT. In this paper, some results of Singh, Gopalakrishna and Kulkarni (1970s) have been extended to higher order derivatives. It has been shown that, if  $\sum_a \Theta(a, f) = 2$  holds for a meromorphic function  $f(z)$  of finite order, then for any positive integer  $k$ ,  $T(r, f) \sim T(r, f^{(k)})$ ,  $r \rightarrow \infty$  if  $\Theta(\infty, f) = 1$  and  $T(r, f^{(k)}) \sim (k+1)T(r, f)$ ,  $r \rightarrow \infty$  if  $\Theta(\infty, f) = 0$ .

### 1. Introduction

Let  $f(z)$  be a meromorphic function in the complex plane  $\mathbb{C}$ . Assume that basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [3], [10] or [12]) are known. We use the following notations of frequent use of value distribution (see [3]) with their usual meaning:

$$m(r, f), N(r, a), \bar{N}(r, a), \delta(a, f), \Theta(a, f), \dots$$

As usual, if  $a = \infty$ , we write  $N(r, \infty) = N(r, f)$ ,  $\bar{N}(r, \infty) = \bar{N}(r, f)$ . We denote by  $S(r, f)$  any quantity such that

$$S(r, f) = o(T(r, f)), \quad r \rightarrow +\infty$$

---

MSC(2010): Primary: 30D30; Secondary: 30D35.

Keywords: Characteristic function, Nevanlinna's deficiency, maximum deficiency sum.

Received: 20 July 2011, Accepted: 21 February 2012.

\*Corresponding author

© 2013 Iranian Mathematical Society.

without restriction if  $f(z)$  is of finite order and otherwise except possibly for a set of values of  $r$  of finite linear measure. The well known Nevanlinna's deficiency relation states that

$$\sum_a \delta(a, f) \leq \sum_a \Theta(a, f) \leq 2.$$

If  $\sum_a \delta(a, f) = 2$ , then we say that  $f(z)$  has maximum deficiency sum (see [2]).

Let  $f(z)$  be a meromorphic scalar valued function in  $\mathbb{C}$ . On the characteristic function of derivative of  $f(z)$  with maximum deficiency sum has been studied by Shan, Singh, Gopalakrishna, Edrei and Weitsman [1], [4]-[8]. For example, Edrei [1] and Weitsman [8] have proved

**THEOREM A** Let  $f(z)$  be a transcendental meromorphic function of finite order and assume  $\sum_{a \in \mathbb{C}} \delta(a, f) = \eta \geq 1$  and  $\delta(\infty) = 2 - \eta$ . Then

$$T(r, f') \sim \eta T(r, f), r \rightarrow +\infty.$$

If  $\sum_a \delta(a, f) = 2$  is replaced by  $\sum_a \Theta(a, f) = 2$ , Singh, Gopalakrishna [6] and Singh, Kulkarni [7] have proved

**THEOREM B** Let  $f(z)$  be a transcendental meromorphic scalar valued function of finite order and assume  $\sum_a \Theta(a, f) = 2$ . Then

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty).$$

Hence

- (1) if  $\Theta(\infty, f) = 1$ ,  $T(r, f) \sim T(r, f')$  as  $r \rightarrow \infty$ ;
- (2) if  $\Theta(\infty, f) = 0$ ,  $T(r, f') \sim 2T(r, f)$  as  $r \rightarrow \infty$ .

We extend the above result to higher order derivatives as follows:

**Theorem 1.1.** Suppose that  $f$  is a transcendental meromorphic function of finite order and  $\sum_a \Theta(a, f) = 2$ . Then for any positive integer  $k$ , we have

- (1) if  $\Theta(\infty, f) = 1$ ,  $T(r, f) \sim T(r, f^{(k)})$  as  $r \rightarrow \infty$ ;

(2) if  $\Theta(\infty, f) = 0$ ,  $T(r, f^{(k)}) \sim (k+1)T(r, f)$  as  $r \rightarrow \infty$ .

From Theorem 1.1, we can get

**Corollary 1.2.** [11] Suppose that  $f$  is a transcendental meromorphic function of finite order and  $\sum_a \delta(a, f) = 2$ . Then for any positive integer  $k$ , we have

(1) if  $\delta(\infty, f) = 1$ ,  $T(r, f) \sim T(r, f^{(k)})$  as  $r \rightarrow \infty$ ;

(2) if  $\delta(\infty, f) = 0$ ,  $T(r, f^{(k)}) \sim (k+1)T(r, f)$  as  $r \rightarrow \infty$ .

## 2. Proof of Theorem 1.1

*Proof.* (1) We prove Theorem 1.1 (1) by induction. Since  $\Theta(\infty, f) = 1$ , by Theorem B, we have  $T(r, f) \sim T(r, f')$  as  $r \rightarrow \infty$ . Assume that

$$(2.1) \quad T(r, f) \sim T(r, f^{(k)}), r \rightarrow \infty.$$

Now we prove  $T(r, f) \sim T(r, f^{(k+1)})$  as  $r \rightarrow \infty$ .

Without loss of generality we can assume that  $q \geq 2$ . Put

$$F(z) = \sum_{i=1}^q \frac{1}{f(z) - a_i}, \quad a_i \in \mathbb{C}.$$

Then (See [6])

$$\sum_{i=1}^q m(r, a_i) \leq m(r, F) + O(1).$$

So

$$\begin{aligned} \sum_{i=1}^q m(r, a_i) &\leq m(r, F) + O(1) \\ &= m\left(r, \frac{1}{f^{(k+1)}} F f^{(k+1)}\right) + O(1) \\ &\leq m\left(r, \frac{1}{f^{(k+1)}}\right) + m\left(r, \sum_{i=1}^q \frac{f^{(k+1)}}{f(z) - a_i}\right) + O(1) \\ &= m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \end{aligned}$$

Hence

$$\begin{aligned}
 qT(r, f) &\leq \sum_{i=1}^q N(r, a_i) + m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\
 &= \sum_{i=1}^q N(r, a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\
 &\leq \sum_{i=1}^q N(r, a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
 &= T\left(r, f^{(k+1)}\right) + \sum_{i=1}^q \bar{N}(r, a_i) - N_0\left(r, \frac{1}{f'}\right) + S(r, f).
 \end{aligned}$$

where  $N_0\left(r, \frac{1}{f'}\right)$  is formed with the zeros of  $f'$  which are not zeros of any of the  $f - a_i, i = 1, 2, \dots, q$ . Since  $N_0\left(r, \frac{1}{f'}\right) \geq 0$ , we have

$$qT(r, f) \leq T\left(r, f^{(k+1)}\right) + \sum_{i=1}^q \bar{N}(r, a_i) + S(r, f).$$

Thus

$$\sum_{i=1}^q \left(1 - \frac{\bar{N}(r, a_i)}{T(r, f)}\right) \leq \frac{T\left(r, f^{(k+1)}\right)}{T(r, f)} + \frac{S(r, f)}{T(r, f)}.$$

So

$$\sum_{i=1}^q \Theta(a_i, f) \leq \liminf_{r \rightarrow \infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f)},$$

holds for any  $q \geq 2$ . Letting  $q \rightarrow \infty$ , we obtain

$$(2.2) \quad 1 = \sum_{a \neq \infty} \Theta(a, f) \leq \liminf_{r \rightarrow \infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f)}$$

Combining (2.1) and (2.2) we have

$$(2.3) \quad 1 \leq \liminf_{r \rightarrow \infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})} \leq \limsup_{r \rightarrow \infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})}.$$

On the other hand, since  $\overline{N}(r, f^{(k)}) = \overline{N}(r, f)$ ,  $\Theta(\infty, f) = 1$  and (2.1), we have

$$\limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f^{(k)})}{T(r, f^{(k)})} \leq \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)} = 0.$$

So

$$\Theta(\infty, f^{(k)}) = 1.$$

Thus

$$\begin{aligned} T(r, f^{(k+1)}) &= m(r, f^{(k+1)}) + N(r, f^{(k+1)}) \\ &\leq m(r, f^{(k)}) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + N(r, f^{(k)}) + \overline{N}(r, f^{(k)}) \\ &= T(r, f^{(k)}) + \overline{N}(r, f^{(k)}) + S(r, f). \end{aligned}$$

Hence

$$(2.4) \quad \limsup_{r \rightarrow \infty} \frac{T(r, f^{(k+1)})}{T(r, f^{(k)})} \leq 2 - \Theta(\infty, f^{(k)}) = 1.$$

(2.1) and (2.3)-(2.4) together imply  $T(r, f) \sim T(r, f^{(k+1)})$  as  $r \rightarrow \infty$ .

We can prove (2) of Theorem 1.1 by using the same method as that in [11]. As [11] may not be abundantly available, we give the following proof. From Nevanlinna's second fundamental theorem, we have

$$(q-1)T(r, f) \leq T(r, f) + \sum_{i=1}^q \overline{N}(r, a_i) + \overline{N}(r, f) + S(r, f).$$

Thus

$$\sum_{i=1}^q \Theta(a_i, f) \leq 1 + \liminf_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2.$$

Letting  $q \rightarrow \infty$ , we obtain

$$2 = \sum_{a \neq \infty} \Theta(a, f) \leq 1 + \liminf_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2$$

So

$$(2.5) \quad T(r, f) \sim N(r, f) \sim \overline{N}(r, f), r \rightarrow \infty.$$

Since

$$\begin{aligned}
 (k+1)\overline{N}(r, f) &\leq N(r, f) + k\overline{N}(r, f) = N\left(r, f^{(k)}\right) \\
 &\leq T\left(r, f^{(k)}\right) \\
 &\leq m(r, f) + m\left(r, \frac{f^{(k)}}{f}\right) + N\left(r, f^{(k)}\right) \\
 &= T(r, f) + k\overline{N}(r, f) + S(r, f).
 \end{aligned}$$

From this and (2.5), we get  $T(r, f^{(k)}) \sim (k+1)T(r, f)$  as  $r \rightarrow \infty$ .  $\square$

### 3. Proof of Corollary 1.2

*Proof.* Since  $\delta(a, f) \leq \Theta(a, f)$  for every  $a \in \mathbb{C} \cup \{\infty\}$ , if  $\sum_a \delta(a, f) = 2$ , then  $\sum_a \Theta(a, f) = 2$  and  $\delta(a, f) = \Theta(a, f)$  for every  $a \in \mathbb{C} \cup \{\infty\}$ . Hence Corollary 1.2 follows by Theorem 1.1.  $\square$

### Acknowledgments

This research was partially supported by the NNSF of China (Grant No. 11201395) and by NSF of Educational Department of the Hubei Province (Grant No. Q20132801, D20132804).

### REFERENCES

- [1] A. Edrei, Sums of deficiencies of meromorphic functions II, *J. Analyse Math.* **19** (1967) 53–74.
- [2] M. L. Fang, A note on a result of Singh and Kulkarni, *Int. J. Math. Math. Sci.* **23** (2000) 285–288.
- [3] W. K. Hayman, Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
- [4] S. M. Shah and S. K. Singh, Borel's theorem on  $a$ -points and exceptional values of entire and meromorphic functions, *Math. Z.* **59** (1953) 88–93.
- [5] S. M. Shah and S. K. Singh, On the derivative of a meromorphic function with maximum defect, *Math. Z.* **65** (1956) 171–174.
- [6] S. K. Singh and H. S. Gopalakrishna, Exceptional values of entire and meromorphic functions, *Math. Ann.* **191** (1971) 121–142.
- [7] S. K. Singh and V. N. Kulkarni, Characteristic function of a meromorphic function and its derivative, *Ann. Polon. Math.* **28** (1973) 123–133.

- [8] A. Weitsman, Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna, *Acta Math.* **123** (1969) 115–139.
- [9] H. Wittich, *Neuere Untersuchungen Über eindeutige Analytische Funktionen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.
- [10] L. Yang, *Value Distribution Theory and its New Research*, Science Press Beijing, Beijing, 1993.
- [11] L. Z. Yang, Sum of deficiencies of meromorphic function and its characteristic function, *J. Shandong Univ. Nat. Sci.* **24** (1989) 7–11.
- [12] J. H. Zheng, *Value Distribution of Meromorphic Functions*, Tsinghua University Press Beijing, Beijing; Springer, Heidelberg, 2010.

**Jia Wu**

Xianning Vocational and Technical College, P.O. Box 437100, Xianning, P. R. China  
Email: 44976882@qq.com

**Zhaojun Wu**

School of Mathematics and Statistics, Hubei University of Science and Technology,  
P.O. Box 437100, Xianning, P. R. China  
Email: wuzj52@hotmail.com

Archive of SID