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# STOCHASTIC BOUNDS FOR A SINGLE SERVER QUEUE WITH GENERAL RETRIAL TIMES

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Abstract. We propose to use a mathematical method based on stochastic comparisons of Markov chains in order to derive performance indice bounds. The main goal of this paper is to investigate various monotonicity properties of a single server retrial queue with first-come-first-served (FCFS) orbit and general retrial times using the stochastic ordering techniques.
**Keywords:** Retrial queues, Markov chain, stochastic bounds, monotonicity, ageing distributions.
**MSC(2010):** Primary: 60E15; Secondary: 60K25.

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## 1. Introduction

Retrial queueing systems are such systems in which arrivals who find the server busy join the retrial queue to try again for their requests or leave the service area immediately. Apart from theoretical interest, these systems have been extensively studied due to their wide applicability. They have been successfully applied in telephone switching systems, telecommunication networks and computers competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models for several computer systems: Packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. There is an extensive literature on retrial queues, see [1, 2, 9, 12, 17, 31] and references therein.

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There is a special type of retrial systems called FCFS retrial queues that was named by Farahmand [10], that is, if a customer finds the server busy, he may join the tail of a retrial queue in accordance with an FCFS discipline. The customer at the head of the retrial queue would then try again to enter the system some time, competing with new customers. In [18] there were introduced retrial systems with strict FCFS policy where all entering customers join a general queue, but there is a restriction on the starting moments of service. Lakatos [18] investigates the number of customers, Koba [15] the waiting time in the system.

Retrial queueing systems with general service times and non-exponential retrial times have also received considerable attention during the last decade. An important characteristic of the general retrial time policy is that we always obtain analytical solutions in terms of closed-form expressions. The named policy arises naturally in problems where the server is required to search for customers [22], that is, this policy is related to many service systems where, after each service completion, the processor will spend a random amount of time in order to find the next item to be processed. The first investigation on the $M/G/1$ retrial queue with general retrial times is due to Kapyrin [13], who assumed that each customer in the orbit generates a stream of repeated attempts that are independent of the number of customers in the orbit and the state of the server. However, this methodology was found to be incorrect by Falin [8]. Subsequently, Yang et al. [30] have developed an approximation method to obtain the steady state performance for the model of Kapyrin. Later, Gómez-Corral [11] discussed extensively an $M/G/1$ retrial queue with FCFS discipline and general retrial times. In recent years, several retrial models with general retrial times have been analyzed, details of which may be found in [1, 3, 7, 20, 16, 28, 29].

Because of complexity of retrial queueing models, analytic results are generally difficult to obtain. In contrast, there are a great number of numerical and approximation methods which are of practical importance. That is, the approximation of a stochastic model, either by a simpler model or by a model with simple constituent components, might lead to convenient bounds and approximations for some particular and desired characteristics of the model. Such circumstances suggest seeking qualitative properties of the real model, that is, the manner in which the model is affected by the changes in its parameters. It is by means of qualitative properties that bounds can be obtained mathematically

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and approximations can be made rigorously [25]. One important approach is monotonicity which can be investigated using the stochastic comparison method based on the general theory of stochastic orders. Stochastic comparison methods have been used to produce bounds and approximations for queue length processes, waiting times and busy period distributions in many queueing systems. For the detailed results and references about the comparison methods and their applications, see [4, 21, 25]

There is a significant body of literature on monotonicity results in retrial queues. Khalil and Falin [14] consider some monotonicity properties of $M/G/1$ retrial queues with exponential retrial times relative to strong stochastic ordering, convex ordering and Laplace ordering. Liang [19] shows that if the hazard rate function of the retrial time distribution is decreasing, then stochastically longer service time or less servers will result in more customers in the system. Boualem et al. [5] investigate some monotonicity properties of an $M/G/1$ queue with constant retrial policy in which the server operates under a general exhaustive service and multiple vacation policy relative to strong stochastic ordering and convex ordering. These results imply in particular simple insensitive bounds for the stationary queue length distribution. Taleb and Aissani [27] show that if the distribution of the retrial time is close to the exponential distribution in Laplace transform, then the exponential bound is closer to the exact value than the deterministic bound. Otherwise, the deterministic bound is better. More recently, Boualem et al. [6] use the tools of a qualitative analysis to investigate various monotonicity properties for an $M/G/1$ retrial queue with classical retrial policy and Bernoulli feedback. The obtained results allow us to place in a prominent position the insensitive bounds for both the stationary distribution and the conditional distribution of the stationary queue of the considered model.

In this paper, we study monotonicity property similar to that of [5], for an $M/G/1$ queue with general service times and non-exponential retrial time distribution under FCFS orbit discipline. The performance characteristics of such a system are available in Gómez-Corral [11]. We prove the monotonicity of the transition operator of the embedded Markov chain relative to strong stochastic ordering and convex ordering. We obtain comparability conditions for the distribution of the number of customers in the system. Bounds are derived for the mean characteristics of the busy period, number of customers served during a busy

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period, number of orbit busy periods and waiting times. Our approach is quite different from those in [11, 30].

The rest of the paper is organized as follows. In the next Section, we describe the considered queueing model. In Section 3, we introduce some pertinent definitions and notions, and present some lemmas that will be used to prove the main results of this paper. Section 4 focusses on monotonicity of the transition operator of the embedded Markov chain and gives comparability conditions of two transition operators. Stochastic bounds for the stationary number of customers in the system are discussed in Section 5. In Section 6, we provide bounds for the mean characteristics of the busy period and waiting time.

## 2. Description of the queueing model

We consider a single server retrial queue without waiting space. Primary customers arrive in a Poisson process with rate $\lambda$. If the server is free, the primary customer is served immediately and leaves the system after service completion. Otherwise, the customer leaves the service area and enters the retrial group in accordance with an FCFS discipline. To this end, we assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer joins the orbit again. Such a process is repeated until the customer finds the server idle and gets the requested service at the time of a retrial. Successive inter-retrial times of any customer follow an arbitrary law with common probability distribution function $A(x)$, Laplace-Stieltjes transform $L_A(s)$ and first moment $\alpha_1$. The service times are independently and identically distributed with probability distribution function $B(x)$, Laplace-Stieltjes transform $L_B(s)$ and first two moments $\beta_1$, $\beta_2$.
We suppose that inter-arrival times, retrial times and service times are mutually independent.

The state of the system can be described by means of the process $\{(C(t), N(t), \xi_0(t), \xi_1(t)), t \geq 0\}$, where $C(t)$ denotes the server state (0 or 1 according to the server is free or busy, respectively) and $N(t)$ is the number of customers in the orbit at time $t$. If $C(t) = 0$ and $N(t) > 0$, then $\xi_0(t)$ represents the elapsed retrial time. If $C(t) = 1$, then $\xi_1(t)$ corresponds to the elapsed service time of the customer being served.

Let $\tau_n$ be the time of the $n$th departure and $Q_n = N(\tau_n)$ be the number of customers in the orbit just after the time $\tau_n$. The sequence of

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random variables $\{Q_n, n \geq 1\}$ forms an embedded Markov chain for our queueing system which is irreducible and aperiodic on the state-space $\mathbb{N}$. Its fundamental equation is defined by

$$Q_{n+1} = Q_n + v^{n+1} - \delta_{Q_{n+1}}, \tag{2.1}$$

where $v^{n+1}$ is the number of primary customers arriving at the system during the service time which ends at $\tau_{n+1}$. Its distribution is given by $k_j = P(v^{n+1} = j) = \int_0^\infty (\lambda x)^j (j!)^{-1} e^{-\lambda x} dB(x),\ j \geq 0$, with generating function $k(z) = \sum_{j \geq 0} k_j z^j = L_B(\lambda(1-z))$.

The Bernoulli random variable $\delta_{Q_{n+1}}$ is equal to 1 or 0 depending on whether the customer, who leaves the system at time $\tau_{n+1}$, proceeds from the orbit or otherwise.

From Gómez-Corral [11], we have that the necessary and sufficient condition for ergodicity of this chain is $\lambda\beta_1 < L_A(\lambda)$.

The transition probabilities of the chain $\{Q_n, n \geq 1\}$ are defined in the following manner

$$\begin{aligned} p_{nm} &= (1 - L_A(\lambda))k_{m-n} + L_A(\lambda)k_{m-n+1}, \text{ for } n \neq 0 \text{ and } m \geq 0, \\ p_{0m} &= k_m, \text{ for } m \geq 0. \end{aligned}$$

This model has been studied by Gómez-Corral [11] (the steady state distribution of the server state and the orbit length, the waiting time distribution, the busy period, and other related quantities). Although the performance characteristics of such a system were obtained, they are cumbersome (they include integrals of Laplace transform, solutions of functional equations, etc.) and are not very exploitable from the application point of view (performance evaluation, ... ). It is why we use, in the rest of this paper, the general theory of stochastic ordering [25] to study monotonicity properties of the considered model relative to the strong stochastic ordering, increasing convex ordering and Laplace ordering.

## 3. Preliminary results

3.1. **Stochastic orders and ageing notions.** Stochastic ordering is useful for studying internal changes of performance due to parameter variations, to compare distinct systems, to approximate a system by a simpler one, and to obtain upper and lower bounds for the main performance measures of systems.

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First, let us recall some stochastic orders and ageing notions which are most pertinent to the main results to be developed in this paper.

**Definition 3.1.** *For two random variables $X$ and $Y$ with densities $f$ and $g$ and cumulative distribution functions $F$ and $G$, respectively, let $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ be the survival functions. $X$ is said to be smaller than $Y$ in:*

- **(a):** *Usual stochastic order (denoted by $X \leq_{st} Y$) if and only if $\overline{F}(x) \leq \overline{G}(x)$, $\forall x \geq 0$.*
- **(b):** *Increasing convex ordering (denoted by $X \leq_{icx} Y$) if and only if $\int_x^{+\infty} \overline{F}(u)d(u) \leq \int_x^{+\infty} \overline{G}(u)d(u)$, $\forall x \geq 0$.*
- **(c):** *Laplace order (denoted by $X \leq_L Y$) if and only if $\int_0^{+\infty} e^{-sx}dF(x) \geq \int_0^{+\infty} e^{-sx}dG(x)$, $\forall s \geq 0$.*

*If the random variables of interest are of discrete type and $\omega = (\omega_n)_{n\geq 0}$, $\nu = (\nu_n)_{n\geq 0}$ are the corresponding distributions, then the above definitions can be given in the following form:*

- **(a):** $\omega \leq_{st} \nu$ *iff* $\overline{\omega}_m = \sum_{n\geq m} \omega_n \leq \overline{\nu}_m = \sum_{n\geq m} \nu_n$, *for all $m$.*
- **(b):** $\omega \leq_{icx} \nu$ *iff* $\overline{\overline{\omega}}_m = \sum_{n\geq m} \sum_{k\geq n} \omega_k \leq \overline{\overline{\nu}}_m = \sum_{n\geq m} \sum_{k\geq n} \nu_k$, *for all $m$.*
- **(c):** $\omega \leq_L \nu$ *iff* $\sum_{n\geq 0} \omega_n z^n \geq \sum_{n\geq 0} \nu_n z^n$, *for all $z \in [0, 1]$.*

For a comprehensive discussion on these stochastic orders and their applications, one may refer to [23, 24, 25, 26].

**Definition 3.2.** *Let $X$ be a positive random variable with distribution function $F$ and mean $m$.*

- **(a):** *The distribution $F$ is HNBUE (Harmonically New Better than Used in Expectation) if*

$$\int_x^{+\infty} \overline{F}(y)dy \leq m\ e^{-\frac{x}{m}},\ \textit{for all } x \geq 0.$$

- **(b):** *$F$ belongs to the $\mathcal{L}$ class if*

$$\int_0^{+\infty} e^{-sy}\ dF(y) \leq \frac{1}{ms+1},\ \textit{for all } s \geq 0.$$

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*If these latter inequalities are reversed, we obtain the HNWUE (Harmonically New Worse than Used in Expectation) and $\overline{\mathcal{L}}$ classes of distributions.*

**(a):** *F is HNBUE (HNWUE) iff* $F \leq_{icx} (\geq_{icx}) F^*$,
**(b):** *F is* $\mathcal{L}$ $(\overline{\mathcal{L}})$ *iff* $F \geq_L \ (\leq_L)\ F^*$,

*where* $F^*$ *is the exponential distribution function with the same mean as* $F$.

3.2. **Some useful lemmas.** This subsection presents several useful lemmas which will be used later in establishing the main results in Section 4.

Consider two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(i)}$ and $B^{(i)}$, $i = 1, 2$. Let $k_j^{(i)} = \int_0^{+\infty} \frac{(\lambda^{(i)}x)^j}{j!} e^{-\lambda^{(i)}x} dB^{(i)}(x)$ be the distribution of the number of primary calls which arrive during the service time of a call in the $i$th system.

**Lemma 3.3.** *If* $\lambda^{(1)} \leq \lambda^{(2)}$ *and* $B^{(1)} \leq_s B^{(2)}$, *then* $\{k_n^{(1)}\} \leq_s \{k_n^{(2)}\}$, *where* $\leq_s$ *is one of the symbols* $\leq_{st}$ *or* $\leq_{icx}$.

*Proof.* By definition,

$$\overline{k}_n^{(i)} = \sum_{j\geq n} k_j^{(i)} = \int_0^{+\infty} \sum_{j\geq n} \frac{(\lambda^{(i)}x)^j}{j!} e^{-\lambda^{(i)}x} dB^{(i)}(x),\ i = 1, 2,$$

$$\overline{\overline{k}}_n^{(i)} = \sum_{j\geq n} \overline{k}_j^{(i)} = \int_0^{+\infty} \sum_{j\geq n} \sum_{l\geq j} \frac{(\lambda^{(i)}x)^l}{l!} e^{-\lambda^{(i)}x} dB^{(i)}(x),\ i = 1, 2.$$

To prove that $\{k_n^{(1)}\} \leq_s \{k_n^{(2)}\}$, we have to establish the usual numerical inequalities

$$\overline{k}_n^{(1)} = \sum_{m\geq n} k_m^{(1)} \leq \overline{k}_n^{(2)},\ (\text{for } \leq_s = \leq_{st}),$$

$$\overline{\overline{k}}_n^{(1)} = \sum_{m\geq n} \overline{k}_m^{(1)} \leq \overline{\overline{k}}_n^{(2)},\ (\text{for } \leq_s = \leq_{icx}).$$

The rest of the proof is known in the more general setting of a random summation. □

**Lemma 3.4.** *If* $\lambda^{(1)} \leq \lambda^{(2)}$ *and* $B^{(1)} \leq_L B^{(2)}$, *then* $\{k_n^{(1)}\} \leq_L \{k_n^{(2)}\}$.

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*Proof.* We have

$$k^{(i)}(z) = \sum_{n\geq 0} k_n^{(i)} z^n = L_{B^{(i)}}(\lambda^{(i)}(1-z)),\ i = 1, 2,$$

where $k^{(1)}(z)$ and $k^{(2)}(z)$ are, respectively, the corresponding distributions of the number of new arrivals in the two systems during a service time.
Let $\lambda^{(1)} \leq \lambda^{(2)}$ and $B^{(1)} \leq_L B^{(2)}$. To prove that $\{k_n^{(1)}\} \leq_L \{k_n^{(2)}\}$, we have to establish that

$$L_{B^{(1)}}(\lambda^{(1)}(1-z)) \geq L_{B^{(2)}}(\lambda^{(2)}(1-z)). \tag{3.1}$$

The inequality $B^{(1)} \leq_L B^{(2)}$ implies that $L_{B^{(1)}}(s) \geq L_{B^{(2)}}(s)$ for all $s \geq 0$.
In particular, for $s = \lambda^{(1)}(1-z)$ we have

$$L_{B^{(1)}}(\lambda^{(1)}(1-z)) \geq L_{B^{(2)}}(\lambda^{(1)}(1-z)). \tag{3.2}$$

Since any Laplace transform is a decreasing function, $\lambda^{(1)} \leq \lambda^{(2)}$ implies that

$$L_{B^{(2)}}(\lambda^{(1)}(1-z)) \geq L_{B^{(2)}}(\lambda^{(2)}(1-z)). \tag{3.3}$$

By transitivity, (3.2) and (3.3) give (3.1). □

## 4. Monotonicity properties of the embedded Markov chain

Let $\mathbf{T}$ be the transition operator of our embedded Markov chain $\{Q_n, n \geq 1\}$ which associates to every distribution $\omega = \{\omega_m\}_{m\geq 0}$ a distribution $\mathbf{T}\omega = \{\nu_m\}_{m\geq 0}$ such that $\nu_m = \sum\limits_{n\geq 0} \omega_n p_{nm}$ (where $p_{nm}$ are the one-step transition probabilities of the considered chain).

**Theorem 4.1.** *The transition operator* $\mathbf{T}$ *is monotone with respect to the orders* $\leq_{st}$ *and* $\leq_{icx}$.

*Proof.* The transition operator $\mathbf{T}$ is monotone with respect to $\leq_{st}$ iff

$$\overline{p}_{nm} - \overline{p}_{n-1m} \geq 0,\ \forall\ n,\ m, \tag{4.1}$$

and is monotone with respect to $\leq_{icx}$ iff

$$\overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m} - 2\overline{\overline{p}}_{nm} \geq 0,\ \forall\ n,\ m. \tag{4.2}$$

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Here, $\overline{p}_{nm} = \sum\limits_{l=m}^{\infty} p_{nl}$ and $\overline{\overline{p}}_{nm} = \sum\limits_{l=m}^{\infty} \overline{p}_{nl}$.
To prove inequalities (4.1) and (4.2), we have

$$\begin{aligned}\overline{p}_{nm} &= (1-L_A(\lambda))\overline{k}_{m-n} + L_A(\lambda)\overline{k}_{m-n+1}\\ &= (1-L_A(\lambda))k_{m-n} + \overline{k}_{m-n+1},\\ \overline{\overline{p}}_{nm} &= (1-L_A(\lambda))\overline{k}_{m-n} + \overline{\overline{k}}_{m-n+1}.\end{aligned}$$

Thus

$$\begin{aligned}\overline{p}_{nm} - \overline{p}_{n-1m} &= (1-L_A(\lambda))k_{m-n} + L_A(\lambda)k_{m-n+1} \geq 0,\\ \overline{\overline{p}}_{n-1m} + \overline{\overline{p}}_{n+1m} - 2\overline{\overline{p}}_{nm} &= (1-L_A(\lambda))k_{m-n-1} + L_A(\lambda)k_{m-n} \geq 0.\end{aligned}$$

Hence, we obtain the stated result. □

**Proposition 4.2.** *If at time $t = 0$ the system was empty, then the number of customers in the orbit would form a monotonically increasing sequence with respect to the orders $\leq_{st}$ and $\leq_{icx}$.*

*Proof.* If $\omega^{(0)} = (1, 0, 0, ...)$ is the initial probability vector, then

$$\omega^{(1)} = \mathbf{T}\omega^{(0)} = (k_0, k_1, k_2, k_3, ...),\ \overline{\omega}_k^{(0)} = \begin{cases} 1, & \text{if } k = 0,\\ 0, & \text{if } k \neq 0,\end{cases}$$

$$\overline{\omega}_k^{(1)} = \begin{cases} 1, & \text{if } k = 0,\\ \sum\limits_{i \geq k} k_i, & \text{if } k \neq 0,\end{cases}\quad \overline{\overline{\omega}}_n^{(0)} = \begin{cases} 1, & \text{if } n = 0,\\ 0, & \text{if } n \neq 0,\end{cases}\quad \text{and } \omega^{(0)} \leq_s \omega^{(1)},$$

where $\leq_s$ is one of the symbols $\leq_{st}$ or $\leq_{icx}$.
By induction and using the monotonicity of $\mathbf{T}$, we show that $\omega^{(n)} \leq_s \omega^{(n+1)}$. Thus $Q_n \leq_s Q_{n+1}$. □

**Remark 4.3.** *The operator $\mathbf{T}$ is not monotone with respect to the order $\leq_L$.*
*Indeed, for $\omega^{(1)} = (1, 0, 0, ...)$, $\omega^{(2)} = (0, 1, 0, ...)$, we have $\omega^{(1)} \leq_L \omega^{(2)}$ but $T\omega^{(1)} \not\leq_L T\omega^{(2)}$.*

In the following two theorems, we give comparability conditions of two transition operators. To this end, we consider two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$, respectively. Let $\mathbf{T}^1$ and $\mathbf{T}^2$ be the transition operators of the corresponding embedded Markov chains.

**Theorem 4.4.** *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\mathbf{T}^1 \leq_s \mathbf{T}^2$, i.e. for any distribution $\omega$, we have $\mathbf{T}^1\omega \leq_s \mathbf{T}^2\omega$, where $\leq_s$ is one of the symbols $\leq_{st}$ or $\leq_{icx}$.*

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*Proof.* From Stoyan [25], we have to show the following numerical inequalities for the one-step transition probabilities $p_{nm}^{(1)},\ p_{nm}^{(2)}$:

(4.3) $\qquad \overline{p}_{nm}^{(1)} \ \leq \ \overline{p}_{nm}^{(2)},\ \forall n,\ m,\ (\text{for } \leq_s = \leq_{st}),$

(4.4) $\qquad \overline{\overline{p}}_{nm}^{(1)} \ \leq \ \overline{\overline{p}}_{nm}^{(2)},\ \forall n,\ m,\ (\text{ for } \leq_s = \leq_{icx}).$

To prove inequality (4.3), we have

$$\overline{p}_{nm}^{(1)} = (1 - L_{A^{(1)}}(\lambda^{(1)}))k_{m-n}^{(1)} + \overline{k}_{m-n+1}^{(1)}.$$

Since $\lambda^{(1)} \leq \lambda^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $L_{A^{(1)}}(\lambda^{(1)}) \geq L_{A^{(2)}}(\lambda^{(2)})$ and

$$\overline{p}_{nm}^{(1)} \leq (1 - L_{A^{(2)}}(\lambda^{(2)}))k_{m-n}^{(1)} + \overline{k}_{m-n+1}^{(1)}.$$

Moreover, we have

$$(1 - L_{A^{(2)}}(\lambda^{(2)}))k_{m-n}^{(1)} + \overline{k}_{m-n+1}^{(1)} = (1 - L_{A^{(2)}}(\lambda^{(2)}))\overline{k}_{m-n}^{(1)} + L_{A^{(2)}}(\lambda^{(2)})\overline{k}_{m-n+1}^{(1)}.$$

By Lemma 3.3 (for $\leq_s = \leq_{st}$), we have $\overline{k}_n^{(1)} \leq \overline{k}_n^{(2)},\ \forall n \geq 0$.
Finally, we get:

$$\overline{p}_{nm}^{(1)} \leq (1 - L_{A^{(2)}}(\lambda^{(2)}))\overline{k}_{m-n}^{(2)} + L_{A^{(2)}}(\lambda^{(2)})\overline{k}_{m-n+1}^{(2)} = \overline{p}_{nm}^{(2)}.$$

Following the above technique and using Lemma 3.3 ( for $\leq_s = \leq_{icx}$), we establish inequality (4.4). $\square$

**Theorem 4.5.** *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_L B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then* $\mathbf{T}^1 \leq_L \mathbf{T}^2$.

*Proof.* Let $\omega = (\omega_m)$ be a distribution and $\mathbf{T}_\omega = \nu = (\nu_m)$, where

$$\nu_m = \sum_{n\geq 0} \omega_n p_{nm} = \omega_0 k_m + \sum_{n \geq 1} \omega_n p_{nm}, \text{ for all } m \geq 0.$$

Let $k(z) = \sum_{n\geq 0} k_n z^n$ and $\omega(z) = \sum_{n \geq 0} \omega_n z^n$ be the generating functions of $(k_n)$ and $(\omega_n)$, respectively.
The generating function of $\nu$ is given by

$$\begin{aligned}
G(z) &= \sum_{m\geq 0} \nu_m z^m = \sum_{m\geq 0}\sum_{n\geq 0} \omega_n p_{nm} z^m = \sum_{m \geq 0}[\omega_0 k_m + \sum_{n\geq 1}\omega_n p_{nm}]z^m \\
&= \omega_0 k(z) + \sum_{n\geq 1}\omega_n \sum_{m\geq 0}\left[(1 - L_A(\lambda))k_{m-n} + L_A(\lambda)k_{m-n+1}\right] z^m \\
&= k(z)\left[\omega_0 + (1 - L_A(\lambda))(\omega(z) - \omega_0) + \frac{1}{z}L_A(\lambda)(\omega(z) - \omega_0)\right] \\
&= \omega_0 k(z) + \frac{k(z)}{z}(\omega(z) - \omega_0)(z + (1 - z)L_A(\lambda)).
\end{aligned}$$

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If the conditions of Theorem 4.5 are fulfilled, then $k^{(1)}(z) \geq k^{(2)}(z)$ by Lemma 3.4 and $(1-z)L_{A^{(1)}}(\lambda^{(1)}) \geq (1-z)L_{A^{(2)}}(\lambda^{(2)})$, $\forall\, z \in [0,1]$. Hence $G^{(1)}(z) \geq G^{(2)}(z)$. □

## 5. Stochastic bounds for the stationary distribution

Assume that we have two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$, respectively, and let $\pi_n^{(1)}$, $\pi_n^{(2)}$ be the corresponding stationary distributions of the number of customers in the two systems.

**Theorem 5.1.** *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $\{\pi_n^{(1)}\} \leq_s \{\pi_n^{(2)}\}$, where $\leq_s$ is one of the symbols $\leq_{st}$ or $\leq_{icx}$.*

*Proof.* The generating function of the number of customers in the system at an arbitrary time point coincides with the generating function of the embedded Markov chain at departure epochs, which can be obtained from the following Kolmogorov equations for the stationary probabilities $\pi_j = P(Q_n = j),\ j \geq 0$:

$$\pi_j \;=\; \pi_0 k_j + (1-\delta_{0j})(1-L_A(\lambda))\sum_{n=1}^{j} \pi_n k_{j-n} + L_A(\lambda)\sum_{n=1}^{j+1} \pi_n k_{j-n+1},\ j \geq 0,$$

where $\delta_{0j}$ denotes Kronecker's delta.
Since the corresponding embedded Markov chain is ergodic, the stationary distribution coincides with the limit distribution [11]. Using Theorems 4.1 and 4.4 which state that $\mathbf{T}^i$ are monotone with respect to the order $\leq_s$ and $\mathbf{T}^1 \leq_s \mathbf{T}^2$, we have by induction $\mathbf{T}^{1,n}\omega \leq_s \mathbf{T}^{2,n}\omega$ for any distribution $\omega$, where $\mathbf{T}^{i,n} = \mathbf{T}^i(\mathbf{T}^{i,n-1}\omega)$. Taking the limit, we obtain the stated result. □

**Theorem 5.2.** *If in the $M/G/1$ retrial queue with general retrial times the service time distribution $B(x)$ is HNBUE and the retrial time distribution is of class $\mathcal{L}$, then $\{\pi_n\} \leq_{icx} \{\pi_n^*\}$, where $\{\pi_n^*\}$ is the stationary distribution of the number of customers in the $M/M/1$ retrial queue with exponential retrial times with the same parameters as those of the $M/G/1$ retrial queue with general retrial times.*

*Proof.* Consider an auxiliary $M/M/1$ retrial queue with the same arrival rate $\lambda$, mean retrial time $\alpha_1$ and mean service time $\beta_1$, as those of the $M/G/1$ retrial queue with general retrial times, but with exponentially distributed retrial times $A^*(x) = 1 - e^{-x/\alpha_1}$ and service times $B^*(x) = 1 - e^{-x/\beta_1}$ for $x > 0$. If $B(x)$ is $HNBUE$ and $A(x)$ is of class $\mathcal{L}$, then

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$B(x) \leq_{icx} B^*(x)$ and $A(x) \leq_L A^*(x)$. Therefore, by using Theorem 5.1, we deduce the statement of this Theorem. □

**Remark 5.3.** *Theorem 5.2 implies that the mean number of customers in the system in steady state satisfies the following inequality*

$$E(N) \leq \frac{\lambda[\lambda\beta_2 + 2(1-\lambda\beta_1)\beta_1](1+\lambda\alpha_1)}{2[1-\lambda\beta_1(1+\lambda\alpha_1)]}.$$

## 6. Application: Bounds for the mean characteristics of the model

In this section, we show how the obtained theoretical results can be used. We are interested in a system busy period and the orbit busy period. These characteristics can be defined as follows: A system busy period is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. In the same time, the orbit busy period is the interval of time from the epoch when a primary customer arrives and finds the server busy and the orbit idle, until the next epoch at which a repeated attempt finds the server idle and the orbit becomes empty. Thus, we have the following characteristics: the length of a system busy period $L$, the number of service completions occurring during $(0, L]$ $I$ and the number of orbit busy periods which take place in $(0, L]$ $N_b$. Another important characteristic is the waiting time $W$.

For the considered model, under the condition $\lambda\beta_1 < L_A(\lambda)$, we have [11]:

$$E(L) = \frac{\beta_1}{L_A(\lambda) - \lambda\beta_1},\ E(I) = \frac{L_A(\lambda)}{L_A(\lambda) - \lambda\beta_1},$$

$$E(N_b) = \frac{1 - L_B(\lambda)}{L_B(\lambda)} \text{ and } E(W) = \frac{\lambda\beta_2 + 2\beta_1(1 - L_A(\lambda))}{2(L_A(\lambda) - \lambda\beta_1)}.$$

Suppose once more that we have two $M/G/1$ retrial queues with non-exponential retrial times with parameters $\lambda^{(1)}$, $A^{(1)}$, $B^{(1)}$ and $\lambda^{(2)}$, $A^{(2)}$, $B^{(2)}$, respectively. Let $L^{(i)}$, $I^{(i)}$, $N_b^{(i)}$ and $W^{(i)}$ be the busy period length, the number of customers served during a busy period, the number of orbit busy periods which take place in $(0, L^{(i)}]$ and the waiting time, respectively, in the $i$th system, $i = 1, 2$.

**Theorem 6.1.** *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$E(L^{(1)}) \leq E(L^{(2)}) \text{ and } E(I^{(1)}) \leq E(I^{(2)}),$$

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*where $\leq_s$ is one of the symbols $\leq_{st}$, $\leq_{icx}$, $\leq_L$.*

*Proof.* The quantities $E(L)$ and $E(I)$ are increasing with respect to $\lambda$ and $\beta_1$, and decreasing with respect to $L_A(.)$. Under conditions of Theorem 6.1, we obtain the desired inequalities. Recall that $X \leq_s Y$ implies $E(X^n) \leq E(Y^n)$ for all $n$. □

**Theorem 6.2.** *For any $M/G/1$ retrial queue,*

$$E(L) \leq \frac{\beta_1}{e^{-\lambda\alpha_1} - \lambda\beta_1} \text{ and } E(I) \leq \frac{e^{-\lambda\alpha_1}}{e^{-\lambda\alpha_1} - \lambda\beta_1}.$$

*If $A$ and $B$ are of class $\mathcal{L}$, then*

$$E(L) \geq \frac{\beta_1(1+\lambda\alpha_1)}{1-\lambda\beta_1(1+\lambda\alpha_1)} \text{ and } E(I) \geq \frac{1}{1-\lambda\beta_1(1+\lambda\alpha_1)}.$$

*Proof.* Consider auxiliary $M/D/1$ and $M/M/1$ retrial queues with the same arrival rate $\lambda$, mean service times $\beta_1$ and mean retrial times $\alpha_1$. $A$ is a Dirac distribution at $\alpha_1$ for the $M/D/1$ system, and is an exponential distribution for the $M/M/1$ system. Using the above theorem, we obtain the stated results. □

**Theorem 6.3.** *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$E(N_b^{(1)}) \leq E(N_b^{(2)}) \text{ and } E(W^{(1)}) \leq E(W^{(2)}).$$

*Proof.* The quantities $E(N_b)$ and $E(W)$ are increasing with respect to $\lambda$, $\beta_1$ and $\beta_2$, decreasing with respect to $L_B(.)$ and $L_A(.)$. Under the conditions of Theorem 6.3, we obtain the desired inequalities. Recall that $X \leq_{st} Y$ implies $E(X^n) \leq E(Y^n)$ for all $n$. □

**Theorem 6.4.** *For any $M/G/1$ retrial queue,*

$$E(N_b) \leq e^{\lambda\beta_1} - 1, \tag{6.1}$$

$$E(W) \leq \frac{\lambda\beta_2 + 2\beta_1(1-e^{-\lambda\alpha_1})}{2(e^{-\lambda\alpha_1} - \lambda\beta_1)}. \tag{6.2}$$

*If $A$ and $B$ are of class $\mathcal{L}$, then*

$$E(N_b) \geq \lambda\beta_1, \tag{6.3}$$

$$\frac{\lambda\beta_2(1+\lambda\alpha_1) + 2\lambda\beta_1\alpha_1}{2(1-\lambda\beta_1(1+\lambda\alpha_1))} \leq E(W) \leq \frac{2\lambda\beta_1^2 + 2\beta_1(1-e^{-\lambda\alpha_1})}{2(e^{-\lambda\alpha_1} - \lambda\beta_1)}. \tag{6.4}$$

*Proof.* The proof is similar to that of Theorem 6.2. Recall that if $B$ is of class $\mathcal{L}$, then $\beta_2 \leq 2\beta_1^2$. □

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**Remark 6.5.** *Inequality (6.2) gives an upper bound for the mean waiting time when the retrial time and service time distributions are unknown, but we have partial information about the first two moments. For the second inequality (6.4), we use the partial information about the ageing class of the retrial time and service time distributions.*

## 7. Conclusion

In this paper, we present a qualitative analysis to establish insensitive bounds for some performance measures of a single-server retrial queue with generally distributed inter-retrial times by using the monotonicity approach relative to the theory of stochastic orderings. Our method is quite different from those in Gómez-Corral [11] and Yang et al. [30], in the sense that our approach provides from the fact that we can come to a compromise between the role of these qualitative bounds and the complexity of resolution of some complicated systems where some parameters are not perfectly known.

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