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AN ITERATIVE METHOD FOR THE HERMITIAN-GENERALIZED HAMILTONIAN SOLUTIONS TO THE INVERSE PROBLEM $AX = B$ WITH A SUBMATRIX CONSTRAINT

J. CAI

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Abstract. In this paper, an iterative method is proposed for solving the matrix inverse problem $AX = B$ for Hermitian-generalized Hamiltonian matrices with a submatrix constraint. By this iterative method, for any initial matrix A_0, a solution A^* can be obtained in finite iteration steps in the absence of roundoff errors, and the solution with least norm can be obtained by choosing a special kind of initial matrix. Furthermore, in the solution set of the above problem, the unique optimal approximation solution to a given matrix can also be obtained. A numerical example is presented to show the efficiency of the proposed algorithm.

1. Introduction

Thought this paper, we adopt the following notation. Let $C^{m\times n}(R^{m\times n})$ and $HC^{n\times n}$ denote the set of $m \times n$ complex (real) matrices and $n \times n$ Hermitian matrices, respectively. For a matrix $A \in C^{m\times n}$, we denote its conjugate transpose, transpose, trace, column space, null space and Frobenius norm by $A^H, A^T, tr(A), R(A), N(A)$ and $\|A\|$, respectively. In space $C^{m\times n}$, we define inner product as: $\langle A, B\rangle = tr(B^H A)$ for all

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$A, B \in C^{m\times n}$, and the symbol $\mathrm{Re}\langle A, B\rangle$ and $\overline{\langle A, B\rangle}$ stand for its real part and conjugate number, respectively. Two matrices A and B are orthogonal if $\langle A, B\rangle = 0$. Let $Q_{s,n} = \{a = (a_1, a_2, \cdots, a_s) : 1 \leq a_1 < a_2 < \cdots < a_s \leq n\}$ denote the strictly increasing sequences of s elements from $1, 2, \cdots, n$. For $A \in C^{m\times n}$, $p \in Q_{s,m}$ and $q \in Q_{t,n}$, let $A[p|q]$ stand for the $s \times t$ submatrix of A determined by rows indexed by p and columns indexed by q.

Let $I_n = (e_1, e_2, \cdots, e_n)$ be the $n\times n$ unit matrix, where e_i denotes its ith column. Let $J_n \in R^{n\times n}$ be the orthogonal skew-symmetric matrix, i.e., $J_n^T J_n = J_n J_n^T = I_n$ and $J_n^T = -J_n$. A matrix $A \in C^{n\times n}$ is called Hermitian-generalized Hamiltonian if $A^H = A$ and $(AJ_n)^H = AJ_n$. The set of all $n \times n$ Hermitian-generalized Hamiltonian matrices is denoted by $HGH^{n\times n}$. Particularly, if $J_n = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, then the set $HGH^{n\times n}$ reduces to the well-known set of Hermitian-Hamiltonian matrices, which have applications in many areas such as linear-quadratic control problem [?, ?], H_∞ optimization [?] and the related problem of solving algebraic Riccati equations [?].

Recently, there have been several papers considering solving the inverse problem $AX = B$ for various matrices by direct methods based on different matrix decompositions. For instance, Xu and Li [?], Peng [?] and Zhou et al. [?] discuss its Hermitian reflexive, anti-reflexive solutions and least-square centrosymmetric solutions, respectively. Then Huang and Yin [?] and Huang et al. [?] generalize the results of the latter to the more general R-symmetric and R-skew symmetric matrices, respectively. Li et al. [?] consider the inverse problem for symmetric P-symmetric matrices with a submatrix constraint. Peng et al. [?, ?] and Gong et al. [?] consider solving the inverse problem for centrosymmetric, bisymmetric and Hermitian-Hamiltonian matrices, respectively, under the leading principal submatrix constraint. Zhao et al. [?] concerns the inverse problem for bisymmetric matrices under a central principal submatrix constraint. However, the inverse problem for the Hermitian-generalized Hamiltonian matrices with general submatrix constraint has not been studied till now.

Hence, in this paper, we consider solving the following problem and its associated best approximation problem which occurs frequently in experimental design ([?, ?, ?, ?]) by iterative methods.

Problem I. Given $X, B \in C^{n\times m}$, $A_S \in HC^{s\times t}$, $p = (p_1, p_2, \cdots, p_s) \in Q_{s,n}$, and $q = (q_1, q_2, \cdots, q_t) \in Q_{t,n}$, find $A \in HGH^{n\times n}$, such that

$$AX = B \text{ and } A[p|q] = A_S. \tag{1.1}$$

Problem II. Let S_E denote the set of solutions of Problem I, for given $\bar{A} \in C^{n\times n}$, find $\hat{A} \in S_E$, such that

$$\|\hat{A} - \bar{A}\| = \min_{A\in S_E} \|A - \bar{A}\|. \tag{1.2}$$

The rest of this paper is organized as follows. In Section 2, we propose an iterative algorithm for solving Problem I and present some basic properties of this algorithm. In Section 3, we consider the iterative method for solving Problem II. A numerical example is given in Section 4 to show the efficiency of the proposed algorithm. Conclusions will be put in Section 5.

2. Iterative algorithm for solving Problem I

Firstly, we present several basic properties of Hermitian-generalized Hamiltonian matrices in the following lemmas.

Lemma 2.1. *Consider a matrix $Y \in C^{n\times n}$. Then $Y + Y^H + J_n(Y + Y^H)J_n \in HGH^{n\times n}$.*

Proof. The proof is easy, thus is omitted. □

Lemma 2.2. *Suppose a matrix $Y \in C^{n\times n}$ and a matrix $D \in HGH^{n\times n}$. Then $4\mathrm{Re}\langle Y, D\rangle = \langle Y + Y^H + J_n(Y + Y^H)J_n, D\rangle$.*

Proof. Since

$$\langle Y^H, D\rangle = tr(D^H Y^H) = tr((YD)^H) = \overline{\langle Y, D^H\rangle} = \overline{\langle Y, D\rangle},$$

we have $\langle Y + Y^H, D\rangle = 2\mathrm{Re}\langle Y, D\rangle$. Then we get

$$\begin{aligned}\langle J_n(Y + Y^H)J_n, D\rangle &= tr(D^H J_n(Y + Y^H)J_n)\\ &= tr(J_n D^H J_n(Y + Y^H)) = tr(D^H(Y + Y^H))\\ &= \langle Y + Y^H, D\rangle = 2\mathrm{Re}\langle Y, D\rangle.\end{aligned}$$

Hence we have $\langle Y + Y^H + J_n(Y + Y^H)J_n, D\rangle = 4\mathrm{Re}\langle Y, D\rangle$. □

Next we propose an iterative algorithm for solving Problem I.

Algorithm 1. *Step 1. Input* $X, B \in C^{n\times m}, A_S \in HC^{s\times t}, p = (p_1, p_2, \cdots, p_s) \in Q_{s,n}$, $q = (q_1, q_2, \cdots, q_t) \in Q_{t,n}$ *and an arbitrary* $A_0 \in HGH^{n\times n}$;
Step 2. Compute

$R_0 = B - A_0X$;
$S_0 = A_S - (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T A_0(e_{q_1}, e_{q_2}, \cdots, e_{q_t})$;
$E_0 = R_0X^H + (e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_0(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T$;
$F_0 = \frac{1}{4}[E_0 + E_0^H + J_n(E_0 + E_0^H)J_n]$; $P_0 = F_0$;
$k := 0$;

Step 3. If $R_k = S_k = 0$ *then stop; else,* $k := k + 1$;
Step 4. Compute

$\alpha_{k-1} = \frac{\|R_{k-1}\|^2 + \|S_{k-1}\|^2}{\|P_{k-1}\|^2}$;
$A_k = A_{k-1} + \alpha_{k-1}P_{k-1}$;
$R_k = B - A_kX$;
$S_k = A_S - (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T A_k(e_{q_1}, e_{q_2}, \cdots, e_{q_t})$;
$E_k = R_kX^H + (e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_k(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T$;
$F_k = \frac{1}{4}[E_k + E_k^H + J_n(E_k + E_k^H)J_n]$;
$\beta_{k-1} = \frac{tr(F_kP_{k-1})}{\|P_{k-1}\|^2}$;
$P_k = F_k - \beta_{k-1}P_{k-1}$;

Step 5. Go to Step 3.

Remark 2.3. *By Lemma 2.1, one can easily see that the matrix sequences* $\{A_k\}$,$\{P_k\}$ *and* $\{F_k\}$ *generated by Algorithm 1 are all the Hermitian-generalized Hamiltonian matrices. And Algorithm 1 implies that if* $R_k = S_k = 0$*, then* A_k *is the solution of Problem I.*

We list some basic properties of Algorithm 1 as follows.

Theorem 2.4. *Assume that* A^* *is a solution of Problem I. Then the sequences* $\{A_i\}, \{P_i\}, \{R_i\}$ *and* $\{S_i\}$ *generated by Algorithm 2.1 satisfy the following equality:*

$$\langle P_i, A^* - A_i\rangle = \|R_i\|^2 + \|S_i\|^2, \ i = 0, 1, 2, \cdots . \tag{2.1}$$

Proof. From Remark 2.3, it follows that $A^* - A_i \in HGH^{n\times n}$, $i = 0, 1, 2, \cdots$. Then according to Lemma 2.2 and Algorithm 1, for $i = 0$, we have

$$\langle P_0, A^* - A_0\rangle = \langle \frac{1}{4}(E_0 + E_0^H + J_n(E_0 + E_0^H)J_n), A^* - A_0\rangle$$
$$= \mathrm{Re}\langle E_0, A^* - A_0\rangle$$

$$
\begin{aligned}
&= \mathrm{Re}\langle R_0X^H + (e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_0(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T, A^* - A_0\rangle \\
&= \mathrm{Re}\langle (e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_0(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T, A^* - A_0\rangle \\
&\quad + \mathrm{Re}\langle R_0X^H, A^* - A_0\rangle \\
&= \mathrm{Re}tr((e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T(A^* - A_0)(e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_0) \\
&\quad + \mathrm{Re}tr(X^H(A^* - A_0)R_0) \\
&= \mathrm{Re}tr(R_0^H R_0) + \mathrm{Re}tr(S_0^H S_0) = tr(R_0^H R_0) + tr(S_0^H S_0) \\
&= \|R_0\|^2 + \|S_0\|^2.
\end{aligned}
$$

Assume that the conclusion holds for $i = k(k > 0)$, i.e., $\langle P_k, A^* - A_k\rangle = \|R_k\|^2 + \|S_k\|^2$, then for $i = k + 1$, we have

$$
\begin{aligned}
&\langle P_{k+1}, A^* - A_{k+1}\rangle = \langle F_{k+1}, A^* - A_{k+1}\rangle - \beta_k\langle P_k, A^* - A_{k+1}\rangle \\
&= \frac{1}{4}\langle E_{k+1} + E_{k+1}^H + J_n(E_{k+1} + E_{k+1}^H)J_n, A^* - A_{k+1}\rangle \\
&\quad - \beta_k\langle P_k, A^* - A_k - \alpha_k P_k\rangle \\
&= \mathrm{Re}\langle E_{k+1}, A^* - A_{k+1}\rangle - \beta_k\langle P_k, A^* - A_k\rangle + \beta_k\alpha_k\|P_k\|^2 \\
&= \langle R_{k+1}X^H + (e_{p_1}, e_{p_2}, \cdots, e_{p_s})S_{k+1}(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T, A^* - A_{k+1}\rangle \\
&\quad - \beta_k(\|R_k\|^2 + \|S_k\|^2) + \beta_k\frac{\|R_k\|^2 + \|S_k\|^2}{\|P_k\|^2}\|P_k\|^2 \\
&= \|R_{k+1}\|^2 + \|S_{k+1}\|^2.
\end{aligned}
$$

This completes the proof by the principle of induction. □

Remark 2.5. *Theorem 2.4 implies that if Problem I is consistent, then $\|R_i\|^2 + \|S_i\|^2 \neq 0$ implies that $P_i \neq 0$. On the other hand, if there exists a positive number k such that $\|R_k\|^2 + \|S_k\|^2 \neq 0$ but $P_k = 0$, then Problem I must be inconsistent.*

Lemma 2.6. *For the sequences $\{R_i\}$, $\{S_i\}$, $\{P_i\}$ and $\{F_i\}$ generated by Algorithm 1, let $\hat{R}_i = \begin{pmatrix} R_i \\ R_i^H \end{pmatrix}$ and $\hat{S}_i = \begin{pmatrix} S_i \\ S_i^H \end{pmatrix}$. Then it follows that*

$$
\langle \hat{R}_{i+1}, \hat{R}_j\rangle + \langle \hat{S}_{i+1}, \hat{S}_j\rangle = \langle \hat{R}_i, \hat{R}_j\rangle + \langle \hat{S}_i, \hat{S}_j\rangle - 2\alpha_i\langle F_j, P_i\rangle. \tag{2.2}
$$

Proof. By Algorithm 1, Remark 2.3 and Lemma 2.2, we have

$$
\begin{aligned}
&\langle \hat{R}_{i+1}, \hat{R}_j\rangle + \langle \hat{S}_{i+1}, \hat{S}_j\rangle = tr(R_j^H R_{i+1} + R_j R_{i+1}^H) + tr(S_j^H S_{i+1} + S_j S_{i+1}^H) \\
&= tr(R_j^H(R_i - \alpha_i P_i X) + R_j(R_i - \alpha_i P_i X)^H) + tr(S_j^H(S_i - \alpha_i(e_{p_1}, e_{p_2}, \\
&\qquad \cdots, e_{p_s})^T P_i(e_{q_1}, e_{q_2}, \cdots, e_{q_t})) + S_j(S_i - \alpha_i(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T \\
&\qquad P_i(e_{p_1}, e_{p_2}, \cdots, e_{p_s})))
\end{aligned}
$$

$$= tr(R_j^H R_i + R_j R_i^H) + tr(S_j^H S_i + S_j S_i^H) - \alpha_i tr(R_j^H P_i X + R_j (P_i X)^H + S_j^H (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T P_i (e_{q_1}, e_{q_2}, \cdots, e_{q_t}) + S_j (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T P_i (e_{p_1}, e_{p_2}, \cdots, e_{p_s}))$$
$$= \langle \hat{R}_i, \hat{R}_j \rangle + \langle \hat{S}_i, \hat{S}_j \rangle - \alpha_i \langle E_i + E_i^H, P_i \rangle$$
$$= \langle \hat{R}_i, \hat{R}_j \rangle + \langle \hat{S}_i, \hat{S}_j \rangle - \frac{\alpha_i}{2} \langle E_i + E_i^H + J_n(E_i + E_i^H) J_n, P_i \rangle$$
$$= \langle \hat{R}_i, \hat{R}_j \rangle + \langle \hat{S}_i, \hat{S}_j \rangle - 2\alpha_i \langle F_i, P_i \rangle.$$

□

Theorem 2.7. *For the sequences $\{\hat{R}_i\}$, $\{\hat{S}_i\}$ and $\{P_i\}$ generated by Algorithm 1, if there exists a positive number k such that $\hat{R}_i \neq 0$ for all $i = 0, 1, 2, \ldots, k$, then we have*

$$\langle \hat{R}_i, \hat{R}_j \rangle + \langle \hat{S}_i, \hat{S}_j \rangle = 0, \ (i, j = 0, 1, 2, \ldots, k, \ i \neq j). \tag{2.3}$$

Proof. Since $\langle \hat{R}_i, \hat{R}_j \rangle = \langle \hat{R}_j, \hat{R}_i \rangle$ and $\langle \hat{S}_i, \hat{S}_j \rangle = \langle \hat{S}_j, \hat{S}_i \rangle$, we only need to prove that (2.3) holds for all $0 \leq j < i \leq k$.

For $k = 1$, it follows from Lemma 2.6 that

$$\langle \hat{R}_1, \hat{R}_0 \rangle + \langle \hat{S}_1, \hat{S}_0 \rangle = \langle \hat{R}_0, \hat{R}_0 \rangle + \langle \hat{S}_0, \hat{S}_0 \rangle - 2\alpha_0 \langle F_0, P_0 \rangle$$
$$= tr(R_0^H R_0 + R_0 R_0^H) + tr(S_0^H S_0 + S_0 S_0^H) - 2\frac{\|R_0\|^2 + \|S_0\|^2}{\|P_0\|^2} \langle P_0, P_0 \rangle$$
$$= 2(\|R_0\|^2 + \|S_0\|^2) - 2\frac{\|R_0\|^2 + \|S_0\|^2}{\|P_0\|^2} \|P_0\|^2 = 0.$$

and

$$\langle P_1, P_0 \rangle = \langle F_1 - \frac{tr(F_1 P_0)}{\|P_0\|^2} P_0, P_0 \rangle = 0,$$

Assume that $\langle \hat{R}_m, \hat{R}_j \rangle + \langle \hat{S}_m, \hat{S}_j \rangle = 0$ and $\langle P_m, P_j \rangle = 0$ hold for all $0 \leq j < m, 0 < m \leq k$. We shall show that $\langle \hat{R}_{m+1}, \hat{R}_j \rangle + \langle \hat{S}_{m+1}, \hat{S}_j \rangle = 0$ and $\langle P_{m+1}, P_j \rangle = 0$ hold for all $0 \leq j < m+1, 0 < m+1 \leq k$.

For $0 \leq j < m$, by Lemma 2.6, we have

$$\langle \hat{R}_{m+1}, \hat{R}_j \rangle + \langle \hat{S}_{m+1}, \hat{S}_j \rangle = \langle \hat{R}_m, \hat{R}_j \rangle + \langle \hat{S}_m, \hat{S}_j \rangle - 2\alpha_m \langle F_j, P_m \rangle$$
$$= \langle \hat{R}_m, \hat{R}_j \rangle + \langle \hat{S}_m, \hat{S}_j \rangle - \alpha_m \langle P_j + \beta_{j-1} P_{j-1}, P_m \rangle$$
$$= -\alpha_m \langle P_j, P_m \rangle = 0,$$

and

$$\langle P_{m+1}, P_j \rangle = \langle F_{m+1}, P_j \rangle - \beta_m \langle P_m, P_j \rangle = \langle F_{m+1}, P_j \rangle$$
$$= \frac{\langle \hat{R}_j, \hat{R}_{m+1} \rangle + \langle \hat{S}_j, \hat{S}_{m+1} \rangle + \langle \hat{R}_{j+1}, \hat{R}_{m+1} \rangle + \langle \hat{S}_{j+1}, \hat{S}_{m+1} \rangle}{2\alpha_{m+1}} = 0.$$

For $j = m$, it follows from Lemma 2.6 and the hypothesis that

$$\begin{aligned}&\langle \hat{R}_{m+1}, \hat{R}_m\rangle + \langle \hat{S}_{m+1}, \hat{S}_m\rangle = \langle \hat{R}_m, \hat{R}_m\rangle + \langle \hat{S}_m, \hat{S}_m\rangle - 2\alpha_s\langle F_m, P_m\rangle\\ &= 2(\|R_m\|^2 + \|S_m\|^2) - 2\alpha_s\langle P_m + \beta_{m-1}P_{m-1}, P_m\rangle\\ &= 2(\|R_m\|^2 + \|S_m\|^2) - 2\frac{\|R_m\|^2 + \|S_m\|^2}{\|P_m\|^2}\langle P_m, P_m\rangle = 0,\end{aligned}$$

and

$$\langle P_{m+1}, P_m\rangle = \langle F_{m+1} - \beta_m P_m, P_m\rangle = \langle F_{m+1}, P_m\rangle - \frac{tr(F_{m+1}P_m)}{\|P_m\|^2}\|P_m\|^2 = 0.$$

Hence $\langle \hat{R}_{m+1}, \hat{R}_j\rangle + \langle \hat{S}_{m+1}, \hat{S}_j\rangle = 0$ and $\langle P_{m+1}, P_j\rangle = 0$ hold for all $0 \le j < m+1, 0 < m+1 \le k$.

This completes the proof by the principle of induction. □

Remark 2.8. *Based on Theorem 2.7, we can further demonstrate the finite termination property of Algorithm 1. Let* $Z_k = \begin{pmatrix} \hat{R}_k \\ \hat{S}_k \end{pmatrix}$. *Theorem 2.7 implies that the matrix sequences* $Z_0, Z_1, \cdots$ *are orthogonal to each other in the finite dimension matrix subspace. Hence there exists a positive integer* t_0 *such that* $Z_{t_0} = 0$. *Then we have* $R_{t_0} = S_{t_0} = 0$. *Thus the iteration will be terminated in finite steps in the absence of roundoff errors.*

Next we consider the least Frobenius norm solution of Problem I.

Lemma 2.9. [?] *Suppose that the consistent system of linear equations* $Ax = b$ *has a solution* $x^* \in R(A^H)$, *then* x^* *is the unique least norm solution of the system of linear equations.*

Theorem 2.10. *Suppose that Problem I is consistent. If we choose the initial bisymmetric matrix as follows:*

(2.4)

$$\begin{aligned}A_0 &= Y_1X^H + XY_1^H + J_n(Y_1X^H + XY_1^H)J_n + (e_{p_1}, e_{p_2}, \cdots, e_{p_s})Y_2\\ &(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T + (e_{q_1}, e_{q_2}, \cdots, e_{q_t})Y_2^H(e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T\\ &+J_n((e_{p_1}, e_{p_2}, \cdots, e_{p_s})Y_2(e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T + (e_{q_1}, e_{q_2}, \cdots, e_{q_t})Y_2^H\\ &(e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T)J_n,\end{aligned}$$

where Y_1, Y_2 *are arbitrary* $n \times n$ *complex matrices, or more especially, if* $A_0 = 0$, *then the solution obtained by Algorithm 1 is the least Frobenius norm solution.*

Proof. Consider the matrix equations as follows:

$$(2.5)\qquad \begin{cases} AX = B, \\ X^H A = B^H, \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T A (e_{q_1}, e_{q_2}, \cdots, e_{q_t}) = A_S, \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T A (e_{p_1}, e_{p_2}, \cdots, e_{p_s}) = A_S^H, \\ J_n A J_n X = B, \\ X^H J_n A J_n = B^H, \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T J_n A J_n (e_{q_1}, e_{q_2}, \cdots, e_{q_t}) = A_S, \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T J_n A J_n (e_{p_1}, e_{p_2}, \cdots, e_{p_s}) = A_S^H. \end{cases}$$

If A is a solution of Problem I, then it must be a solution of (2.5). Conversely, if (2.5) has a solution A, let $\tilde{A} = \dfrac{A + A^H + J_n(A + A^H)J_n}{4}$, then it is easy to verify that $\tilde{A}$ is a solution of Problem I. Therefore, the consistency of Problem I is equivalent to that of (2.5).

By using Kronecker products, (2.5) can be equivalently written as

$$(2.6)\qquad \begin{pmatrix} X^T \otimes A \\ I \otimes X^H \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T \otimes (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T \otimes (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T \\ X^T J_n^T \otimes J_n \\ J_n^T \otimes X^H J_n \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T J_n^T \otimes (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T J_n \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T J_n^T \otimes (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T J_n \end{pmatrix} \mathrm{vec}(A) = \mathrm{vec} \begin{pmatrix} B \\ B^H \\ A_S \\ A_S^H \\ B \\ B^H \\ A_S \\ A_S^H \end{pmatrix}.$$

Let the initial matrix A_0 be of the form (2.4), then by Algorithm 1 and Remark 2.5, we can obtain the solution A^* of Problem I within finite iteration steps, which can be represented in the same form. Hence we have

$$\mathrm{vec}(A^*) \in R \left(\begin{pmatrix} X^T \otimes A \\ I \otimes X^H \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T \otimes (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T \otimes (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T \\ X^T J_n^T \otimes J_n \\ J_n^T \otimes X^H J_n \\ (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T J_n^T \otimes (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T J_n \\ (e_{p_1}, e_{p_2}, \cdots, e_{p_s})^T J_n^T \otimes (e_{q_1}, e_{q_2}, \cdots, e_{q_t})^T J_n \end{pmatrix}^H \right).$$

According to Lemma 2.9, $\mathrm{vec}(A^*)$ is the least norm solution of (2.6), i.e., A^* is the least norm solution of the (2.5). Since the solution set of Problem I is a subset of that of (2.5), A^* also is the least norm solution of Problem I.

This completes the proof. □

3. **Iterative algorithm for solving Problem II**

In this section, we consider iterative algorithm for solving Problem II. For given $\bar{A} \in C^{n\times n}$ and arbitrary $A \in S_E$, we have

$$\begin{aligned}&\|A-\bar{A}\|^2\\ &=\|A-\tfrac{\bar{A}+\bar{A}^H}{2}\|^2+\|\tfrac{\bar{A}-\bar{A}^H}{2}\|^2\\ &=\|A-\tfrac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4}\|^2+\|\tfrac{\bar{A}+\bar{A}^H-J_n(\bar{A}+\bar{A}^H)J_n}{4}\|^2+\|\tfrac{\bar{A}-\bar{A}^H}{2}\|^2\end{aligned},$$

which implies that $\min\limits_{A\in S_E}\|A-\bar{A}\|$ is equivalent to

$$\min_{A\in S_E}\|A-\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4}\|.$$

When Problem I is consistent, for $A \in S_E$, it follows that

$$\begin{cases}(A-\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4})X=B-\frac{(\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n)X}{4},\\ (A-\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4})[p|q]=A_S-(\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4})[p|q].\end{cases}$$

Hence Problem II is equivalent to finding the least norm Hermitian-generalized Hamiltonian solution of the following problem:

$$\tilde{A}X=\tilde{B},\ \tilde{A}[p|q]=\tilde{A}_S,$$

where $\tilde{A}=A-\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4}$, $\tilde{B}=B-\frac{(\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n)X}{4}$ and $\tilde{A}_S=A_S-(\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4})[p|q]$.

Once the unique least norm solution $\widetilde{A}^*$ of the above problem is obtained by applying Algorithm 1 with the initial matrix A_0 having the representation assumed in Theorem 2.10, the unique solution $\hat{A}$ of Problem II can then be obtained, where $\hat{A}=\tilde{A}^*+\frac{\bar{A}+\bar{A}^H+J_n(\bar{A}+\bar{A}^H)J_n}{4}$.

4. **A numerical example**

In this section, we give a numerical example to illustrate the efficiency of the proposed iterative algorithm. All the tests are performed by MATLAB 7.4 with machine precision around 10^{-16}. Let $zeros(n)$ denote the $n \times n$ matrix whose all elements are zero. Because of the influence of the error of calculation, we shall regard a matrix X as zero matrix if $\|X\| < 1.0e-010$.

Example 4.1. Given matrices X and B as follows:

$$X = \begin{pmatrix} 0 & 2 & -1 & 1 & 4 \\ -1 & 2 & 1 & -1 & 1 \\ 4 & -2 & 5 & -6 & 0 \\ 2 & 6 & -1 & 0 & -3 \\ -1 & 2 & 0 & 0 & -4 \\ 3 & -1 & 1 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 25 & 4 & 4 & -5 & -1 \\ 19 & -25 & 12 & -2 & -10 \\ 7 & 25 & 0 & 4 & -11 \\ -3 & 4 & 5 & -10 & -6 \\ 21 & 21 & 22 & -33 & -29 \\ 4 & -6 & 20 & -21 & 23 \end{pmatrix}.$$

Let $p = (1,3,5), q = (1,2,6) \in Q_{3,6}$ and

$$A_S = \begin{pmatrix} 1 & -4 & 2 \\ 2 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix}.$$

Consider the least Frobenius norm Hermitian-generalized Hamiltonian solution of the following inverse problem with submatrix constraint:

$$AX = B \text{ and } A[1,3,5|1,2,6] = A_S. \tag{4.1}$$

If we choose the initial matrix $A_0 = zeros(6)$, then by Algorithm 1 and iterating 11 steps, we obtain the least Frobenius norm solution of the problem (4.1) as follows:

$$A_{11} = \begin{pmatrix} 1.0000 & -4.0000 & 2.0000 & 3.0000 & -1.0000 & 2.0000 \\ -4.0000 & -3.0000 & 1.0000 & -1.0000 & 1.0000 & 5.0000 \\ 2.0000 & 1.0000 & -0.0000 & 2.0000 & 5.0000 & 3.0000 \\ 3.0000 & -1.0000 & 2.0000 & -1.0000 & 4.0000 & -2.0000 \\ -1.0000 & 1.0000 & 5.0000 & 4.0000 & 3.0000 & -1.0000 \\ 2.0000 & 5.0000 & 3.0000 & -2.0000 & -1.0000 & 0.0000 \end{pmatrix}$$

with

$$\|R_{11}\|^2 + \|S_{11}\|^2 = 5.8839e-012.$$

5. **Conclusions**

In this paper, we construct an iterative method to solve the inverse problem $AX = B$ of the Hermitian-generalized Hamiltonian matrices with general submatrix constraint. In the solution set of the matrix equations, the optimal approximation solution to a given matrix can also be found by this iterative method. The given numerical example show that the proposed iterative method is quite efficient.

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J. Cai
School of Science, Huzhou Teachers College, Huzhou, P.R. China
Email: caijing@hutc.zj.cn