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A LOWER ESTIMATE OF HARMONIC FUNCTIONS

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Abstract. We shall give a lower estimate of harmonic functions of order greater than one in a half space, which generalize the result obtained by B. Ya. Levin in a half plane.

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1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : \ x_n > 0\}$, whose boundary is ∂H.

For a set E, $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H : |x| \in E\}$ and $\{x \in \partial H : |x| \in E\}$ by HE and ∂HE, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x,\ y \in \mathbf{R}^n$ as $x = (x', x_n),\ y = (y', y_n)$, where $y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'},$$

$$|x'| = |x| \cos\theta \text{ and } x_n = |x| \sin\theta \ (0 < \theta \leq \pi/2).$$

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Let B_r denote the open ball with center at the origin and radius r (> 0) in $\mathbf{R}^n$. We use the standard notations $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. In the sense of Lebesgue measure $dy' = dy_1 \cdots dy_{n-1}$ and $dy = dy' dy_n$. Let σ denote $(n-1)$-dimensional surface area measure and $\partial/\partial n$ denote differentiation along the inward normal into H.

The estimate we deal with has a long history which can be traced back to Levin's estimate of harmonic functions from below (see, for example, Levin [6, p. 209]).

Theorem 1.1. *Let A_1 be a constant, $u(z)$ harmonic in the upper half space $\mathbf{C}_+$ and continuous on $\partial\mathbf{C}_+$. Suppose that*

$$u(z) \le A_1 R^\rho, \quad z \in \mathbf{C}_+,\ R = |z| > 1,\ \rho > 1$$

and

$$|u(z)| \le A_1, \quad |z| \le 1,\ Imz \ge 0.$$

Then

$$u(Re^{i\varphi}) \ge -A_2 A_1 (1 + R^\rho) \sin^{-1}\varphi, \quad Re^{i\varphi} \in \mathbf{C}_+,$$

where A_2 is a constant independent of A_1, R, φ and the function $u(z)$.

Further versions and refinements of Theorem 1.1 may be found in the monograph Nikol'skiĭ [7, Ch. 1] and in the paper Krasichkov-Ternovskiĭ [3].

In this article, we will consider functions $u(x)$ harmonic in H and continuous on $\overline{H}$. In what follows we shall denote by M various values which does not depend on K, R $(= |x|)$, θ and the function $u(x)$.

In this note we prove analogous estimates for $u(x)$ in H.

Theorem 1.2. *Suppose that*

$$u(x) \le K R^{\rho(R)}, \quad x \in H,\ R = |x| > 1,\ \rho(R) > 1 \tag{1.1}$$

and

$$u(x) \ge -K, \quad |x| \le 1,\ x_n \ge 0. \tag{1.2}$$

Then

$$u(x) \ge -MK\left(1 + (2R)^{\rho(2R)}\right) \sin^{1-n}\theta,$$

where $x \in H$ and $\rho(R)$ is nondecreasing on $[1, +\infty)$.

Remark 1.3. *If $n = 2$ and $\rho(R) \equiv \rho$, Theorem 1.2 is just a consequence of Theorem 1.1.*

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Theorem 1.4. *If* (1.1) *and* (1.2) *hold, then*

$$u(x) \geq -MK\left(1+(\frac{N+1}{N}R)^{\rho(\frac{N+1}{N}R)}\right)\sin^{1-n}\theta,$$

where $x \in H$, $N(\geq 1)$ *is a sufficiently large number and* $\rho(R)$ *is as defined in Theorem 1.2.*

2. Lemmas

Carleman's formula [2] connects the modulus and the zeros of a function analytic in $\mathbf{C}_+$ (see, for example, [5, p. 224]). Nevanlinna's formula (see [6, p. 193]) refers to a harmonic function in a half disk. Armitage and Kuran obtained a generalized Nevanlinna-type formula in a half space and Poisson integral forumla for half balls resepectively, which play important roles in our discussions.

Lemma 2.1. ([1]). *If* $R > 1$, *then we have*

$$\int_{\{x\in H:\ |x|=R\}} u(x)\frac{nx_n}{R^{n+1}}d\sigma(x)+\int_{\partial H(1,R)} u(x')(\frac{1}{|x'|^n}-\frac{1}{R^n})dx' = c_1+\frac{c_2}{R^n},$$

where

$$c_1 = \int_{\{x\in H:|x|=1\}}\left((n-1)x_n u(x)+x_n\frac{\partial u(x)}{\partial n}\right)d\sigma(x)$$

and

$$c_2 = \int_{\{x\in H:|x|=1\}}\left(x_n u(x)-x_n\frac{\partial u(x)}{\partial n}\right)d\sigma(x).$$

Lemma 2.2. ([4]). *Let* $R > 1$, $u(x)$ *be a function in* $B_R^+ = B_R \cap H$ *and continuous in* $\overline{B}_R^+$. *Then*

$$\begin{aligned} u(x) &= \int_{\{y\in H:\ |y|=R\}} \frac{R^2-|x|^2}{\omega_n R}(\frac{1}{|y-x|^n}-\frac{1}{|y-x^*|^n})u(y)d\sigma(y) \\ &\quad +\frac{2x_n}{\omega_n}\int_{\partial H[0,R)}(\frac{1}{|y'-x|^n}-\frac{R^n}{|x|^n}\frac{1}{|y'-\widetilde{x}|^n})u(y')dy', \end{aligned}$$

where $x \in B_R^+$, $\widetilde{x} = R^2x/|x|^2$, $x^* = (x', -x_n)$ *and* $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ *is the volume of the unit* n*-ball in* $\mathbf{R}^n$.

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3. Proof of Theorem 1

By applying Lemma 2.1 to $u(x)$, we have

$$(3.1)\quad \int_{\{x\in H:\,|x|=R\}} u^+(x)\frac{nx_n}{R^{n+1}}d\sigma(x)+\int_{\partial H(1,R)} u^+(x')(\frac{1}{|x'|^n}-\frac{1}{R^n})dx'$$
$$= \int_{\{x\in H:\,|x|=R\}} u^-(x)\frac{nx_n}{R^{n+1}}d\sigma(x)$$
$$+\int_{\partial H(1,R)} u^-(x')(\frac{1}{|x'|^n}-\frac{1}{R^n})dx'+c_1+\frac{c_2}{R^n}.$$

It immediately follows from (1.1) that

$$(3.2)\quad \int_{\{x\in H:\,|x|=R\}} u^+(x)\frac{nx_n}{R^{n+1}}d\sigma(x)\le MKR^{\rho(R)-1}$$

and

$$(3.3)\quad \int_{\partial H(1,R)} u^+(x')(\frac{1}{|x'|^n}-\frac{1}{R^n})dx'\le MKR^{\rho(R)-1}.$$

Hence from (3.1), (3.2) and (3.3) we have

$$(3.4)\quad \int_{\{x\in H:\,|x|=R\}} u^-(x)\frac{nx_n}{R^{n+1}}d\sigma(x)\le MKR^{\rho(R)-1}$$

and

$$(3.5)\quad \int_{\partial H(1,R)} u^-(x')(\frac{1}{|x'|^n}-\frac{1}{R^n})dx'\le MKR^{\rho(R)-1}.$$

And (3.5) gives

$$\int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n}dx'$$
$$\le \frac{2^n}{2^n-1}\int_{\partial H(1,R)} u^-(x')\left(\frac{1}{|x'|^n}-\frac{1}{(2R)^n}\right)dx'$$
$$(3.6)\quad \le MK(2R)^{\rho(2R)-1}.$$

Since $-u(x)\le u^-(x)$, by applying Lemma 2.2 to $-u(x)$ we have

$$(3.7)\quad -u(x)\le I_1(x)+I_2(x),$$

where

$$I_1(x)=\int_{\{y\in H:\,|y|=R\}} \frac{R^2-|x|^2}{\omega_n R}(\frac{1}{|y-x|^n}-\frac{1}{|y-x^*|^n})u^-(y)d\sigma(y)$$

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and

$$I_2(x) = \frac{2x_n}{\omega_n} \int_{\partial H[0,R)} (\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\widetilde{x}|^n}) u^-(y') dy'.$$

We remark that

$$\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \leq \frac{2nx_n y_n}{|y-x|^{n+2}} \tag{3.8}$$

and

$$|y-x|^n \geq x_n^n = |x|^n \sin^n \theta, \quad x \in H,\ y_n = 0. \tag{3.9}$$

If we put $|x| = r > 1/2$ and $R = 2r$ in (3.7), then we finally have from (3.4), (3.8) and (3.9)

$$\begin{aligned} I_1(x) &\leq \int_{\{y \in H:\ |y|=R\}} \frac{R^2 - r^2}{\omega_n R} \frac{2nx_n y_n}{\omega_n |y-x|^{n+2}} u^-(y) d\sigma(y) \\ &\leq MKR^{\rho(R)} \end{aligned} \tag{3.10}$$

and

$$I_2(x) \leq I_{21}(x) + I_{22}(x), \tag{3.11}$$

where

$$I_{21}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} u^-(y') dy'$$

and

$$I_{22}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} u^-(y') dy'.$$

We obtain that

$$\begin{aligned} I_{21}(x) &\leq \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} \frac{u^-(y')}{|y'|^n} dy' \\ &\leq MK(2R)^{\rho(2R)} \sin^{1-n} \theta \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_{22}(x) &\leq \frac{2K}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} dy' \\ &\leq MK \sin^{1-n} \theta \end{aligned} \tag{3.13}$$

from (3.6) and (1.2), respectively.

From (3.7), (3.10), (3.11), (3.12) and (3.13), we have for $|x| > 1/2$

$$-u(x) \leq MK \left(1 + (2R)^{\rho(2R)}\right) \sin^{1-n} \theta. \tag{3.14}$$

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For $|x| \leq 1/2$, we have from (1.2)

$$-u(x) \leq K \leq K\left(1+(2R)^{\rho(2R)}\right)\sin^{1-n}\theta. \tag{3.15}$$

Thus the conclusion immediately follows from (3.14) and (3.15).

4. **Proof of Theorem 2**

By modifying (3.6), we have

$$\begin{aligned}
&\int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n}dx' \\
\leq\ & \frac{(N+1)^n}{(N+1)^n-N^n}\int_{\partial H(1,R)} u^-(x')\left(\frac{1}{|x'|^n}-\frac{1}{(\frac{N+1}{N}R)^n}\right)dx' \\
\leq\ & MK(\frac{N+1}{N}R)^{\rho(\frac{N+1}{N}R)-1}.
\end{aligned}$$

Then (3.12), (3.14) and (3.15) are replaced by the following estimates

$$I_{21}(x) \leq MK\left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)-1}\sin^{1-n}\theta. \tag{4.1}$$

$$-u(x) \leq MK\left(1+(\frac{N+1}{N}R)^{\rho(\frac{N+1}{N}R)}\right)\sin^{1-n}\theta. \tag{4.2}$$

$$-u(x) \leq K \leq MK\left(1+(\frac{N+1}{N}R)^{\rho(\frac{N+1}{N}R)}\right)\sin^{1-n}\theta. \tag{4.3}$$

All (3.7), (3.10), (3.11), (4.1), (3.12), (4.2) and (4.3) give

$$u(x) \geq -MK\left(1+(\frac{N+1}{N}R)^{\rho(\frac{N+1}{N}R)}\right)\sin^{1-n}\theta$$

from which the conclusion immediately follows.

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