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CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

J. WU AND Z. WU *

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Abstract. In this paper, some results of Singh, Gopalakrishna and Kulkarni (1970s) have been extended to higher order derivatives. It has been shown that, if $\sum_a \Theta(a,f) = 2$ holds for a meromorphic function $f(z)$ of finite order, then for any positive integer k, $T(r,f) \sim T(r,f^{(k)}), r \to \infty$ if $\Theta(\infty,f) = 1$ and $T(r,f^{(k)}) \sim (k+1)T(r,f), r \to \infty$ if $\Theta(\infty,f) = 0$.

1. Introduction

Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. Assume that basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [3], [10] or [12]) are known. We use the following notations of frequent use of value distribution (see [3]) with their usual meaning:

$$m(r,f), N(r,a), \overline{N}(r,a), \delta(a,f), \Theta(a,f), \cdots$$

As usual, if $a = \infty$, we write $N(r,\infty) = N(r,f), \overline{N}(r,\infty) = \overline{N}(r,f)$. We denote by $S(r,f)$ any quantity such that

$$S(r,f) = o(T(r,f)), \quad r \to +\infty$$

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∗Corresponding author
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without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure. The well known Nevanlinna's deficiency relation states that

$$\sum_a \delta(a,f) \le \sum_a \Theta(a,f) \le 2.$$

If $\sum\limits_a \delta(a,f) = 2$, then we say that $f(z)$ has maximum deficiency sum (see [2]).

Let $f(z)$ be a meromorphic scalar valued function in $\mathbb{C}$. On the characteristic function of derivative of $f(z)$ with maximum deficiency sum has been studied by Shan, Singh, Gopalakrishna, Edrei and Weitsman [1], [4]-[8]. For example, Edrei [1] and Weitsman [8] have proved

Theorem A Let $f(z)$ be a transcendental meromorphic function of finite order and assume $\sum\limits_{a\in\mathbb{C}} \delta(a,f) = \eta \ge 1$ and $\delta(\infty) = 2 - \eta$. Then

$$T(r,f') \sim \eta T(r,f), r \to +\infty.$$

If $\sum\limits_a \delta(a,f) = 2$ is replaced by $\sum\limits_a \Theta(a,f) = 2$, Singh, Gopalakrishna [6] and Singh, Kulkarni [7] have proved

Theorem B Let $f(z)$ be a transcendental meromorphic scalar valued function of finite order and assume $\sum\limits_a \Theta(a,f) = 2$. Then

$$\lim_{r\to+\infty} \frac{T(r,f')}{T(r,f)} = 2 - \Theta(\infty).$$

Hence

(1) if $\Theta(\infty,f) = 1$, $T(r,f) \sim T(r,f')$ as $r \to \infty$;

(2) if $\Theta(\infty,f) = 0$, $T(r,f') \sim 2T(r,f)$ as $r \to \infty$.

We extend the above result to higher order derivatives as follows:

Theorem 1.1. *Suppose that f is a transcendental meromorphic function of finite order and $\sum\limits_a \Theta(a,f) = 2$. Then for any positive integer k, we have*

(1) *if $\Theta(\infty,f) = 1$, $T(r,f) \sim T(r,f^{(k)})$ as $r \to \infty$;*

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(2) *if* $\Theta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k+1)T(r, f)$ *as* $r \to \infty$.

From Theorem 1.1, we can get

Corollary 1.2. [11] *Suppose that* f *is a transcendental meromorphic function of finite order and* $\sum_a \delta(a, f) = 2$. *Then for any positive integer* k, *we have*

(1) *if* $\delta(\infty, f) = 1$, $T(r, f) \sim T\left(r, f^{(k)}\right)$ *as* $r \to \infty$;

(2) *if* $\delta(\infty, f) = 0$, $T(r, f^{(k)}) \sim (k+1)T(r, f)$ *as* $r \to \infty$.

2. Proof of Theorem 1.1

Proof. (1) We prove Theorem 1.1 (1) by induction. Since $\Theta(\infty, f) = 1$, by Theorem B, we have $T(r, f) \sim T(r, f')$ as $r \to \infty$. Assume that

$$T(r, f) \sim T(r, f^{(k)}), r \to \infty. \tag{2.1}$$

Now we prove $T(r, f) \sim T(r, f^{(k+1)})$ as $r \to \infty$.

Without loss of generality we can assume that $q \geq 2$. Put

$$F(z) = \sum_{i=1}^{q} \frac{1}{f(z) - a_i}, \quad a_i \in \mathbb{C}.$$

Then (See [6])

$$\sum_{i=1}^{q} m(r, a_i) \leq m(r, F) + O(1).$$

So

$$\begin{aligned}
\sum_{i=1}^{q} m(r, a_i) &\leq m(r, F) + O(1) \\
&= m\left(r, \frac{1}{f^{(k+1)}} F f^{(k+1)}\right) + O(1) \\
&\leq m\left(r, \frac{1}{f^{(k+1)}}\right) + m\left(r, \sum_{i=1}^{q} \frac{f^{(k+1)}}{f(z) - a_i}\right) + O(1) \\
&= m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).
\end{aligned}$$

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Hence

$$\begin{aligned} qT(r,f) &\leq \sum_{i=1}^{q} N(r,a_i) + m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &= \sum_{i=1}^{q} N(r,a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \sum_{i=1}^{q} N(r,a_i) + T\left(r, f^{(k+1)}\right) - N\left(r, \frac{1}{f'}\right) + S(r,f) \\ &= T\left(r, f^{(k+1)}\right) + \sum_{i=1}^{q} \overline{N}(r,a_i) - N_0\left(r, \frac{1}{f'}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f'}\right)$ is formed with the zeros of f' which are not zeros of any of the $f - a_i, i = 1, 2, \cdots, q$. Since $N_0\left(r, \frac{1}{f'}\right) \geq 0$, we have

$$qT(r,f) \leq T\left(r, f^{(k+1)}\right) + \sum_{i=1}^{q} \overline{N}(r,a_i) + S(r,f).$$

Thus

$$\sum_{i=1}^{q}\left(1 - \frac{\overline{N}(r,a_i)}{T(r,f)}\right) \leq \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

So

$$\sum_{i=1}^{q} \Theta(a_i, f) \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)},$$

holds for any $q \geq 2$. Letting $q \to \infty$, we obtain

$$1 = \sum_{a\neq\infty} \Theta(a,f) \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r,f)} \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$1 \leq \liminf_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T(r, f^{(k)})}. \tag{2.3}$$

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On the other hand, since $\overline{N}\left(r, f^{(k)}\right) = \overline{N}(r, f), \Theta(\infty, f) = 1$ and (2.1), we have

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r, f^{(k)}\right)}{T\left(r, f^{(k)}\right)} \leq \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 0.$$

So

$$\Theta\left(\infty, f^{(k)}\right) = 1.$$

Thus

$$\begin{aligned} T\left(r, f^{(k+1)}\right) = & \ m\left(r, f^{(k+1)}\right) + N\left(r, f^{(k+1)}\right) \\ \leq & \ m\left(r, f^{(k)}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + N\left(r, f^{(k)}\right) + \overline{N}\left(r, f^{(k)}\right) \\ = & \ T\left(r, f^{(k)}\right) + \overline{N}\left(r, f^{(k)}\right) + S(r, f). \end{aligned}$$

Hence

$$\limsup_{r\to\infty} \frac{T\left(r, f^{(k+1)}\right)}{T\left(r, f^{(k)}\right)} \leq 2 - \Theta(\infty, f^{(k)}) = 1. \tag{2.4}$$

(2.1) and (2.3)-(2.4) together imply $T(r, f) \sim T\left(r, f^{(k+1)}\right)$ as $r \to \infty$.

We can prove (2) of Theorem 1.1 by using the same method as that in [11]. As [11] may not be abundantly available, we give the following proof. From Nevanlinna's second fundamental theorem, we have

$$(q-1)T(r, f) \leq T(r, f) + \sum_{i=1}^{q} \overline{N}(r, a_i) + \overline{N}(r, f) + S(r, f).$$

Thus

$$\sum_{i=1}^{q} \Theta(a_i, f) \leq 1 + \liminf_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2.$$

Letting $q \to \infty$, we obtain

$$2 = \sum_{a\neq\infty} \Theta(a, f) \leq 1 + \liminf_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 1 + \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2$$

So

$$T(r, f) \sim N(r, f) \sim \overline{N}(r, f), r \to \infty. \tag{2.5}$$

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Since

$$\begin{aligned}(k+1)\overline{N}(r,f) &\leq N(r,f)+k\overline{N}(r,f)=N\left(r,f^{(k)}\right)\\ &\leq T\left(r,f^{(k)}\right)\\ &\leq m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+N\left(r,f^{(k)}\right)\\ &= T(r,f)+k\overline{N}(r,f)+S(r,f).\end{aligned}$$

From this and (2.5), we get $T(r,f^{(k)}) \sim (k+1)T(r,f)$ as $r \to \infty$. □

3. Proof of Corollary 1.2

Proof. Since $\delta(a,f) \leq \Theta(a,f)$ for every $a \in \mathbb{C} \cup \{\infty\}$, if $\sum_a \delta(a,f) = 2$, then $\sum_a \Theta(a,f) = 2$ and $\delta(a,f) = \Theta(a,f)$ for every $a \in \mathbb{C} \cup \{\infty\}$. Hence Corollary 1.2 follows by Theorem 1.1. □

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Jia Wu
Xianning Vocational and Technical College, P.O. Box 437100, Xianning, P. R. China
Email: `44976882@qq.com`

Zhaojun Wu
School of Mathematics and Statistics, Hubei University of Science and Technology, P.O. Box 437100, Xianning, P. R. China
Email: `wuzj52@hotmail.com`

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