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COMMON FIXED POINTS OF A FINITE FAMILY OF MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we introduce a one-step iterative scheme for finding a common fixed point of a finite family of multivalued quasi-nonexpansive mappings in a real uniformly convex Banach space. We establish weak and strong convergence theorems of the proposed iterative scheme under some appropriate conditions.

1. Introduction

Let X be a real Banach space. A subset K of X is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K),$$

where $d(x,K) := \inf\{\|x-y\| : y \in K\}$. It is clear that every closed convex subset of a uniformly convex Banach space is proximinal. We denote by $C(X)$, $P(X)$ and $CB(X)$ the collection of all nonempty compact subsets of X, nonempty proximinal bounded subsets and nonempty closed bounded subsets of X, respectively. The *Hausdorff metric* on $CB(X)$ is

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defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad \forall A,B \in CB(X),$$

where $d(x,B) = \inf\{\|x-y\| : y \in B\}$ is the distance from the point x to the set B. An element $p \in K$ is called a *fixed point* of a single valued or multivalued mapping T of K into itself if $p = Tp$ or $p \in Tp$, respectively. The set of all fixed points of T is denoted by $F(T)$. Let K be a nonempty closed convex subset of a real Banach space X and let $CB(K)$ be a family of nonempty closed bounded subsets of K. A single valued mapping $T : K \to K$ is said to be *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$. A multivalued mapping $T : K \to CB(K)$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$. The multivalued mapping $T : K \to CB(K)$ is called *nonexpansive* if $H(Tx, Ty) \leq \|x-y\|$ for all $x, y \in K$. It is well-known that every nonexpansive multivalued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there is a quasi-nonexpansive mapping which is not nonexpansive. It is also known that if T is a quasi-nonexpansive multivalued mapping, then $F(T)$ is closed.

In 1969, Nadler [6] combined the ideas of multivalued mapping and Lipschitz mapping and proved some fixed point theorems for multivalued contraction mappings. These results place no severe restrictions on the images of points and all that is required of the space is that it is a complete metric space.

In 1997, Hu et al. [5] obtained a common fixed point of two nonexpansive multivalued mappings satisfying certain contractive condition.

In 2005, Sastry and Babu [10] extended the convergence results from single valued mappings to multivalued mappings by defining Ishikawa and Mann iterates for multivalued mappings with a fixed point. They also gave an example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0.

In 2007, Panyanak [8] generalized results of Sastry and Babu [10] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain.

Later in 2008, Song and Wang [15] proved strong convergence theorems of Mann and Ishikawa iterates for multivalued nonexpansive mappings under some appropriate control conditions. Furthermore, they also gave an affirmative answer to Panyanak's open question in [8].

In 2009, Shahzad and Zegeye [16] proved some strong convergence theorems of the Ishikawa iterative scheme for a quasi-nonexpansive multivalued mapping T. They also relaxed compactness of the domain of T and constructed an iterative scheme which removes the restriction of T, namely, $Tp = \{p\}$ for any $p \in F(T)$.

On the other hand, Song and Cho [11] proved strong convergence theorems of the Halpern type iteration for a multivalued nonexpansive mapping T with $Tp = \{p\}$ for any $p \in F(T)$ in a reflexive Banach space with weakly sequentially continuous duality mapping. Later in 2010, Hussain, Amini-Harandi and Cho [4] proved existence of approximate fixed points and approximate endpoints of the multivalued almost I-contractions.

In 2011, Song and Cho [12] modified and improved the proofs of the main results given by Shahzad and Zegeye [16]. They also proved strong convergence theorems of Ishikawa iterative scheme for a multivalued mapping with P_T quasi-nonexpansive.

Next, Abbas et al. [2] introduced a new one-step iterative process for approximating a common fixed point of two multivalued nonexpansive mappings in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let $S, T : K \to CB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \quad n \in \mathbb{N} \end{cases}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ whenever p is a fixed point of any one of the mappings S and T, and $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences of numbers in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$.

In this paper, we generalize and modify the iteration of Abbas et al. [2] from two mappings to a finite family of multivalued quasi-nonexpansive mappings $\{T_i : i = 1, 2, \ldots, m\}$ in a real uniformly convex Banach space.

For finite multivalued quasi-nonexpansive mapping T_i and $x_1 \in K$, we define

$$x_{n+1} = a_{n,0}x_n + a_{n,1}x_{n,1} + a_{n,2}x_{n,2} + \ldots + a_{n,m}x_{n,m}, \tag{1.1}$$

where the sequences $\{a_{n,i}\} \subset [0, 1)$ satisfies $\sum_{i=0}^{m} a_{n,i} = 1$ and let $x_{n,i} \in T_i x_n$ such that $d(p, x_{n,i}) = d(p, T_i x_n)$ for all $i = 1, 2, \ldots, m$ and $p \in \cap_{i=1}^{m} F(T_i)$. The main purpose of this paper is to prove weak and strong convergence of the iterative scheme (1.1) to a common fixed point of $\{T_i : i = 1, 2, \ldots, m\}$.

A Banach space X is said to satisfy *Opial's property* [7] if for each $x \in X$ and each sequence $\{x_n\}$ weakly converging to x, the following condition holds for all $y \neq x$:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 1.1. [13] *Let X be a Banach space which satisfies Opial's property and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 1.2. [9] *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$. Then, $\limsup_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Main results

We first prove that the sequence $\{x_n\}$ generated by (1.1) is an approximating fixed point sequence of each T_i $(i = 1, 2, \ldots, m)$.

Theorem 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\{T_i : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.1). Then*

(1) $\lim_{n\to\infty} \|x_n - x_{n,i}\| = 0$ *for all* $i = 1, 2, \ldots, m$,
(2) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ *for all* $i = 1, 2, \ldots, m$.

Proof. First, we show that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.
Let $p \in F$. By (1.1) and quasi-nonexpansiveness of T_i, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq a_{n,0}\|x_n - p\| + a_{n,1}\|x_{n,1} - p\| + a_{n,2}\|x_{n,2} - p\| + \ldots \\
&\quad + a_{n,m}\|x_{n,m} - p\| \\
&= a_{n,0}\|x_n - p\| + a_{n,1}d(T_1x_n, p) + a_{n,2}d(T_2x_n, p) + \ldots \\
&\quad + a_{n,m}d(T_mx_n, p) \\
&\leq a_{n,0}\|x_n - p\| + a_{n,1}H(T_1x_n, T_1p) + a_{n,2}H(T_2x_n, T_2p) + \ldots \\
&\quad + a_{n,m}H(T_mx_n, T_mp) \\
&\leq a_{n,0}\|x_n - p\| + a_{n,1}\|x_n - p\| + a_{n,2}\|x_n - p\| + \ldots \\
&\quad + a_{n,m}\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.

Next, we show that $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$ for all $i = 1, 2, \ldots, m$.
Suppose that $\lim_{n\to\infty}\|x_n - p\| = c$ for some $c \geq 0$. Then

$$\begin{aligned}
\lim_{n\to\infty}\|x_{n+1} - p\| &= \lim_{n\to\infty}\|a_{n,0}(x_n - p) + a_{n,1}(x_{n,1} - p) + a_{n,2}(x_{n,2} - p) \\
&\quad + \ldots + a_{n,m}(x_{n,m} - p)\| \\
&= \lim_{n\to\infty}\Bigg\|(1 - a_{n,m})\Bigg[\frac{a_{n,0}}{1 - a_{n,m}}(x_n - p) + \frac{a_{n,1}}{1 - a_{n,m}}(x_{n,1} - p) \\
&\quad + \frac{a_{n,2}}{1 - a_{n,m}}(x_{n,2} - p) + \ldots + \frac{a_{n,m-1}}{1 - a_{n,m}}(x_{n,m-1} - p)\Bigg] \\
&\quad + a_{n,m}(x_{n,m} - p)\Bigg\| \\
&= c.
\end{aligned}$$

By quasi-nonexpansiveness of each T_i, we have $\|x_{n,i} - p\| = d(T_ix_n, p) \leq H(T_ix_n, T_ip) \leq \|x_n - p\|$ for each $p \in F$ and $i = 1, 2, \ldots, m$.
Taking $\limsup$ on both sides, we get

$$\limsup_{n\to\infty}\|x_{n,i} - p\| \leq \limsup_{n\to\infty}\|x_n - p\| = c$$

for all $i = 1, 2, \ldots, m$. We also have

$$
\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{a_{n,0}}{1-a_{n,m}}(x_n-p)+\frac{a_{n,1}}{1-a_{n,m}}(x_{n,1}-p)+\ldots+\frac{a_{n,m-1}}{1-a_{n,m}}(x_{n,m-1}-p)\right\|\\
&\le\limsup_{n\to\infty}\left[\frac{a_{n,0}}{1-a_{n,m}}\|x_n-p\|+\frac{a_{n,1}}{1-a_{n,m}}\|x_{n,1}-p\|+\ldots+\right.\\
&\quad\left.\frac{a_{n,m-1}}{1-a_{n,m}}\|x_{n,m-1}-p\|\right]\\
&\le\limsup_{n\to\infty}\left[\frac{a_{n,0}}{1-a_{n,m}}\|x_n-p\|+\frac{a_{n,1}}{1-a_{n,m}}\|x_n-p\|+\ldots+\right.\\
&\quad\left.\frac{a_{n,m-1}}{1-a_{n,m}}\|x_n-p\|\right]\\
&=\limsup_{n\to\infty}\left(\frac{a_{n,0}+a_{n,1}+\ldots+a_{n,m-1}}{1-a_{n,m}}\right)\|x_n-p\|\\
&=\limsup_{n\to\infty}\|x_n-p\|\\
&=c.
\end{aligned}
$$

It follows from Lemma 1.2 that

$$
\begin{aligned}
\lim_{n\to\infty}\Bigg\|&\frac{a_{n,0}}{1-a_{n,m}}(x_n-p)+\frac{a_{n,1}}{1-a_{n,m}}(x_{n,1}-p)+\ldots+\\
&\frac{a_{n,m-1}}{1-a_{n,m}}(x_{n,m-1}-p)-(x_{n,m}-p)\Bigg\|=0.
\end{aligned}
$$

This yields

$$
\begin{aligned}
0&=\lim_{n\to\infty}\left\|\frac{a_{n,0}}{1-a_{n,m}}x_n+\frac{a_{n,1}}{1-a_{n,m}}x_{n,1}+\ldots+\frac{a_{n,m-1}}{1-a_{n,m}}x_{n,m-1}-x_{n,m}\right\|\\
&=\lim_{n\to\infty}\left(\frac{1}{1-a_{n,m}}\right)\|a_{n,0}x_n+a_{n,1}x_{n,1}+\ldots+a_{n,m-1}x_{n,m-1}\\
&\quad-(1-a_{n,m})x_{n,m}\|\\
&=\lim_{n\to\infty}\left(\frac{1}{1-a_{n,m}}\right)\|a_{n,0}x_n+a_{n,1}x_{n,1}+\ldots+a_{n,m-1}x_{n,m-1}\\
&\quad+a_{n,m}x_{n,m}-x_{n,m}\|\\
&=\lim_{n\to\infty}\left(\frac{1}{1-a_{n,m}}\right)\|x_{n+1}-x_{n,m}\|.
\end{aligned}
$$

It implies that $\lim_{n\to\infty}\|x_{n+1}-x_{n,m}\|=0$. In the same way, we can show that $\lim_{n\to\infty}\|x_{n+1}-x_{n,i}\|=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ for all $i=1,2,\ldots,m-1$. Since $\|x_n-x_{n,i}\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-x_{n,i}\|$, we obtain $\lim_{n\to\infty}\|x_n - x_{n,i}\| = 0$ for all $i = 1, 2, \ldots, m$. Since $d(x_n, T_i x_n) \le \|x_n - x_{n,i}\|$, we get $d(x_n, T_i x_n) \to 0$ as $n \to \infty$ for all $i = 1, 2, \ldots, m$. $\square$

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial's property. Let $\{T_i : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive and continuous mappings from K into $C(K)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a common fixed point of $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof. From Theorem 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i = 1, 2, \ldots, m$. Hence $\{x_n\}$ is bounded. Since X is uniformly convex, by passing to a subsequence we can assume that $x_n \rightharpoonup q$ as $n \to \infty$ for some $q \in K$.

First, we show that $q \in T_1 q$.

Since $T_1 q$ is compact, for each $n \ge 1$, we can choose $y_n \in T_1 q$ such that $\|x_n - y_n\| = d(x_n, T_1 q)$ and the sequence $\{y_n\}$ has a convergent subsequence $\{z_n\}$ with $\lim_{n\to\infty} z_n = z \in T_1 q$. Suppose that $z \neq q$. Then

$$\begin{aligned}
\limsup_{n\to\infty}\|x_n - z\| &\le \limsup_{n\to\infty}\|x_n - z_n\| + \limsup_{n\to\infty}\|z_n - z\| \\
&= \limsup_{n\to\infty}\|x_n - z_n\| \\
&= \limsup_{n\to\infty} d(x_n, T_1 q) \\
&\le \limsup_{n\to\infty} d(x_n, T_1 x_n) + \limsup_{n\to\infty} H(T_1 x_n, T_1 q) \\
&\le \limsup_{n\to\infty}\|x_n - q\| \\
&< \limsup_{n\to\infty}\|x_n - z\|,
\end{aligned}$$

which is a contradiction and hence $z = q \in T_1 q$. Similarly, we can show that $q \in T_i q$ for all $i = 2, 3, \ldots, m$.

It follows by Lemma 1.1 that $\{x_n\}$ has a unique weakly subsequential limit in F. $\square$

Theorem 2.3. *Let X be a real Banach space and K a closed convex subset of X. Let $\{T_i : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. Conversely, assume that

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.1}$$

From the proof of Theorem 2.1, we get $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $p \in F$. Hence $d(x_{n+1}, F) \le d(x_n, F)$. Thus, $\lim_{n\to\infty} d(x_n, F)$ exists. By our hypothesis, we get $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence in K.
Let ϵ be arbitrary. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists n_0 such that for all $n \ge n_0$, $d(x_n, F) < \frac{\epsilon}{3}$. Thus, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{3}$. Then there exists a $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$. For $m, n \ge n_0$, we get

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\le 2\|x_{n_0} - p^*\| \\ &< \epsilon.\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in K. Hence $\lim_{n\to\infty} x_n = q$ for $q \in K$. This implies by Theorem 2.1 (i) that for each $i = 1, 2, \ldots, m$,

$$\begin{aligned}d(q, T_i q) &\le d(q, x_n) + d(x_n, T_i x_n) + H(T_i x_n, T_i q) \\ &\le d(q, x_n) + d(x_n, x_{n,i}) + d(x_n, q) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $d(q, T_i q) = 0$ which implies that $q \in T_i q$ for all $i = 1, 2, \ldots, m$. Thus, $q \in F$. □

Corollary 2.4. *Let X be a real Banach space and K a closed convex subset of X. Let $\{T_i : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$. Assume that there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $f(r) > 0$ for all $r > 0$ such that for some $i = 1, 2, \ldots, m$,*

$$d(x_n, T_i(x_n)) \ge f(d(x_n, F)).$$

Then the sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of $\{T_i\}$.

Proof. Assume that $d(x_n, T_i x_n) \ge f(d(x_n, F))$ for some $i = 1, 2, \ldots, m$. By Theorem 2.1 (ii), we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, m$. It follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.3, we get the result. □

Suzuki [14] introduced a condition on mappings, called condition (C) which is weaker than nonexpansiveness. A multivalued mapping T :

$X \to CB(X)$ is said to satisfy the *condition (C)* provided that

$$\frac{1}{2}d(x,Tx) \le \|x-y\| \Rightarrow H(Tx,Ty) \le \|x-y\|, \quad x,y \in X.$$

The following known results can be found in [1] and [3].

Lemma 2.5. [1] *Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (C).*

Lemma 2.6. [3] *Let $T : X \to CB(X)$ be a multivalued mapping which satisfies the condition (C) and has a fixed point. Then T is a quasi-nonexpansive mapping.*

Recently, Eslamian and Abkar [3] introduced the following iterative process. Let $P(E)$ be nonempty proximinal bounded subsets of E and let $\{T_i : E \to P(E) : i = 1, 2, \dots, m\}$ be a finite family of multivalued mappings and

$$P_{T_i}(x) := \{y \in T_i(x) : \|x-y\| = d(x, T_i(x))\}.$$

For a fixed $x_0 \in E$, they considered an iterative process defined by

$$x_{n+1} = a_{n,0}x_n + a_{n,1}z_{n,1} + a_{n,2}z_{n,2} + \ldots + a_{n,m}z_{n,m}, \quad n \ge 0, \tag{2.2}$$

where $z_{n,i} \in P_{T_i}(x_n)$ and $\{a_{n,k}\}$ are sequences of numbers in $[0,1]$ such that for every natural number n,

$$\sum_{k=0}^{m} a_{n,k} = 1.$$

They obtained the following result:

Theorem 2.7. [3] *Let E be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : E \to P(E)$, $(i = 1, 2, \dots, m)$ be a finite family of multivalued mappings with $F = \cap_{i=1}^{m} F(T_i) \neq \emptyset$ and such that each $P_{T_i} (i = 1, 2, \dots, m)$ satisfies the condition (C). Let $\{x_n\}$ be the iterative process defined by (2.2) and $a_{n,k} \in [a, 1] \subset (0, 1)$ for $k = 0, 1, \dots, m$. Assume that there exists an increasing function $f : [0,\infty) \to [0,\infty)$ with $f(r) > 0$ for all $r > 0$ such that for some $i = 1, 2, \dots, m$,*

$$d(x_n, T_i(x_n)) \ge f(d(x_n, F)).$$

Then the sequence $\{x_n\}$ defined by (2.2) converges strongly to a common fixed point of $\{T_i\}$.

Remark 2.8. *In Theorem 2.7, we observe that the sequence* $\{x_n\}$ *generated by* (2.2) *converges strongly to a common fixed point of* $T_i (i = 1, 2, \ldots, m)$ *under the condition that each* P_{T_i} *satisfies the condition (C). But in Corollary 2.4, the sequence* $\{x_n\}$ *generated by* (1.1) *converges strongly to a common fixed point of* T_i $(i = 1, 2, \ldots, m)$ *without any condition imposed on* P_{T_i}*. However, the iterative schemes* (1.1) *and* (2.2) *are different.*

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