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COUPLED FIXED POINT RESULTS FOR WEAKLY RELATED MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In the present paper, we show the existence of a coupled fixed point for a non-decreasing mapping in partially ordered complete metric space using a partial order induced by an appropriate function ϕ. We also define the concept of weakly related mappings on an ordered space. Moreover common coupled fixed points for two and three weakly related mappings are also proved in the same space.
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MSC(2010): Primary: 47H10; Secondary: 54H25.

1. Introduction and Preliminaries

Fixed point theory plays a major role in many applications including variational and linear inequalities, optimization and applications in the field of approximation theory and minimum norm problem. The Banach contraction principle which is the most celebrated metrical fixed point theorem, plays a very important role in nonlinear analysis. It has been generalized in several directions, see for example [3] and [10]. Another recent direction of such generalizations (see [2, 6, 9]), has been obtained by weakening the requirements in the contractive condition and in compensation, by simultaneously enriching the metric space structure with

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a partial order. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity ([2, 5–8]).

Bhashkar and Lakshmikantham [4] introduced the concept of a coupled fixed point of a mapping $F : X \times X \longrightarrow X$ (a nonempty set) and established some coupled fixed point theorems in partially ordered complete metric spaces. They noted that their theorem can be used to investigate a large class of problems and discussed the existence and uniqueness of solution for a periodic boundary value problem. Later, Lakshmikantham and Ciric [6] proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ satisfying certain contractive conditions in partially ordered complete metric spaces. Abbas et. al. [1] proved some coupled fixed point results for nonlinear contraction mappings having a mixed monotone property in a partially ordered G-metric space. Sintunavarat et. al. [11] extended the results of the coupled fixed point theorems by weakening the concept of the mixed monotone property. Also Sintunavarat et. al. [12] established the coupled fixed point theorems for nonlinear contraction mappings which have a mixed monotone property by using the cone ball-metric. Recently Sintunavarat et. al. [13] shown the existence of a coupled fixed point theorem of nonlinear contraction mappings in complete metric spaces without the mixed monotone property.

Our technique of proof is essentially different and more natural. In our work, we show the existence of a coupled fixed point for a non-decreasing mapping in partially ordered complete metric space using a partial order induced by an appropriate function ϕ without using any contractive condition. Some examples are also given in order to illustrate our results. Moreover common coupled fixed point for two and three mappings satisfying the weakly related property are also proved in the same space.

Recall that if $(X, \preceq)$ is a partially ordered set and $T : X \longrightarrow X$ is a map such that for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$ then T is said to be non-decreasing. Similarly, a mapping $T : X \times X \longrightarrow X$ is said to be non-decreasing, if for (x_1, y_1), $(x_2, y_2) \in X \times X$ and $x_1 \preceq x_2$, $y_1 \preceq y_2$ implies $T(x_1, y_1) \preceq T(x_2, y_2)$.

Definition 1.1. *[3]. An element $(x, y) \in X \times X$ is called:*
(C1) a coupled fixed point of mapping $T : X \times X \longrightarrow X$ if $x = T(x, y)$, $y = T(y, x)$;
(C2) a coupled coincidence point of mappings $T : X \times X \longrightarrow X$ and $f : X \longrightarrow X$ if $f(x) = T(x, y)$ and $f(y) = T(y, x)$, and in this case (fx, fy) is called coupled point of coincidence;
(C3) a common coupled fixed point of mappings $T : X \times X \longrightarrow X$ and $f : X \longrightarrow X$ if $x = f(x) = T(x, y)$ and $y = f(y) = T(y, x)$.

Definition 1.2. *An element $(x, y) \in X \times X$ is called:*
(CC1) a common coupled coincidence point of the mappings $T : X \times X \longrightarrow X$ and $f, g : X \longrightarrow X$ if $T(x, y) = fx = gx$ and $T(y, x) = fy = gy$;
(CC2) a common coupled fixed point of mappings $T : X \times X \longrightarrow X$ and $f, g : X \longrightarrow X$ if $T(x, y) = fx = gx = x$ and $T(y, x) = fy = gy = y$.

Our approach brings at least some new features to the coupled fixed point theory:

1. We need not use any contractive condition satisfied by the mapping.
2. Our technique of proof is simpler and essentially different from the ones used in the numerous papers devoted to coupled fixed point.
3. We have used more natural mappings (monotonically non-decreasing) than that of in the previous papers (mixed monotone property) of coupled fixed points.

2. Coupled fixed point

We first prove the following lemma:

Lemma 2.1. *Let (X, d) be a metric space and $\phi : X \longrightarrow R$ a map. Define the relation " $\preceq$ " on X as follows:*

$$x \preceq y \Leftrightarrow d(x, y) \leq \phi(y) - \phi(x).$$

Then " $\preceq$ " is partial order on X, called the partial order induced by ϕ.

Proof. For all $x \in X$, $d(x, x) = 0 = \phi(x) - \phi(x)$ then $x \preceq x$ that is " $\preceq$ " is reflexive. Now for $x, y \in X$ s.t. $x \preceq y$ and $y \preceq x$, we have

$$d(x, y) \leq \phi(y) - \phi(x)$$

and

$$d(y, x) \leq \phi(x) - \phi(y).$$

This shows that $d(x, y) = 0$ i.e. $x = y$. Thus " $\preceq$ " is antisymmetric. Again for $x, y, z \in X$ s.t. $x \preceq y$ and $y \preceq z$, we have

$$d(x, y) \leq \phi(y) - \phi(x)$$

and

$$d(y, z) \leq \phi(z) - \phi(y)$$

and so

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &\leq \phi(y) - \phi(x) + \phi(z) - \phi(y) \\ &= \phi(z) - \phi(x). \end{aligned}$$

Then $x \preceq z$. Thus " $\preceq$ " is transitive, and so the relation " $\preceq$ " is a partial order on X. □

Example 2.2. *Let $X = [0, \infty)$ and $d(x, y) = |x - y|$, then (X, d) is a metric space. Let $\phi : X \longrightarrow R$, $\phi(x) = 2x$. Then for $x, y \in X$*

$$\begin{aligned} x \leq y &\Leftrightarrow d(x, y) \leq \phi(y) - \phi(x) \\ &\Leftrightarrow |x - y| \leq 2y - 2x. \end{aligned}$$

It follows that $1 \preceq 2$, $1/2 \preceq 1$, $1 \preceq 1$, $3 \preceq 5$ whereas 3 is not comparable to 1 and 6 is not comparable to 5 etc. Therefore X is a partially ordered space.

Now we prove the following theorem.

Theorem 2.3. *Let (X, d) be a complete metric space, $\phi : X \longrightarrow R$ be a bounded from above function and " $\preceq$ " be the partial order induced by ϕ. Let $F : X \times X \longrightarrow X$ be a nondecreasing continuous mapping on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \preceq F(y_0, x_0).$$

Then F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{2.1}$$

We shall show that

$$x_n \preceq x_{n+1} \; for \; all \; n \geq 0, \tag{2.2}$$

and

$$y_n \preceq y_{n+1} \; for \; all \; n \geq 0. \tag{2.3}$$

We shall use the mathematical induction. Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$ we have $x_0 \preceq x_1$ and $y_0 \preceq y_1$. Thus (2.2) and (2.3) hold for $n = 0$. Suppose now that (2.2) and (2.3) hold for some fixed $n \geq 0$. Then since $x_n \preceq x_{n+1}$ and $y_n \preceq y_{n+1}$ and F is nondecresing, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{2.4}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \succeq F(y_n, x_n) = y_{n+1}. \tag{2.5}$$

Thus by mathematical induction we conclude that (2.2) and (2.3) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \ldots x_n \preceq x_{n+1} \cdots$$

and

$$y_0 \preceq y_1 \preceq y_2 \preceq y_3 \ldots y_n \preceq y_{n+1} \cdots .$$

That is the sequences $\{x_n\}$ and $\{y_n\}$ are non-decreasing in X. By the definition of " $\preceq$ ", we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \phi(x_3) \cdots$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \phi(y_3) \cdots .$$

In other words, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are non-decreasing sequences in the set of real numbers. Since ϕ is bounded from above, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are convergent and hence are Cauchy. So for all $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$, we have $|\phi(x_m) - \phi(x_n)| = \phi(x_m) - \phi(x_n) < \epsilon$ and $|\phi(y_m) - \phi(y_n)| = \phi(y_m) - \phi(y_n) < \epsilon$. Since $x_n \preceq x_m$, it follows that

$$d(x_n, x_m) \leq \phi(x_m) - \phi(x_n) < \epsilon$$

and since $y_n \preceq y_m$, it follows that

$$d(y_n, y_m) \leq \phi(y_m) - \phi(y_n) < \epsilon.$$

This shows that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in X and since X is complete, there exist points $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Consequently, taking the limit as $n \to \infty$ in (2.1) and using the continuity of F, we get

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) = F(y, x).$$

Thus we have proved that $x = F(x, y)$ and $y = F(y, x)$. Hence (x, y) is a coupled fixed point of F. □

Example 2.4. *Let $X = [0, \infty)$ and $d(x, y) = |x - y|$, then (X, d) is a complete metric space and " $\leq$ " is the usual ordering. We define $\phi : X \longrightarrow R$ as follows:*

$$\phi(x) = 2x$$

and $F : X \times X \longrightarrow X$ as follows:

$$F(x, y) = x(1 + y)$$

which is obviously a non-decreasing function on X.

If we let $x_0 = 1$ and $y_0 = 0$ then $F(x_0, y_0) = 1 \cdot (1 + 0) = 1$ and $F(y_0, x_0) = 0 \cdot (1 + 1) = 0$.

So we see that $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$. Also $F(0, y) = 0 \cdot (1 + y) = 0$ and $F(0, x) = 0 \cdot (1 + x) = 0$. Hence $(0, 0)$ is a coupled fixed point of F.

3. Common coupled fixed point

Now we define the concept of weakly related mappings on ordered spaces as follows:

Definition 3.1. *Let $(X, \preceq)$ be a partially ordered space, and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. Then the pair $\{F, g\}$ is said to be weakly related if $F(x, y) \preceq gF(x, y)$ and $gx \preceq F(gx, gy)$ also $F(y, x) \preceq gF(y, x)$ and $gy \preceq F(gy, gx)$ for all $(x, y) \in X \times X$.*

We prove the common coupled fixed point existence theorem for the weakly related mappings.

Theorem 3.2. *Let (X, d) be a complete metric space, $\phi : X \longrightarrow R$ be a bounded from above function and " $\preceq$ " be the partial order induced by ϕ. Let $F : X \times X \longrightarrow X$ and $G : X \longrightarrow X$ be two continuous mappings such that the pair $\{F, G\}$ is weakly related on X. If there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \, and \, y_0 \preceq F(y_0, x_0)$$

then F and G have a common coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ *and* $y_0 \preceq F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{2n+1} = F(x_{2n}, y_{2n}) \;\&\; x_{2n+2} = Gx_{2n+1} \tag{3.1}$$

and

$$y_{2n+1} = F(y_{2n}, x_{2n}) \;\&\; y_{2n+2} = Gy_{2n+1} \tag{3.2}$$

for all $n \geq 0$.

We shall show that

$$x_n \preceq x_{n+1} \, for \, all \, n \geq 0, \tag{3.3}$$

and

$$y_n \preceq y_{n+1} \, for \, all \, n \geq 0. \tag{3.4}$$

Since $x_0 \preceq F(x_0, y_0)$, using (3.1) we have $x_0 \preceq x_1$. Again since the pair $\{F, G\}$ is weakly related, we have $F(x_0, y_0) \preceq GF(x_0, y_0)$ i.e. $x_1 \preceq Gx_1$ and using (3.1) we get $x_1 \preceq x_2$. Also, since $Gx_1 \preceq F(Gx_1, Gy_1)$, using (3.1) we have $x_2 \preceq x_3$. Similarly using weakly related property for $\{F, G\}$ and repeated use of (3.1), we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \ldots x_n \preceq x_{n+1} \cdots .$$

Also, since $y_0 \preceq F(y_0, x_0)$, using (3.2) we have $y_0 \preceq y_1$. Again since the pair $\{F, G\}$ is weakly related, we have $F(y_0, x_0) \preceq GF(x_0, y_0)$ i.e. $y_1 \preceq Gy_1$ and using (3.2) we get $y_1 \preceq y_2$. Also, since $Gy_1 \preceq F(Gy_1, Gx_1)$, using (3.2) we have $y_2 \preceq y_3$. Similarly using the weakly related property for $\{F, G\}$ and repeated use of (3.2), we get

$$y_0 \preceq y_1 \preceq y_2 \preceq y_3 \ldots y_n \preceq y_{n+1} \cdots .$$

That is the sequence $\{x_n\}$ and $\{y_n\}$ are nondecreasing sequences in X. By the definition of " $\preceq$ ", we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \phi(x_3) \cdots$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \phi(y_3) \cdots$$

In other words, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are nondecreasing sequences in the set of real numbers. Since ϕ is bounded, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are convergent and hence are Cauchy. So for any $\epsilon > 0$, there exist $n_0 \in N$ such that for any $m > n > n_0$, we have $|\phi(x_m) - \phi(x_n)| = \phi(x_m) - \phi(x_n) < \epsilon$ and $|\phi(y_m) - \phi(y_n)| = \phi(y_m) - \phi(y_n) < \epsilon$. Since $x_n \preceq x_m$, it follows that

$$d(x_n, x_m) \leq \phi(x_m) - \phi(x_n) < \epsilon$$

and since $y_n \preceq y_m$, it follows that

$$d(y_n, y_m) \leq \phi(y_m) - \phi(y_n) < \epsilon.$$

This shows that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in X and since X is complete, there exist points $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since the sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ are subsequences of $\{x_n\}$, $x_{2n} \longrightarrow x$, $x_{2n+1} \longrightarrow x$ and $x_{2n+2} \longrightarrow x$. Also the sequences $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ are subsequences of $\{y_n\}$, therefore $y_{2n} \longrightarrow y$, $y_{2n+1} \longrightarrow y$ and $y_{2n+2} \longrightarrow y$.

Consequently, taking the limit as $n \longrightarrow \infty$ in (3.1) and using the continuity of F and G, we get

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} F(x_{2n}, y_{2n}) = F(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}) = F(x, y)$$

and

$$x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Gx_{2n+1} = G(\lim_{n\to\infty} x_{2n+1}) = Gx.$$

Similarly, taking the limit as $n \longrightarrow \infty$ in (3.2) and using the continuity of F and G, we get

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} F(y_{2n}, x_{2n}) = F(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}) = F(y, x),$$

and

$$y = \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Gy_{2n+1} = G(\lim_{n\to\infty} y_{2n+1}) = Gy$$

Thus we have proved that $Gx = x = F(x, y)$ and $Gy = y = F(y, x)$.

Hence (x, y) is a coupled common fixed point of H and G. □

Now we give an example to furnish the weakly related property as well as the above theorem:

Example 3.3. *Let $X = [0, \infty)$ and $d(x, y) = |x - y|$, then (X, d) is a complete metric space and " $\leq$ " is the usual ordering.*

We define $\phi(x) = 2x$ and $F : X \times X \longrightarrow X$ and $G : X \longrightarrow X$ as follows

$$F(x, y) = x + |sin(xy)| \text{ and } Gx = 5x.$$

If we let $x_0 = 1$ and $y_0 = 0$ then $F(x_0, y_0) = 1 + 0 = 1$ and $F(y_0, x_0) = 0 + 0 = 0$. So $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$. We have $GF(x, y) = 5(x + |sin(xy)|)$ i.e. $F(x, y) \leq GF(x, y)$ and $F(Gx, Gy) = F(5x, 5y) = 5x + |sin(25xy)|$ i.e. $Gx \leq F(Gx, Gy)$.

Again $F(y, x) = y + |sin(xy)|$ and $Gy = 5y$ so we get $F(y, x) \leq GF(y, x)$ and $Gy \leq F(Gy, Gx)$. And so the pair $\{F, G\}$ is weakly related.

Hence we see that all the conditions of our theorem are satisfied. Also we have $F(0, y) = 0$, $G0 = 0$ and $F(0, x) = 0$ implying $F(0, y) = 0 = G0$ and $F(0, x) = 0 = G0$. Thus $(0, 0)$ is a common coupled fixed point of the pair $\{F, G\}$.

Theorem 3.4. *Let (X, d) be a complete metric space , $\phi : X \longrightarrow R$ be a bounded from above function and " $\preceq$ " be the partial order induced by ϕ .Let $F : X \times X \longrightarrow X$ and $G, H : X \longrightarrow X$ be three continuous mappings such that the pair $\{F, G\}$ and $\{F, H\}$ are weakly related on X. Then F, G and H have a common coupled fixed point in X.*

Proof. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_{3n} = Gx_{3n-1}, x_{3n-1} = F(x_{3n-2}, y_{3n-2})x_{3n-2} = Hx_{3n-3} \tag{3.5}$$

and

$$y_{3n} = Gy_{3n-1}, y_{3n-1} = F(y_{3n-2}, x_{3n-2})y_{3n-2} = Hy_{3n-3} \tag{3.6}$$

for all $n \geq 0$.

We shall show that

$$x_n \preceq x_{n+1} \, for \, all \, n \geq 0, \tag{3.7}$$

and

$$y_n \preceq y_{n+1} \, for \, all \, n \geq 0. \tag{3.8}$$

We have $x_1 = Hx_0$. Since the pair $\{F, H\}$ is weakly related, we have $Hx_0 \preceq F(Hx_0, Hy_0)$ i.e. $x_1 \preceq F(x_1, y_1)$ and using (3.5) we get $x_1 \preceq x_2$. Again since the pair $\{F, G\}$ is weakly related, we have $F(x_1, y_1) \preceq GF(x_1, y_1)$ and using (3.5) we get $x_2 \preceq x_3$. Similarly using the weakly related property for $\{F, G\}$ and $\{F, H\}$ and repeated use of (3.5) we get

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \, ... \, x_n \preceq x_{n+1} \, \cdots .$$

We have $y_1 = Hy_0$. Since the pair $\{F, H\}$ is weakly related, we have $Hy_0 \preceq F(Hy_0, Hx_0)$ i.e. $y_1 \preceq F(y_1, x_1)$ and using (3.6) we get $y_1 \preceq y_2$. Again since the pair $\{F, G\}$ is weakly related, we have $F(y_1, x_1) \preceq GF(y_1, x_1)$ and using (3.6) we get $y_2 \preceq y_3$. Similarly using the weakly related property for $\{F, G\}$ and $\{F, H\}$ and repeated use of (3.6) we get

$$y_0 \preceq y_1 \preceq y_2 \preceq y_3 \, ... \, y_n \preceq y_{n+1} \, \cdots .$$

That is the sequence, $\{x_n\}$ and $\{y_n\}$ are nondecreasing sequences in X. By the definition of " $\preceq$ ", we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \phi(x_3) \, \cdots$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \phi(y_3) \, \cdots .$$

In other words, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are nondecreasing sequences in the set of real numbers. Since ϕ is bounded, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are convergent and hence are Cauchy. So for any $\epsilon > 0$, there exist $n_0 \in N$ such that for all $m > n > n_0$, we have $|\phi(x_m) - \phi(x_n)| = \phi(x_m) - \phi(x_n) < \epsilon$ and $|\phi(y_m) - \phi(y_n)| = \phi(y_m) - \phi(y_n) < \epsilon$. Since $x_n \preceq x_m$, it follows that

$$d(x_n, x_m) \leq \phi(x_m) - \phi(x_n) < \epsilon$$

and since $y_n \preceq y_m$, it follows that

$$d(y_n, y_m) \leq \phi(y_m) - \phi(y_n) < \epsilon$$

This shows that the sequences $\{(x_n)\}$ and $\{(y_n)\}$ are Cauchy in X and since X is complete, there exist points $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since the sequences $\{x_{3n}\}$, $\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are subsequences of $\{x_n\}$, $x_{3n} \longrightarrow x$, $x_{3n-1} \longrightarrow x$ and $x_{3n-2} \longrightarrow x$

also the sequences $\{y_{3n}\}$,$\{y_{3n-1}\}$ and $\{y_{3n-2}\}$ are subsequences of $\{y_n\}$ therefore $y_{3n} \longrightarrow y$, $y_{3n-1} \longrightarrow y$ and $y_{3n-2} \longrightarrow y$.

Consequently, taking the limit as $n \longrightarrow \infty$ in (3.5) and using the continuity of F, G and H, we get

$$\begin{aligned} x &= lim_{n\to\infty} x_{3n-1} \\ &= \lim_{n\to\infty} F(x_{3n-2}, y_{3n-2}) \\ &= F(\lim_{n\to\infty} x_{3n-2}, \lim_{n\to\infty} y_{3n-2}) \\ &= F(x,y) \end{aligned}$$

and

$$x = \lim_{n\to\infty} x_{3n} = \lim_{n\to\infty} Gx_{3n-1} = G(\lim_{n\to\infty} x_{3n-1}) = Gx,$$

also

$$x = \lim_{n\to\infty} x_{3n-2} = \lim_{n\to\infty} Hx_{3n-3} = H(\lim_{n\to\infty} x_{3n-3}) = Hx$$

Similarly, taking the limit as $n \longrightarrow \infty$ in (3.6) and using the continuity of F, G and H, we get

$$\begin{aligned} y = \lim_{n\to\infty} y_{3n-1} = \lim_{n\to\infty} F(y_{3n-2}, x_{3n-2}) = F(\lim_{n\to\infty} y_{3n-2}, \lim_{n\to\infty} x_{3n-2}) \\ = F(y,x) \end{aligned}$$

and

$$y = \lim_{n\to\infty} y_{3n} = \lim_{n\to\infty} Gy_{3n-1} = G(\lim_{n\to\infty} y_{3n-1}) = Gy,$$

also

$$y = \lim_{n\to\infty} y_{3n-2} = \lim_{n\to\infty} Hy_{3n-3} = H(\lim_{n\to\infty} y_{3n-3}) = Hy$$

Thus we proved that $Hx = Gx = x = F(x,y)$ and $Hy = Gy = y = F(y,x)$.

Hence (x,y) is a coupled common fixed point of H, G and F. □

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