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SOME PROPERTIES OF MARGINAL AUTOMORPHISMS OF GROUPS

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Abstract. Let W be a non-empty subset of a free group. The automorphism α of a group G is said to be a marginal automorphism (with respect to W), if for all $x \in G$, $x^{-1}\alpha(x) \in W^*(G)$, where $W^*(G)$ is the marginal subgroup of G.

In this paper, we study the concept of marginal automorphisms of a given group and we obtain a necessary and sufficient condition for a purely non-abelian finite p-group G, such that the set of all marginal automorphisms of G forms an elementary abelian p-group.

1. Introduction

Let G be a group, and let G' and $Z(G)$ denote the derived subgroup and the center of G, respectively. The subgroup of automorphisms of G which acts trivially on $G/Z(G)$ is called the group of *central* automorphisms and is denoted by $Aut_c(G)$, which is a normal subgroup of the full automorphisms group of G. Hence, if α belongs to $Aut_c(G)$, then $x^{-1}\alpha(x)$ lies in $Z(G)$, for every x in G. The concept of central automorphisms have been already studied by many authors, such as Adney and Yen (1965), Sanders (1969), Curran and McCaughan (1986), S. Franciosi, F. D. Giovanni and M. L. Newell (1994) and Jamali and Mousavi

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(2002).

Let W be a non-empty subset of a free group $F = \langle x_1, x_2, \ldots \rangle$ and let G be an arbitrary group. Then

$$W(G) = \langle w(g_1, \ldots, g_r) |\ w = w(x_1, \ldots, x_r) \in W \ \text{ and } \ g_1, \ldots, g_r \in G \rangle,$$

and

$$W^*(G) = \big\{g \in G |\ w(g_1, \ldots, g_{i-1}, g_i g, g_{i+1}, \ldots, g_r) = w(g_1, \ldots, g_r), \\ \text{for all } w \in W,\ g_1, \ldots, g_r \in G \ \text{ and } \ i \in \{1, \ldots, r\}\big\},$$

are the *verbal* and *marginal* subgroups of G, with respect to W. Clearly $W(G)$ and $W^*(G)$ are fully-invariant and characteristic subgroups of G, respectively (see [6, 7] for more information).

The automorphism $\alpha \in Aut(G)$ is called *marginal* (with respect to W), if $x^{-1}\alpha(x) \in W^*(G)$, for all $x \in G$. Clearly, the set of all marginal automorphisms of G is a normal subgroup of $Aut(G)$ which is denoted by $Aut_{W^*}(G)$ (see also [9]). In particular, if $W = \{[x_1, x_2]\}$ then $W(G) = G'$ and $W^*(G) = Z(G)$ and hence $Aut_{W^*}(G) = Aut_c(G)$.

The group of central automorphisms of a finite group G is very important in studying its full automorphisms group. However, in some cases the group of central automorphisms of G does not have the desired properties because of the special structures of $Z(G)$ or G'. But, one may choose a suitable subset W of the free group F, for which its corresponding marginal automorphism group carries the required properties (see, for instance Example 3.6). This justifies our study of considering marginal subgroups instead of the center of the group.

In [5], Jamali and Mousavi proved that, if G is a purely non-abelian finite p-group of class two (p odd), then $Aut_c(G)$ is elementary abelian if and only if
(i) $\Omega_1(Z(G)) = \Phi(G)$, and
(ii) $\exp(Z(G)) = p$ or $\exp(G/G') = p$, where $\Phi(G)$ is the *Frattini* subgroup of G.
Recall that a group G is called *purely non-abelian* if it has no non-trivial abelian direct factor.

In the present article, we improve the above and obtain a similar result by omitting condition (i) and nilpotency condition of the group G. In fact, we prove the following main theorem, which is a wide generalization of the above result and one of the main theorems of [4].

Main Theorem. *Let G be a purely non-abelian finite p-group (p odd)* *and $\emptyset \neq W \subseteq F$ with $W^*(G) \leq Z(G)$ and $G/W(G)$ is abelian. Then $Aut_{W^*}(G)$ is an elementary abelian p-group if and only if* $\exp(W^*(G)) = p$ *or* $\exp(G/W(G)) = p$.

In the next section, we discuss and collect all the preliminary facts needed in proving our main theorem.

2. Preliminary results

Let G be a finite group. If $W^*(G)$ is contained in the center of G and $\sigma \in Aut_{W^*}(G)$, then for all $x \in G$, one can easily see that the map $f_\sigma : x \longmapsto x^{-1}\sigma(x)$ is a homomorphism from G into $W^*(G)$. On the other hand, for every $f \in Hom(G, W^*(G))$ the map $\sigma_f : x \longmapsto xf(x)$ is a marginal endomorphism of G. Note that, the endomorphism σ_f is an automorphism if and only if $f(x) \neq x^{-1}$, for all $1 \neq x \in G$. The following results of Attar [9] are useful in our investigations.

Proposition 2.1. ([9]). *Let G be any group and $\emptyset \neq W \subseteq F$. If $W^*(G)$ is abelian, then*
(i) $f(t) = 1$, for all $f \in Hom(G, W^*(G))$ and all $t \in W(G)$;
(ii) $Hom(G, W^*(G)) \cong Hom(G/W(G), W^*(G))$.

One can easily verify that every marginal endomorphism fixes the verbal subgroup $W(G)$, element-wise.

Theorem 2.2. ([9]). *Let G be a purely non-abelian finite group and $\emptyset \neq W \subseteq F$ with $W^*(G) \leq Z(G)$. Then $|Aut_{W^*}(G)| = |Hom(G, W^*(G))|$.*

Remark 2.3. *By Theorem 2.2, it is obvious that if G is a purely non-abelian finite p-group and $\emptyset \neq W \subseteq F$ with $W^*(G) \leq Z(G)$, then $Aut_{W^*}(G)$ is also a finite p-group.*

Theorem 2.4. ([9]). *Let G be any group and $\emptyset \neq W \subseteq F$ with $W^*(G) \leq W(G) \cap Z(G)$, then $Aut_{W^*}(G) \cong Hom(G/W(G), W^*(G))$.*

Remark 2.5. *One should note that, in Theorem 2.4 if we only assume that $W^*(G) \leq Z(G)$ and $Imf \leq W(G)$, for all $f \in Hom(G, W^*(G))$, then the conclusion is still held. Also from this theorem one may deduce that $Aut_{W^*}(G)$ is abelian.*

Let G be a finite abelian p-group and $1 \neq x \in G$, then the *height* of x (denoted by $ht(x)$) is defined to be the largest p-power p^n say, for which $x \in G^{p^n}$. The following lemma is useful in the next section. We recall that $o(x)$ denotes the order of the element x in a group G.

Lemma 2.6. ([4]). *Let x be an element of a finite p-group G and N a normal subgroup of G containing G' such that $o(x) = o(xN) = p$. If the cyclic subgroup $\langle x \rangle$ is normal in G such that $ht(xN) = 1$, then $\langle x \rangle$ is a direct factor of G.*

The next section is devoted to proving our main theorem.

3. Elementary abelian p-groups as marginal automorphism groups

Using the results and discussions of the previous section, we are able to prove the necessity of our main theorem with no restriction on the prime p.

Theorem 3.1. *Let G be a purely non-abelian finite p-group and $\emptyset \neq W \subseteq F$ so that $W^*(G) \leq Z(G)$ and $G/W(G)$ is abelian. If either the exponent of $G/W(G)$ or of $W^*(G)$ is equal to p, then $Aut_{W^*}(G)$ is an elementary abelian p-group.*

Proof. We first assume that $\exp(G/W(G)) = p$ and $f \in Hom(G, W^*(G))$, then by Proposition 2.1(ii), $\bar{f} \in Hom(G/W(G), W^*(G))$. So for any element $x \in G$, put $\bar{f}(xW(G)) = a$. If $aW(G) \neq 1$, then it follows that $o(aW(G)) = o(a) = p$. Clearly, $\langle a \rangle \leq W^*(G) \leq Z(G)$ and hence the cyclic subgroup $\langle a \rangle$ is normal in G. We also have $ht(aW(G)) = 1$. Now, by Lemma 2.6 the cyclic subgroup $\langle a \rangle$ is an abelian direct factor of G, which contradicts the assumption. Therefore, $a \in W(G)$ which implies that $Im(f) \leq W(G)$. Hence, by Remark 2.5, $Aut_{W^*}(G) \cong Hom(G/W(G), W^*(G))$, which follows that $Aut_{W^*}(G)$ is abelian and since $\exp(G/W(G)) = p$, it implies that $Aut_{W^*}(G)$ is elementary abelian p-group.

Now, assume that $\exp(W^*(G)) = p$ and consider $f, g \in Hom(G, W^*(G))$. We first show that $gof(x) = 1_G$, for all $x \in G$. Assume that $\bar{f}(xW(G)) = b \in W^*(G)$, for $x \in G$. Since $\exp(W^*(G)) = p$, it implies that $o(b)|p$. If $b = 1_G$ then $gof(x) = g(\bar{f}(xW(G)) = 1_G$. So take $o(b) = p$. If $b \in W(G)$ then we have $g(f(x)) = g(\bar{f}(xW(G)) = g(b) = 1_G$. Thus assume $bW(G) \neq 1$ and as $b^p = 1_G$, it follows that $o(bW(G)) = p$. Also, since $b \in W^*(G) \leq Z(G)$, it implies that $\langle b \rangle$ is normal in G. Now, if $ht(bW(G)) = 1$, then by Lemma 2.6 the cyclic subgroup $\langle b \rangle$ is an abelian direct factor of G, which gives a contradiction. So assume $ht(bW(G)) = p^m$, for some $m \in \mathbb{N}$. Hence, there exists an element $yW(G)$ in $G/W(G)$ such that $bW(G) = (yW(G))^{p^m}$. Therefore, since $\exp(W^*(G)) = p$ we have $g \circ f(x) = g(b) = (\bar{g}(yW(G)))^{p^m} = 1_G$. Thus, for all $f, g \in Hom(G, W^*(G))$ and each $x \in G$, $g(f(x)) = 1_G$. Similarly, $f(g(x)) = 1_G$ and so $f \circ g = g \circ f$, which implies that $\sigma_f \circ \sigma_g = \sigma_g \circ \sigma_f$. This shows that $Aut_{W^*}(G)$ is abelian. Now, we show that each non-trivial element of $Aut_{W^*}(G)$ has order p. So, if $\alpha \in Aut_{W^*}(G)$ then by Theorem 2.2, there exists a homomorphism $f \in Hom(G, W^*(G))$, such that $\alpha = \sigma_f$. We then show that $\circ(\sigma_f)|p$. Clearly, taking $f = g$ and using $f(f(x)) = 1_G$, then for all $x \in G$, we have ${\sigma_f}^2(x) = \sigma_f(xf(x)) = x(f(x))^2$. By repeating application of the above procedure, we obtain ${\sigma_f}^p(x) = x(f(x))^p$. As $\exp(W^*(G)) = p$ and $f(x) \in W^*(G)$ we have ${\sigma_f}^p(x) = x$, which implies that ${\sigma_f}^p = 1_{Aut_{W^*}(G)}$. This shows that $\circ(\sigma_f)|p$ and so $Aut_{W^*}(G)$ is an elementary abelian p-group. $\square$

Proof of the main theorem. For the odd prime p, let $Aut_{W^*}(G)$ be an elementary abelian p-group. Contrary to the claim, assume that the exponents of $W^*(G)$ and $G/W(G)$ are both strictly greater than p. Since $G/W(G)$ is finite abelian, it has a cyclic direct summand $\langle xW(G)\rangle$ say, of order p^n $(n \geq 2)$ and hence $G/W(G) \cong \langle xW(G)\rangle \times K/W(G)$. On the other hand, there exists an element $a \in W^*(G)$ of order p^m, where $2 \leq m \leq n$. Now, we define the homomorphism $\bar{f} \in Hom(G/W(G), W^*(G))$, given by

$$\begin{aligned} \bar{f} : \langle xW(G)\rangle \times K/W(G) &\longrightarrow W^*(G) \\ (x^iW(G), kW(G)) &\longmapsto a^i \ . \end{aligned}$$

Note that, $o(a)|o(xW(G))$ as $m \leq n$, it implies that $\bar{f}$ is well-defined. If $aW(G) = (x^sW(G), kW(G))$, then we show that $p|s$. Assume the contrary, then $\langle xW(G)\rangle = \langle x^sW(G)\rangle$ and hence $G/W(G) = \langle aW(G)\rangle \frac{K}{W(G)}$. Now, we have

$$o(a) \geq o(aW(G)) \geq o(x^sW(G)) = o(xW(G)) \geq o(\bar{f}(xW(G))) = o(a),$$

which implies that $o(a) = o(aW(G))$ and so $\langle a\rangle \cap W(G) = 1$. On the other hand, $o(aW(G)) = o(xW(G))$ and so we get $G/W(G) \cong \langle aW(G)\rangle \times K/W(G)$ and hence $G \cong \langle a\rangle \times K$, which is a contradiction, since G is purely non-abelian. By Theorem 2.2, $\sigma_f \in Aut_{W^*}(G)$ and by assumption we have $o(\sigma_f) = p$. Now, we have $\sigma_f(x) = xf(x) = xa$ and since $f(a) = \bar{f}((xW(G))^s, kW(G)) = a^s$, it implies that ${\sigma_f}^2(x) = xa^{s+2} = xa^{(\frac{(s+1)^2-1}{s})}$, and hence ${\sigma_f}^3(x) = xa^{(\frac{(s+1)^3-1}{s})}$. Using the above procedure, we obtain ${\sigma_f}^t(x) = xa^{(\frac{(s+1)^t-1}{s})}$, for every $t \in \mathbb{N}$. As the order

of σ_f is p, we get $a^{(\frac{(s+1)^p-1}{s})} = 1$. Since p is odd and $p|s$, one can see that $p^2|(\frac{(s+1)^p-1}{s}) - p$. Therefore $qp^2 + p = \frac{(s+1)^p-1}{s}$, for some $q \in \mathbb{Z}$, and so $(a^p)^{qp+1} = 1$. Since $o(a) = p^m$, we must have $o(a^p) = p^{m-1}$. Now,
(i) if $a^p \neq 1$, then $p^{m-1}|qp+1$ and this is impossible, as $m \geq 2$.
(ii) If $a^p = 1$, then since $o(a) = p^m$ and $m \geq 2$ we obtain a contradiction.

The converse was already proved in Theorem 3.1. So the proof of the main theorem is completed. □

Now, taking the set $W = \{[x_1, x_2]\}$, we obtain the following result.

Corollary 3.2. ([4]). *Let G be a purely non-abelian finite p-group (p odd). Then $Aut_c(G)$ is an elementary abelian p-group if and only if $\exp(Z(G)) = p$ or $\exp(G/G') = p$.*

One can easily observe that, when taking the set of words $W = \{[x_1, x_2], {x_3}^p\}$ then the exponent of the marginal subgroup is p and $Aut_{W^*}(G)$ is an elementary abelian p-group.

In the following we discuss the problem further and show that the main theorem does not hold for $p = 2$.

Proposition 3.3. *Let G be an arbitrary group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$, then $C_{Aut_{W^*(G)}}(Z(G)) \cong Hom(\frac{G}{W(G)Z(G)}, W^*(G))$.*

Proof. For each $f \in C_{Aut_{W^*(G)}}(Z(G))$ the map $\sigma_f : \frac{G}{W(G)Z(G)} \to W^*(G)$ given by $xW(G)Z(G) \longmapsto x^{-1}f(x)$ is a homomorphism. It is straightforward to see that $\sigma : f \mapsto \sigma_f$ is an isomorphism. □

Remark 3.4. *Note that, the above proposition was proved in* [9] *with extra conditions that G must be purely non-abelian finite group.*

We also remark that, Theorem A in [9] has some conditions. In fact, if the group H is only assumed to be torsion-free abelian, then one can easily show that $Hom(G, H)$ is also torsion-free.

Now, in the following we give an example showing that our main theorem does not hold for $p = 2$, while Theorem 3.1 is true for all prime numbers p.

Example 3.5. *Consider the group*

$$G = \langle x, y|\ x^8 = y^2 = 1,\ x^y = x^{-3}\rangle,$$

which is the semidirect product of $\mathbb{Z}_8$ by $\mathbb{Z}_2$. One can easily check that $Z(G) = \langle x^2\rangle \cong \mathbb{Z}_4$ and $G' = \{1, x^4\} \cong \mathbb{Z}_2$. If we take $W = \{[x_1, x_2]\}$, *then* $W^*(G) = Z(G)$ *and* $W(G) = G'$. *Clearly,* $\exp(Z(G)) = 4$ *and* $\exp(G/G') = 4$. *Now, we prove that* $Aut_{W^*}(G) = Aut_c(G)$ *is an elementary abelian 2-group, which shows that the main theorem does not hold for* $p = 2$. *Clearly,* G *is a purely non-abelian 2-group and so using Theorem* 2.2 *and Proposition* 2.1, $|Aut_c(G)| = |Hom(G/G', Z(G))| = 8$. *Now, using Proposition* 3.3 *and the property that* G' *is central, we have* $C_{Aut_c(G)}(Z(G)) \cong Hom(G/Z(G), Z(G)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. *So* $\alpha^2 \in C_{Aut_c(G)}(Z(G))$, *for all* $\alpha \in Aut_c(G)$. *Hence* $\alpha^2(z) = z$, *for all* $z \in Z(G)$ *and so* $\alpha^2(x^2) = x^2$. *Since* $\alpha^2 \in Aut_c(G)$, *we may write* $(x^{-1}\alpha^2(x))^2 = 1_G$. *Now, since* $x^{-1}\alpha^2(x) \in Z(G)$ *we have either* $x^{-1}\alpha^2(x) = 1_G$ *or* $x^{-1}\alpha^2(x) = x^4$ $(*)$. *For, let* $\alpha^2(x) = x^5$. *Since* $\alpha \in Aut_c(G)$, *it implies that* $\alpha(x) = xz$, *where* $z = x^{2i}$ *for some* $i \in \{0, 1, 2, 3\}$. *On the other hand, we have* $x^5 = \alpha(\alpha(x)) = x^{(2i+5)^2} = x$, *which is a contradiction. Thus the equality in* $(*)$ *is impossible and so we must have* $\alpha^2(x) = x$. *Similarly, one may show that* $\alpha^2(y) = y$ *and hence* $\alpha^2 = id_G$. *It follows that* $Aut_c(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

One observes that in the above example, G is a purely non-abelian 2-group, so that its central automorphism group is elementary abelian, while the exponent of its center and the derived subgroup are not equal to 2.

In the following example the group of central automorphisms is not elementary abelian. However, one can consider a suitable subset W of the free group F, for which its corresponding marginal automorphism group is an elementary abelian group.

Example 3.6. *Consider the group*

$$G = \langle x, y |\ x^{16} = y^4 = 1,\ x^y = x^{-3} \rangle,$$

which is a semidirect product of $\mathbb{Z}_{16}$ *by* $\mathbb{Z}_4$. *It is easily checked that* $G' = Z(G) = \langle x^4 \rangle \cong \mathbb{Z}_4$. *By Proposition* 3.3, *we have* $Aut_c(G) = C_{Aut_c(G)}(Z(G)) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$. *Therefore,* $Aut_c(G)$ *is not elementary abelian. Now, take* $W = \{x^2\}$. *Then clearly* $W^*(G) = \{z \in Z(G) |\ z^2 = 1\} = \{1, x^8\}$. *Since* $G/W(G)$ *is abelian, by Theorem* 3.1, $Aut_{W^*}(G)$ *is elementary abelian 2-group. On the other hand,* $W^*(G) \leq Z(G) \cap W(G)$. *Therefore, by Proposition* 2.1(ii) *and Theorem* 2.4, $Aut_{W^*}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, our final result gives some sufficient conditions that the group of marginal automorphisms of G is trivial.

Proposition 3.7. *Let G be a group. Assume that $\emptyset \neq W \subseteq F$, $W^*(G)$ is torsion-free central subgroup of G and $G/W(G)$ is torsion. Then $Aut_{W^*}(G) = \langle 1 \rangle$.*

Proof. Let $\alpha \in Aut_{W^*}(G)$ and $x \in G$, then by the assumption $x^n \in W(G)$, for some $n \in \mathbb{N}$. As every marginal automorphism fixes the verbal subgroup element-wise, we have $\alpha(x)^n = \alpha(x^n) = x^n$, and so $x^{-n}\alpha(x)^n = 1$. On the other hand, α is a marginal automorphism and hence $x^{-1}\alpha(x) \in W^*(G) \leq Z(G)$, which implies that $(x^{-1}\alpha(x))^n = 1$. Since $W^*(G)$ is torsion-free, it follows that $x^{-1}\alpha(x) = 1$ and so $Aut_{W^*}(G) = \langle 1 \rangle$, which completes the proof. □

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