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FIXED POINTS FOR E-ASYMPTOTIC CONTRACTIONS AND BOYD-WONG TYPE E-CONTRACTIONS IN UNIFORM SPACES

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Abstract. In this paper we discuss the fixed points of asymptotic contractions and Boyd-Wong type contractions in uniform spaces equipped with an E-distance. A new version of Kirk's fixed point theorem is given for asymptotic contractions and Boyd-Wong type contractions is investigated in uniform spaces.

1. Introduction and preliminaries

In 2003, Kirk [5] discussed the existence of fixed points for (not necessarily continuous) asymptotic contractions in complete metric spaces. Jachymski and Jóźwik [4] constructed an example to show that continuity of the self-mapping is essential in Kirk's theorem. They also established a fixed point result for uniformly continuous asymptotic φ-contractions in complete metric spaces.

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Motivated by [5, Theorem 2.1] and [4, Example 1], we aim to give a more general form of [5, Theorem 2.1] in uniform spaces where the self-mappings are assumed to be continuous. We also generalize the Boyd-Wong fixed point theorem [3, Theorem 1] to uniform spaces equipped with an E-distance.

We begin with some basics in uniform spaces which are needed in this paper. The reader can find an in-depth discussion in, e.g., [6].

A uniformity on a nonempty set X is a nonempty collection $\mathcal{U}$ of subsets of $X \times X$ (called the entourages of X) satisfying the following conditions:

(1) Each entourage of X contains the diagonal $\{(x,x) : x \in X\}$;
(2) $\mathcal{U}$ is closed under finite intersections;
(3) For each entourage U in $\mathcal{U}$, the set $\{(x,y) : (y,x) \in U\}$ is in $\mathcal{U}$;
(4) For each $U \in \mathcal{U}$, there exists an entourage V such that $(x,y),(y,z) \in V$ implies $(x,z) \in U$ for all $x,y,z \in X$;
(5) $\mathcal{U}$ contains the supersets of its elements.

If $\mathcal{U}$ is a uniformity on X, then $(X,\mathcal{U})$ (shortly denoted by X) is called a uniform space.

If d is a metric on a nonempty set X, then it induces a uniformity, called the uniformity induced by the metric d, in which the entourages of X are all the supersets of the sets

$$\big\{(x,y) \in X \times X : d(x,y) < \varepsilon\big\},$$

where $\varepsilon > 0$.

It is well-known that a uniformity $\mathcal{U}$ on a nonempty set X is separating if the intersection of all entourages of X coincides with the diagonal $\{(x,x) : x \in X\}$. In this case, X is called a separated uniform space.

We next recall some basic concepts about E-distances. For more details and examples, the reader is referred to [1].

Definition 1.1. [1] *Let X be a uniform space. A function $p : X \times X \to \mathbb{R}^{\geq 0}$ is called an E-distance on X if*

(1) *for each entourage U in $\mathcal{U}$, there exists a $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $(x,y) \in U$ for all $x,y,z \in X$;*
(2) *p satisfies the triangular inequality, i.e.,*

$$p(x,y) \leq p(x,z) + p(z,y) \qquad (x,y,z \in X).$$

If p is an E-distance on a uniform space X, then a sequence $\{x_n\}$ in X is said to be p-convergent to a point $x \in X$, denoted by $x_n \xrightarrow{p} x$, whenever $p(x_n,x) \to 0$ as $n \to \infty$, and X is p-Cauchy whenever

$p(x_m, x_n) \to 0$ as $m, n \to \infty$. The uniform space X is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X.

The next lemma contains an important property of E-distances on separated uniform spaces. The proof is straightforward and it is omitted here.

Lemma 1.2. [1] *Let $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences in a separated uniform space X equipped with an E-distance p. If $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p} y$, then $x = y$. In particular, $p(z, x) = p(z, y) = 0$ for some $z \in X$ implies $x = y$.*

Using E-distances, p-boundedness and p-continuity are defined in uniform spaces.

Definition 1.3. [1] *Let p be an E-distance on a uniform space X. Then*

(1) *X is called p-bounded if*

$$\delta_p(X) = \sup\{p(x, y) : x, y \in X\} < \infty.$$

(2) *A mapping $T : X \to X$ is called p-continuous on X if $x_n \xrightarrow{p} x$ implies $Tx_n \xrightarrow{p} Tx$ for all sequences $\{x_n\}$ and all points x in X.*

2. E-asymptotic contractions

In this section, we denote by Φ the class of all functions $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with the following properties:

- φ is continuous on $\mathbb{R}^{\geq 0}$;
- $\varphi(t) < t$ for all $t > 0$.

It is worth mentioning that if $\varphi \in \Phi$, then

$$0 \leq \varphi(0) = \lim_{t \to 0^+} \varphi(t) \leq \lim_{t \to 0^+} t = 0,$$

that is, $\varphi(0) = 0$.

Following [5, Definition 2.1], we define E-asymptotic contractions.

Definition 2.1. *Let p be an E-distance on a uniform space X. We say that a mapping $T : X \to X$ is an E-asymptotic contraction if*

$$p(T^n x, T^n y) \leq \varphi_n\big(p(x, y)\big) \quad \text{for all } x, y \in X \text{ and } n \geq 1, \tag{2.1}$$

where $\{\varphi_n\}$ is a sequence of nonnegative functions on $\mathbb{R}^{\geq 0}$ converging uniformly to some $\varphi \in \Phi$ on the range of p.

If (X, d) is a metric space, then replacing the E-distance p by the metric d in Definition 2.1, we get the concept of an asymptotic contraction introduced by Kirk [5, Definition 2.1]. So each asymptotic contraction on a metric space is an E-asymptotic contraction on the uniform space induced by the metric. But in the next example, we see that the converse is not generally true.

Example 2.2. *Uniformize the set* $X = [0, 1]$ *with the uniformity induced from the Euclidean metric and put* $p(x, y) = y$ *for all* $x, y \in X$. *It is easily verified that* p *is an* E*-distance on* X. *Define* $T : X \to X$ *and* $\varphi_1 : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ *by*

$$Tx = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{8} & x = 1 \end{cases} \quad \textit{and} \quad \varphi_1(t) = \begin{cases} \frac{1}{16} & 0 \leq t < 1 \\ \frac{1}{8} & t \geq 1 \end{cases}$$

for all $x \in X$ *and all* $t \geq 0$, *and set* $\varphi_n = \varphi$ *for* $n \geq 2$, *where* φ *is any arbitrary fixed function in* Φ. *Clearly,* $\varphi_n \to \varphi$ *uniformly on* $\mathbb{R}^{\geq 0}$ *and* $T^n = 0$ *for all* $n \geq 2$. *To see that* T *is an* E*-asymptotic contraction on* X, *it suffices to check* (2.1) *for* $n = 1$. *To this end, given* $x, y \in [0, 1]$, *if* $y = 1$, *then we have*

$$p(Tx, T1) = T1 = \frac{1}{8} = \varphi_1(1) = \varphi_1\big(p(x, 1)\big),$$

and for $0 \leq y < 1$, *we have*

$$p(Tx, Ty) = Ty = 0 \leq \frac{1}{16} = \varphi_1(y) = \varphi_1\big(p(x, y)\big).$$

But T *fails to be an asymptotic contraction on the metric space* X *with the functions* φ_n *since*

$$\Big|T1 - T\frac{1}{2}\Big| = \frac{1}{8} > \frac{1}{16} = \varphi_1\big(\frac{1}{2}\big) = \varphi_1\Big(\big|1 - \frac{1}{2}\big|\Big).$$

In the next example, we see that an E-asymptotic contraction need not be p-continuous.

Example 2.3. *Let* X *and* p *be as in Example 2.2. Define a mapping* $T : X \to X$ *by* $Tx = 0$ *if* $0 < x \leq 1$ *and* $T0 = 1$. *Note that* T *is fixed point free. Now, let* φ_1 *be the constant function* 1 *and* $\varphi_2 = \varphi_3 = \cdots = \varphi$, *where* φ *is an arbitrary fixed function in* Φ. *Then* T *satisfies* (2.1) *and since* $T0 \neq 0$, *it follows that* T *fails to be* p*-continuous on* X.

Theorem 2.4. *Let p be an E-distance on a separated uniform space X such that X is p-complete and let $T : X \to X$ be a p-continuous E-asymptotic contraction for which the functions φ_n in Definition 2.1 are all continuous on $\mathbb{R}^{\geq 0}$ for large indices n. Then T has a unique fixed point $u \in X$, and $T^n x \xrightarrow{p} u$ for all $x \in X$.*

Proof. We divide the proof into three steps.

Step 1: $p(T^n x, T^n y) \to 0$ as $n \to \infty$ for all $x, y \in X$.
Let $x, y \in X$ be given. Letting $n \to \infty$ in (2.1), we get

$$0 \leq \limsup_{n\to\infty} p(T^n x, T^n y) \leq \lim_{n\to\infty} \varphi_n\big(p(x,y)\big) = \varphi\big(p(x,y)\big) \leq p(x,y) < \infty.$$

Now, if

$$\limsup_{n\to\infty} p(T^n x, T^n y) = \varepsilon > 0,$$

then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $p(T^{n_k} x, T^{n_k} y) \to \varepsilon$, and so by the continuity of φ, one obtains

$$\varphi\big(p(T^{n_k} x, T^{n_k} y)\big) \to \varphi(\varepsilon) < \varepsilon.$$

Therefore, there is an integer $k_0 \geq 1$ such that $\varphi(p(T^{n_{k_0}} x, T^{n_{k_0}} y)) < \varepsilon$. So (2.1) yields

$$\begin{aligned}\varepsilon &= \limsup_{n\to\infty} p(T^n x, T^n y)\\ &= \limsup_{n\to\infty} p\big(T^n(T^{n_{k_0}} x), T^n(T^{n_{k_0}} y)\big)\\ &\leq \lim_{n\to\infty} \varphi_n\big(p(T^{n_{k_0}} x, T^{n_{k_0}} y)\big)\\ &= \varphi\big(p(T^{n_{k_0}} x, T^{n_{k_0}} y)\big) < \varepsilon,\end{aligned}$$

which is a contradiction. Hence

$$\limsup_{n\to\infty} p(T^n x, T^n y) = 0.$$

Consequently,

$$0 \leq \liminf_{n\to\infty} p(T^n x, T^n y) \leq \limsup_{n\to\infty} p(T^n x, T^n y) = 0,$$

that is, $p(T^n x, T^n y) \to 0$.

Step 2: The sequence $\{T^n x\}$ is p-Cauchy for all $x \in X$.
Suppose that $x \in X$ is arbitrary. If $\{T^n x\}$ is not p-Cauchy, then there exist $\varepsilon > 0$ and positive integers m_k and n_k such that

$$m_k > n_k \geq k \quad \text{and} \quad p(T^{m_k} x, T^{n_k} x) \geq \varepsilon \qquad k = 1, 2, \ldots .$$

Keeping the integer n_k fixed for sufficiently large k, say $k \geq k_0$, and using Step 1, we may assume without loss of generality that $m_k > n_k$ is the smallest integer with $p(T^{m_k}x, T^{n_k}x) \geq \varepsilon$, that is,

$$p(T^{m_k-1}x, T^{n_k}x) < \varepsilon.$$

Hence for each $k \geq k_0$, we have

$$\begin{aligned}\varepsilon &\leq p(T^{m_k}x, T^{n_k}x)\\ &\leq p(T^{m_k}x, T^{m_k-1}x) + p(T^{m_k-1}x, T^{n_k}x)\\ &< p(T^{m_k}x, T^{m_k-1}x) + \varepsilon.\end{aligned}$$

If $k \to \infty$, since $p(T^{m_k}x, T^{m_k-1}x) \to 0$, it follows that $p(T^{m_k}x, T^{n_k}x) \to \varepsilon$.

We next show by induction that

$$\limsup_{k\to\infty} p(T^{m_k+i}x, T^{n_k+i}x) \geq \varepsilon, \qquad i = 1, 2, \ldots . \tag{2.2}$$

To this end, note first that from Step 1,

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} p(T^{m_k}x, T^{n_k}x) = \limsup_{k\to\infty} p(T^{m_k}x, T^{n_k}x)\\ &\leq \limsup_{k\to\infty} \Big[p(T^{m_k}x, T^{m_k+1}x) + p(T^{m_k+1}x, T^{n_k+1}x)\\ &\quad + p(T^{n_k+1}x, T^{n_k}x)\Big]\\ &\leq \limsup_{k\to\infty} p(T^{m_k}x, T^{m_k+1}x) + \limsup_{k\to\infty} p(T^{m_k+1}x, T^{n_k+1}x)\\ &\quad + \limsup_{k\to\infty} p(T^{n_k+1}x, T^{n_k}x)\\ &= \limsup_{k\to\infty} p(T^{m_k+1}x, T^{n_k+1}x),\end{aligned}$$

that is, (2.2) holds for $i = 1$. If (2.2) is true for some i, then

$$\begin{aligned}\varepsilon &\leq \limsup_{k\to\infty} p(T^{m_k+i}x, T^{n_k+i}x)\\ &\leq \limsup_{k\to\infty} \Big[p(T^{m_k+i}x, T^{m_k+i+1}x) + p(T^{m_k+i+1}x, T^{n_k+i+1}x)\\ &\quad + p(T^{n_k+i+1}x, T^{n_k+i}x)\Big]\\ &\leq \limsup_{k\to\infty} p(T^{m_k+i+1}x, T^{n_k+i+1}x).\end{aligned}$$

Consequently, we have

$$\varphi(\varepsilon) = \lim_{k\to\infty} \varphi\big(p(T^{m_k}x, T^{n_k}x)\big)$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\lim_{i\to\infty}\varphi_i\big(p(T^{m_k}x,T^{n_k}x)\big)\\
&= \lim_{i\to\infty}\lim_{k\to\infty}\varphi_i\big(p(T^{m_k}x,T^{n_k}x)\big)\\
&\geq \limsup_{i\to\infty}\limsup_{k\to\infty} p(T^{m_k+i}x,T^{n_k+i}x)\\
&\geq \varepsilon,
\end{aligned}
$$

where the first equality holds because φ is continuous, the second equality holds because $\{\varphi_i\}$ is pointwise convergent to φ on the range of p, the third equality holds because $\{\varphi_i\}$ is uniformly convergent to φ on the range of p, and the last two inequalities hold by (2.1) and (2.2), respectively. Hence $\varphi(\varepsilon)\geq\varepsilon$, which is a contradiction. Therefore $\{T^n x\}$ is p-Cauchy.

Step 3: T has a unique fixed point.

Because X is p-complete, it is concluded from Steps 1 and 2 that the family $\{\{T^n x\} : x\in X\}$ of Picard iterates of T is p-equiconvergent, that is, there exists $u\in X$ such that $T^n x \xrightarrow{p} u$ for all $x\in X$. In particular, $T^n u\xrightarrow{p} u$. We claim that u is the unique fixed point for T. To this end, first note that since T is p-continuous on X, it follows that $T^{n+1}u\xrightarrow{p} Tu$, and so, by Lemma 1.2, we have $u=Tu$. And if $v\in X$ is a fixed point for T, then

$$p(u,v)=\lim_{n\to\infty}p(T^n u,T^n v)\leq\lim_{n\to\infty}\varphi_n\big(p(u,v)\big)=\varphi\big(p(u,v)\big),$$

which is impossible unless $p(u,v)=0$. Similarly $p(u,u)=0$ and using Lemma 1.2 once more, we get $v=u$. □

It is worth mentioning that the boundedness of some orbit of T is not necessary in Theorem 2.4 unlike [5, Theorem 2.1] or [2, Theorem 4.1.15].

As a consequence of Theorem 2.4, we have the following version of [1, Theorem 3.1].

Corollary 2.5. *Let p be an E-distance on a separated uniform space X such that X is p-complete and p-bounded and let a mapping $T:X\to X$ satisfy*

$$p(Tx,Ty)\leq\varphi\big(p(x,y)\big)\quad \text{for all } x,y\in X, \tag{2.3}$$

where $\varphi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ is nondecreasing and continuous with $\varphi^n(t)\to 0$ for all $t>0$. Then T has a unique fixed point $u\in X$, and $T^n x\xrightarrow{p} u$ for all $x\in X$.

Proof. Note first that $\varphi(0) = 0$; for if $0 < t < \varphi(0)$ for some t, then the monotonicity of φ implies that $0 < t < \varphi(0) \leq \varphi^n(t)$ for all $n \geq 1$, which contradicts the fact that $\varphi^n(t) \to 0$.

Next, since φ is nondecreasing, it follows that T satisfies

$$p(T^n x, T^n y) \leq \varphi^n\big(p(x,y)\big) \quad \text{for all } x, y \in X \text{ and } n \geq 1.$$

Setting $\varphi_n = \varphi^n$ for each $n \geq 1$ in Definition 2.1, it is seen that $\{\varphi_n\}$ converges pointwise to the constant function 0 on $[0, +\infty)$, and since

$$\sup\Big\{\varphi^n\big(p(x,y)\big) : x, y \in X\Big\} = \varphi^n\big(\delta_p(X)\big) \to 0,$$

it follows that $\{\varphi_n\}$ converges uniformly to 0 on the range of p. Because the constant function 0 belongs to Φ, it is concluded that T is an E-asymptotic contraction on X. Moreover, φ_n's are all continuous on $\mathbb{R}^{\geq 0}$ and (2.3) ensures that T is p-continuous on X. Consequently, the result follows immediately from Theorem 2.4. □

The next corollary is a partial modification of Kirk's theorem [5, Theorem 2.1] in uniform spaces. One can find it with an additional assumption, e.g., in [2, Theorem 4.1.15].

Corollary 2.6. *Let X be a complete metric space and let $T : X \to X$ be a continuous asymptotic contraction for which the functions φ_n in Definition 2.1 are all continuous on $\mathbb{R}^{\geq 0}$ for large indices n. Then T has a unique fixed point $u \in X$, and $T^n x \to u$ for all $x \in X$.*

3. Boyd-Wong type E-contractions

In this section, we denote by Ψ the class of all functions $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with the following properties:

- ψ is upper semicontinuous on $\mathbb{R}^{\geq 0}$ from the right, i.e.,

$$t_n \downarrow t \geq 0 \quad \text{implies} \quad \limsup_{n\to\infty} \psi(t_n) \leq \psi(t);$$

- $\psi(t) < t$ for all $t > 0$, and $\psi(0) = 0$.

It might be interesting for the reader to be mentioned that the family Φ defined and used in Section 2 is contained in the family Ψ but these two families do not coincide. To see this, consider the function $\psi(t) = 0$ if $0 \leq t < 1$, and $\psi(t) = \frac{1}{2}$ if $t \geq 1$. Then ψ is upper semicontinuous from the right but it is not continuous on $\mathbb{R}^{\geq 0}$. Furthermore, the upper semicontinuity of ψ on $\mathbb{R}^{\geq 0}$ from the right and the condition that $\psi(t) <$

t for all $t > 0$, do not imply that ψ vanishes at zero in general. In fact, the function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ defined by the rule

$$\psi(t) = \begin{cases} a & t = 0 \\ \frac{t}{2} & 0 < t < 1 \\ \frac{1}{2t} & t \geq 1 \end{cases}$$

for all $t \geq 0$, where a is an arbitrary positive real number, confirms this claim.

Theorem 3.1. *Let p be an E-distance on a separated uniform space X such that X is p-complete and let $T : X \to X$ satisfy*

$$p(Tx, Ty) \leq \psi\big(p(x, y)\big) \quad \text{for all } x, y \in X, \tag{3.1}$$

where $\psi \in \Psi$. Then T has a unique fixed point $u \in X$, and $T^n x \overset{p}{\longrightarrow} u$ for all $x \in X$.

Proof. We divide the proof into three steps as Theorem 2.4.

Step 1: $p(T^n x, T^n y) \to 0$ as $n \to \infty$ for all $x, y \in X$.

Let $x, y \in X$ be given. Then for each nonnegative integer n, by the contractive condition (3.1) we have

$$p(T^{n+1}x, T^{n+1}y) \leq \psi\big(p(T^n x, T^n y)\big) \leq p(T^n x, T^n y). \tag{3.2}$$

Thus, $\{p(T^n x, T^n y)\}$ is a nonincreasing sequence of nonnegative numbers and so it converges decreasingly to some $\alpha \geq 0$. Letting $n \to \infty$ in (3.2), by the upper semicontinuity of ψ from the right, we get

$$\alpha = \lim_{n\to\infty} p(T^{n+1}x, T^{n+1}y) \leq \limsup_{n\to\infty} \psi\big(p(T^n x, T^n y)\big) \leq \psi(\alpha),$$

which is a contradiction unless $\alpha = 0$. Consequently, $p(T^n x, T^n y) \to 0$.

Step 2: The sequence $\{T^n x\}$ is p-Cauchy for all $x \in X$.

Let $x \in X$ be arbitrary and suppose on the contrary that $\{T^n x\}$ is not p-Cauchy. Then similar to the proof of Step 2 of Theorem 2.4, it is seen that there exist an $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k$ for each k and $p(T^{m_k}x, T^{n_k}x) \to \varepsilon$. On the other hand, for each k, by (3.1) we have

$$\begin{aligned} p(T^{m_k}x, T^{n_k}x) \leq\ & p(T^{m_k}x, T^{m_k+1}x) + p(T^{m_k+1}x, T^{n_k+1}x) \\ & + p(T^{n_k+1}x, T^{n_k}x) \end{aligned}$$

$$\leq p(T^{m_k}x, T^{m_k+1}x) + \psi\big(p(T^{m_k}x, T^{n_k}x)\big)$$
$$+ p(T^{n_k+1}x, T^{n_k}x).$$

Letting $k \to \infty$ and using Step 1 and the upper semicontinuity of ψ from the right, we obtain

$$\begin{aligned}
\varepsilon &= \lim_{k\to\infty} p(T^{m_k}x, T^{n_k}x) = \limsup_{k\to\infty} p(T^{m_k}x, T^{n_k}x) \\
&\leq \limsup_{k\to\infty} \Big[p(T^{m_k}x, T^{m_k+1}x) + \psi\big(p(T^{m_k}x, T^{n_k}x)\big) \\
&\quad + p(T^{n_k+1}x, T^{n_k}x)\Big] \\
&\leq \limsup_{k\to\infty} p(T^{m_k}x, T^{m_k+1}x) + \limsup_{k\to\infty} \psi\big(p(T^{m_k}x, T^{n_k}x)\big) \\
&\quad + \limsup_{k\to\infty} p(T^{n_k+1}x, T^{n_k}x) \\
&= \limsup_{k\to\infty} \psi\big(p(T^{m_k}x, T^{n_k}x)\big) \\
&\leq \psi(\varepsilon),
\end{aligned}$$

which is a contradiction. Therefore, $\{T^n x\}$ is p-Cauchy.

Step 3: T has a unique fixed point.

Since X is p-complete, it follows from Steps 1 and 2 that the family $\{\{T^n x\} : x \in X\}$ is p-equiconvergent to some $u \in X$. In particular, $T^n u \xrightarrow{p} u$. Since (3.1) implies the p-continuity of T on X, it follows that $T^{n+1}u \xrightarrow{p} Tu$ and so, by Lemma 1.2, we have $u = Tu$, that is, u is a fixed point for T. If $v \in X$ is a fixed point for T, then

$$p(u, v) = p(Tu, Tv) \leq \psi\big(p(u, v)\big),$$

which is impossible unless $p(u, v) = 0$. Similarly $p(u, u) = 0$. Therefore, using Lemma 1.2 once more, one gets $v = u$. □

As an immediate consequence of Theorem 3.1, we have the Boyd-Wong's theorem [3] in metric spaces:

Corollary 3.2. *Let X be a complete metric space and let a mapping $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big) \quad \textit{for all } x, y \in X, \tag{3.3}$$

where $\psi \in \Psi$. Then T has a unique fixed point $u \in X$, and $T^n x \to u$ for all $x \in X$.

In the following example, we see that Theorem 3.1 guarantees the existence and uniqueness of a fixed point while Corollary 3.2 cannot be applied.

Example 3.3. *Let the set* $X = [0,1]$ *be endowed with the uniformity induced by the Euclidean metric and define a mapping* $T : X \to X$ *by* $Tx = 0$ *if* $0 \le x < 1$, *and* $T1 = \frac{1}{4}$. *Then* T *does not satisfy* (3.3) *for any* $\psi \in \Psi$ *since it is not continuous on* X. *In fact, if* $\psi \in \Psi$ *is arbitrary, then*

$$\Big|T1 - T\frac{3}{4}\Big| = \frac{1}{4} > \psi(\frac{1}{4}) = \psi\Big(\big|1 - \frac{3}{4}\big|\Big).$$

Now set $p(x,y) = \max\{x,y\}$. *Then* p *is an* E*-distance on* X *and* T *satisfies* (3.1) *for the function* $\psi : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ *defined by the rule* $\psi(t) = \frac{t}{4}$ *for all* $t \ge 0$. *It is easy to check that this* ψ *belongs to* Ψ, *and the hypotheses of Theorem 3.1 are fulfilled.*

Remark 3.4. *In Theorem 2.4 (Corollary 2.6), assume that for some index* k *the function* φ_k *belongs to* Φ. *Then Theorem 3.1 (Corollary 3.2) implies that* T^k *and so* T *has a unique fixed point* u *and* $T^{kn}x \xrightarrow{p} u$ *for all* $x \in X$. *So, it is concluded by the* p*-continuity of* T *that the family* $\{\{T^n x\} : x \in X\}$ *is* p*-equiconvergent to* u. *Hence the significance of Theorem 2.4 (Corollary 2.6) is whenever none of* φ_n*'s satisfy* $\varphi_n(t) < t$ *for all* $t > 0$, *that is, whenever for each* $n \ge 1$ *there exists a* $t_n > 0$ *such that* $\varphi_n(t_n) \ge t_n$.

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