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RELATIVE VOLUME COMPARISON THEOREMS IN FINSLER GEOMETRY AND THEIR APPLICATIONS

B. Y. WU

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Abstract. We establish some relative volume comparison theorems for extremal volume forms of Finsler manifolds under suitable curvature bounds. As their applications, we obtain some results on curvature and topology of Finsler manifolds. Our results remove the usual assumption on S-curvature that is needed in the literature.

Keywords: Extreme volume form, Finsler manifold, Gromov precompactness, first Betti number, fundamental group.
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1. Introduction

Comparison technique is a powerful tool in global analysis in differential geometry, and it has been well developed in Riemannian geometry. Volume, as the important geometric invariant, plays a key role in comparison technique. Recently comparison technique has been developed for Finsler manifolds and the relationship between curvature and topology of Finsler manifolds has also been investigated [2, 8–11, 15]. It should be pointed out here that volume form is uniquely determined by the given Riemannian metric, while there are different choices of volume forms for Finsler metrics. As the result, we usually need to control the S-curvature in order to obtain volume comparison theorems as well as results on curvature and topology. This additional assumption on S-curvature has been removed by author recently by using the extreme volume forms (the maximal and minimal volume forms) [14].

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In [14] we obtain some volume comparison theorems for extreme volume forms without any assumption on S-curvature and obtain some applications. For further study on curvature and topology of Finsler manifolds we need the relative volume comparison theorems. The main purpose of the present paper is to establish some relative volume comparison theorems for extreme volume forms and then to investigate the curvature and topology of Finsler manifolds further. Our results remove the usual assumption on S-curvature that is needed in the literature.

2. **Finsler geometry**

Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be local coordinates on TM, and $\pi : TM\backslash 0 \to M$ the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. The *fundamental tensor* g_{ij} and the *Cartan tensor* C_{ijk} are defined by

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad C_{ijk}(x,y) := \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k}.$$

Let $\Gamma^i_{jk}(x, y)$ be the Chern connection coefficients. Then the *first Chern curvature tensor* $R_{j\ kl}^{\ i}$ can be expressed by

$$R_{j\ kl}^{\ i} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls},$$

where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - y^k \Gamma^j_{ik} \frac{\partial}{\partial y^j}$. Let $R_{ijkl} := g_{js} R_{i\ kl}^{\ s}$, and write $\mathbf{g}_y = g_{ij}(x,y)dx^i \otimes dx^j, \mathbf{R}_y = R_{ijkl}(x,y)dx^i \otimes dx^j \otimes dx^k \otimes dx^l$. For a tangent plane $P \subset T_xM$, let

$$\mathbf{K}(P, y) = \mathbf{K}(y; u) := \frac{\mathbf{R}_y(y,u,u,y)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2},$$

where $y, u \in P$ are tangent vectors such that $P = \text{span}\{y, u\}$. We call $\mathbf{K}(P, y)$ the *flag curvature of* P *with flag pole* y. Let

$$\mathbf{Ric}(y) = \sum_i \mathbf{K}(y; e_i),$$

here $e_1, \cdots, e_n$ is a $\mathbf{g}_y$-orthogonal basis for the corresponding tangent space. We call $\mathbf{Ric}(y)$ the *Ricci curvature of* y.

Let $V = v^i \partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric $\widetilde{g} = \mathbf{g}_V$ and a linear connection ∇^V (called *Chern connection*) on the tangent bundle over $\mathcal{U}$ as follows:

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x,v)\frac{\partial}{\partial x^k}.$$

From the torsion freeness and almost $\mathbf{g}$-compatibility of Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X,Y], \tag{2.1}$$

$$X \cdot \mathbf{g}_V(Y,Z) = \mathbf{g}_V(\nabla^V_X Y, Z) + \mathbf{g}_V(Y, \nabla^V_X Z) + 2\mathbf{C}_V(\nabla^V_X V, Y, Z), \tag{2.2}$$

here $\mathbf{C}_V = C_{ijk}(x,v)dx^i \otimes dx^j \otimes dx^k$.

Given a Finsler manifold (M,F), the *dual Finsler metric* F^* on M is defined by

$$F^*(\xi_x) := \sup_{Y \in T_xM\backslash 0} \frac{\xi(Y)}{F(Y)}, \quad \forall \xi \in T^*M,$$

and the corresponding fundamental tensor is defined by

$$g^{*kl}(\xi) = \frac{1}{2}\frac{\partial^2 F^{*2}(\xi)}{\partial \xi_k \partial \xi_l}.$$

The *Legendre transformation* $l : TM \to T^*M$ is defined by

$$l(Y) = \begin{cases} \mathbf{g}_Y(Y,\cdot), & Y \neq 0 \\ 0, & Y = 0. \end{cases}$$

It is well-known that for any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM\backslash 0$ onto $T^*_xM\backslash 0$, and it is norm-preserving, namely, $F(Y) = F^*(l(Y)), \forall Y \in TM$. Consequently, $g^{ij}(Y) = g^{*ij}(l(Y))$.

Now let $f : M \to \mathbb{R}$ be a smooth function on M. The *gradient* of f is defined by $\nabla f = l^{-1}(df)$. Thus we have

$$df(X) = \mathbf{g}_{\nabla f}(\nabla f, X), \quad X \in TM.$$

Let $\mathcal{U} = \{x \in M : \nabla f \mid_x \neq 0\}$. We define the *Hessian* $H(f)$ of f on $\mathcal{U}$ as follows:

$$H(f)(X,Y) := XY(f) - \nabla^{\nabla f}_X Y(f), \quad \forall X, Y \in TM \mid_{\mathcal{U}}.$$

It is known that $H(f)$ is symmetric, and it can be rewritten as (see [15])

$$H(f)(X,Y) = \mathbf{g}_{\nabla f}(\nabla^{\nabla f}_X \nabla f, Y). \tag{2.3}$$

It should be noted that the notion of Hessian defined here is different from that in [8]. In that case $H(f)$ is in fact defined by

$$H(f)(X,X) = X \cdot X \cdot (f) - \nabla^X_X X(f),$$

and there is no definition for $H(f)(X,Y)$ if $X \neq Y$. The advantage of our definition is that $H(f)$ is a symmetric bilinear form and we can treat it by using the theory of symmetric matrix.

3. Volume form

A *volume form* $d\mu$ on Finsler manifold (M,F) is nothing but a global non-degenerate n-form on M. In local coordinates we can express $d\mu$ as $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. The frequently used volume forms in Finsler geometry are so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form. In [14] we introduce the maximal and minimal volume forms for Finsler manifolds which play the important role in comparison technique in Finsler geometry. They are defined as following. Let

$$dV_{max} = \sigma_{max}(x)dx^1 \wedge \cdots \wedge dx^n$$

and

$$dV_{min} = \sigma_{min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{max}(x) := \max_{y \in T_xM\setminus 0} \sqrt{\det(g_{ij}(x,y))},\ \sigma_{min}(x) := \min_{y \in T_xM\setminus 0} \sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the n-forms dV_{max} and dV_{min} as well as the function $\nu := \frac{\sigma_{max}}{\sigma_{min}}$ are well-defined on M. dV_{max} and dV_{min} are called the *maximal volume form* and the *minimal volume form* of (M,F), respectively. Both maximal volume form and minimal volume form are called *extreme volume form*, and we shall denote by dV_{ext} the maximal or minimal volume form. The volume with respect to dV_{max} (respectively dV_{min}) is called the *maximal volume* (respectively *minimal volume*). Maximal volume and minimal volume are both called *extreme volume*.

The *uniformity function* $\mu : M \to \mathbb{R}$ is defined by

$$\mu(x) = \max_{y,z,u \in T_xM\setminus 0} \frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

$\mu_F = \max_{x\in M} \mu(x)$ is called the *uniformity constant* [3]. It is clear that

$$\mu^{-1}F^2(u) \leq \mathbf{g}_y(u,u) \leq \mu F^2(u,u).$$

Similarly, the *reversible function* $\lambda : M \to \mathbb{R}$ is defined by

$$\lambda(x) = \max_{y \in T_xM\setminus 0} \frac{F(y)}{F(-y)}.$$

$\lambda_F = \max_{x\in M}\lambda(x)$ is called the *reversibility* of (M,F) [7], and (M,F) is called *reversible* if $\lambda_F = 1$. It is clear that $\lambda(x)^2 \le \mu(x)$.

Proposition 3.1. *Let (M,F) be an n-dimensional Finsler manifold. Then*
(1) F is Riemannian $\Leftrightarrow \nu = 1 \Leftrightarrow \mu = 1$;
(2) $\nu \le \mu^{\frac{n}{2}}$.

Proof. (1) is obvious, so we only prove (2). For fixed $x \in M$, let $y, z \in T_xM\backslash 0$ be two vectors so that $\sigma_{max}(x) = \sqrt{\det(g_{ij}(x,y))}$ and $\sigma_{min}(x) = \sqrt{\det(g_{ij}(x,z))}$. Let $e_1, \cdots, e_n$ be an $\mathbf{g}_z$-orthogonal basis for T_xM such that they are eigenvectors of $(g_{ij}(x,y))$ with eigenvalues $\rho_1, \cdots, \rho_n$. Then

$$\rho_i = \mathbf{g}_y(e_i,e_i) \le \mu(x)\mathbf{g}_z(e_i,e_i) = \mu(x),$$

and consequently,

$$\nu(x) = \sqrt{\rho_1\rho_2\cdots\rho_n} \le \mu(x)^{\frac{n}{2}}.$$

□

Remark 3.2. *In [14] we wrongly write $\nu(x) = \rho_1\rho_2\cdots\rho_n$ and obtain $\nu \le \mu^n$ accordingly. Of course $\nu \le \mu^n$ is correct but obviously not optimal since $\mu(x) \ge 1$. As the result, the quantity Λ^{n+1} in Theorems 7.1-7.3 of [14] can be improved into $\Lambda^{\frac{n}{2}+1}$.*

Fix $x \in M$, let $I_x = \{v \in T_xM : F(v) = 1\}$ be the indicatrix at x. For $v \in I_x$, the *cut-value* $c(v)$ is defined by

$$c(v) := \sup\{t > 0 : d(x, \exp_x(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(x)$ of x by $\mathbf{C}(x) := \{c(v)v : c(v) < \infty, v \in I_x\}$, the *cut locus* $C(x)$ of x by $C(x) = \exp_x \mathbf{C}(x)$, and the *injectivity radius* i_x at x by $i_x = \inf\{c(v) : v \in I_x\}$, respectively. It is known that $C(x)$ has zero Hausdorff measure in M. Also, we set $\mathbf{D}_x = \{tv : 0 \le t < c(v), v \in I_x\}$ and $D_x = \exp_x \mathbf{D}_x$. It is known that $\mathbf{D}_x$ is the largest domain, which is starlike with respect to the origin of T_xM for which $\exp_x$ restricted to that domain is a diffeomorphism, and $D_x = M\backslash C(x)$.

In the following we consider the polar coordinates on $D(x)$. For any $q \in D(x)$, the polar coordinates of q are defined by $(r,\theta) = (r(q), \theta^1(q), \cdots, \theta^{n-1}(q))$, where $r(q) = F(v), \theta^\alpha(q) = \theta^\alpha(u)$, here $v = \exp_x^{-1}(q)$ and $u = v/F(v)$. Then by the Gauss lemma (see [2], page 140), the unit radial coordinate vector $T = d(\exp_x)\left(\frac{\partial}{\partial r}\right)$ is $\mathbf{g}_T$-orthogonal to coordinate vectors ∂_α which is defined by

$$\partial_\alpha|_{\exp_x(ru)} = d(\exp_x)\left(\frac{\partial}{\partial\theta^\alpha}\right)\bigg|_{\exp_x(ru)}$$

$$= d(\exp_x)_{ru}\left(r\frac{\partial}{\partial\theta^\alpha}\right) = rd(\exp_x)_{ru}\left(\frac{\partial}{\partial\theta^\alpha}\right)$$

for $\alpha = 1, \cdots, n-1$, and consequently, $T = \nabla r$. Consider the singular Riemannian metric $\widetilde{g} = \mathbf{g}_{\nabla r}$ on $D(x)$, then it is clear that

$$\widetilde{g} = dr^2 + \widetilde{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta, \quad \widetilde{g}_{\alpha\beta} = \mathbf{g}_{\nabla r}(\partial_\alpha, \partial_\beta).$$

4. Relative volume comparison theorems

In order to study the volume we need the following result which can be verified directly.

Lemma 4.1. *Let f, g are two positive integrable functions of r. If f/g is monotone increasing (respectively decreasing), then the function*

$$\frac{\int_0^r f(t)dt}{\int_0^r g(t)dt}$$

is also monotone increasing (respectively decreasing).

Let $B_p(R)$ be the forward geodesic ball of M with radius R centered at p, and $d\mu$ a volume form of (M, F). By definition, $B_p(R) = r^{-1}([0, R))$, here $r = d(p, \cdot) : M \to \mathbb{R}$ is the distance function from p. The volume of $B_p(R)$ with respect to $d\mu$ is defined by

$$\mathrm{vol}(B_p(R)) = \int_{B_p(R)} d\mu \ .$$

For $r > 0$, let $\mathbf{D}_p(r) \subset I_p$ be defined by

$$\mathbf{D}_p(r) = \{v \in I_p : rv \in \mathbf{D}_p\}.$$

It is easy to see that $\mathbf{D}_p(r_1) \subset \mathbf{D}_p(r_2)$ for $r_1 > r_2$ and $\mathbf{D}_p(r) = I_p$ for $r < i_p$. Consider the Riemannian metric $\widetilde{g} = \mathbf{g}_{\nabla r}$ on $\dot{B}_p(R) = B_p(R) \cap D_p \backslash \{p\}$ as defined in §3. It is clear that the corresponding volume form is $dV_{\widetilde{g}} = \widetilde{\sigma}(r, \theta)dr \wedge d\theta^1 \wedge \cdots \wedge \theta^{n-1} := \widetilde{\sigma}(r, \theta)dr \wedge d\theta$, here

$\widetilde{\sigma}(r,\theta)=\sqrt{\det(\widetilde{g}_{\alpha\beta})}$. Since $C(p)$ has zero Hausdorff measure in M, we have

$$\mathrm{vol}_{\widetilde{g}}(B_p(R))=\int_{B_p(R)}dV_{\widetilde{g}}=\int_{B_p(R)\cap D_p}dV_{\widetilde{g}}$$

$$=\int_{\exp_p^{-1}(B_p(R))\cap\mathbf{D}_p}\exp_p^*(dV_{\widetilde{g}})=\int_0^R dr\int_{\mathbf{D}_p(r)}\widetilde{\sigma}(r,\theta)d\theta. \tag{4.1}$$

Let

$$\mathrm{s}_c(t)=\begin{cases}\frac{\sin(\sqrt{c}t)}{\sqrt{c}}, & c>0\\ t, & c=0\\ \frac{\sinh(\sqrt{-c}t)}{\sqrt{-c}}, & c<0\end{cases}, \tag{4.2}$$

$$V_{c,n}(R)=\mathrm{vol}(\mathbb{S}^{n-1}(1))\int_0^R\mathrm{s}_c(t)^{n-1}dt.$$

The geometric meaning of $V_{c,n}(R)$ is that it equals to $\mathrm{vol}(\mathbb{B}_c^n(R))$ when $R\le i_c$, here $\mathbb{B}_c^n(R)$ denotes the geodesic ball of radius R in space form of constant c, and i_c the corresponding injectivity radius. Now we are ready to prove the following relative volume comparison theorem with flag curvature bound.

Theorem 4.2. *Let (M,F) be a forward complete Finsler n-manifold which satisfies $\mathbf{K}(V;W)\le c$. Then*

$$\frac{\mathrm{vol}_{ext}(B_p(r))}{\mathrm{vol}(\mathbb{B}_c^n(r))}\le\max_{x\in B_p(R)}\mu(x)^{\frac{n}{2}}\cdot\frac{\mathrm{vol}_{ext}(B_p(R))}{\mathrm{vol}(\mathbb{B}_c^n(R))}$$

for any $r<R\le i_p$, here vol_{ext} denotes the extreme volume (i.e., the maximal volume vol_{max} or minimal volume vol_{min}), and i_p the injectivity radius of p.

Proof. Recall that $T=\nabla r$ is a geodesic field, and

$$[T,\partial_\alpha]=\left[d(\exp_p)\left(\frac{\partial}{\partial r}\right),d(\exp_p)\left(\frac{\partial}{\partial\theta^\alpha}\right)\right]=0,$$

by (2.1)-(2.3) we have

$$\frac{\partial\widetilde{g}_{\alpha\beta}}{\partial r}=T\cdot\mathbf{g}_T(\partial_\alpha,\partial_\beta)=\mathbf{g}_T(\nabla_T^T\partial_\alpha,\partial_\beta)+\mathbf{g}_T(\partial_\alpha,\nabla_T^T\partial_\beta)$$

$$=\mathbf{g}_T(\nabla_{\partial_\alpha}^T T,\partial_\beta)+\mathbf{g}_T(\partial_\alpha,\nabla_{\partial_\beta}^T T)=2H(r)(\partial_\alpha,\partial_\beta).$$

Consequently,

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} = \frac{1}{2}\widetilde{g}^{\alpha\beta}\frac{\partial\widetilde{g}_{\alpha\beta}}{\partial r} = \mathrm{tr}_{\mathbf{g}_T}H(r). \tag{4.3}$$

Since $\mathbf{K}(V;W) \leq c$, by Hessian comparison theorem [15] it follows that

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} \geq (n-1)\mathrm{ct}_c(r) = \frac{d}{dr}\log\left(\mathrm{s}_c(r)^{n-1}\right), \tag{4.4}$$

here s_c is given by (4.2), and

$$\mathrm{ct}_c(r) = \begin{cases} \sqrt{c}\cdot\mathrm{cotan}(\sqrt{c}r), & c>0 \\ \frac{1}{r}, & c=0 \\ \sqrt{-c}\cdot\mathrm{cotanh}(\sqrt{-c}r), & c<0 \end{cases}.$$

From (4.4) we see that the function

$$\frac{\int_{I_p}\widetilde{\sigma}(r,\theta)d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{s}_c(r)^{n-1}}$$

is monotone increasing about $r(\leq i_p)$, and thus by Lemma 4.1 and (4.1) the function

$$\frac{\int_0^R\int_{I_p}\widetilde{\sigma}(r,\theta)drd\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\int_0^R \mathrm{s}_c(r)^{n-1}dr} = \frac{\mathrm{vol}_{\widetilde{g}}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))}$$

is also monotone increasing for $R \leq i_p$. On the other hand, by Proposition 3.1 it is clear that $dV_{min} \leq dV_{\widetilde{g}} \leq dV_{max} = \nu(x)\cdot dV_{min} \leq \mu(x)^{\frac{n}{2}}\cdot dV_{min}$, and consequently,

$$\frac{\mathrm{vol}_{min}(B_p(r))}{\mathrm{vol}(\mathbb{B}^n_c(r))} \leq \frac{\mathrm{vol}_{\widetilde{g}}(B_p(r))}{\mathrm{vol}(\mathbb{B}^n_c(r))} \leq \frac{\mathrm{vol}_{\widetilde{g}}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))} \leq \max_{x\in B_p(R)}\mu(x)^{\frac{n}{2}}\cdot\frac{\mathrm{vol}_{min}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))}$$

holds for any $r < R \leq i_p$. Similarly,

$$\frac{\mathrm{vol}_{max}(B_p(r))}{\mathrm{vol}(\mathbb{B}^n_c(r))} \leq \max_{x\in B_p(R)}\mu(x)^{\frac{n}{2}}\cdot\frac{\mathrm{vol}_{max}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))}$$

for any $r < R \leq i_p$, and the theorem is proved. □

We also have the following relative volume comparison theorem with Ricci curvature bound.

Theorem 4.3. *Let (M,F) be a forward complete Finsler n-manifold which satisfies $\mathbf{Ric}_M \geq (n-1)c$. Then*

$$\frac{\mathrm{vol}_{ext}(B_p(r))}{\mathrm{vol}(\mathbb{B}^n_c(r))} \geq \max_{x\in B_p(R)} \mu(x)^{-\frac{n}{2}} \cdot \frac{\mathrm{vol}_{ext}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))}, \quad \forall r < R.$$

Proof. First we note that from (4.3) and the proof of Theorem 5.3 of [15] one has

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} = \mathrm{tr}_{\mathbf{g}_T} H(r) \leq (n-1)\mathrm{ct}_c(r) = \frac{d}{dr}\log\left(\mathrm{s}_c(r)^{n-1}\right),$$

namely, the function

$$\frac{\widetilde{\sigma}(r,\theta)}{\mathrm{s}_c(r)^{n-1}}$$

is monotone decreasing for r where it is smooth. Noting that $\mathbf{D}_p(R) \subset \mathbf{D}_p(r)$ for $R > r > 0$, we have for $R > r > 0$,

$$\frac{\int_{\mathbf{D}_p(r)}\widetilde{\sigma}(r,\theta)d\theta}{\mathrm{s}_c(r)^{n-1}} = \int_{\mathbf{D}_p(r)}\frac{\widetilde{\sigma}(r,\theta)}{\mathrm{s}_c(r)^{n-1}}d\theta \geq \int_{\mathbf{D}_p(R)}\frac{\widetilde{\sigma}(r,\theta)}{\mathrm{s}_c(r)^{n-1}}d\theta$$

$$\geq \int_{\mathbf{D}_p(R)}\frac{\widetilde{\sigma}(R,\theta)}{\mathrm{s}_c(R)^{n-1}}d\theta = \frac{\int_{\mathbf{D}_p(R)}\widetilde{\sigma}(R,\theta)d\theta}{\mathrm{s}_c(R)^{n-1}},$$

which together with (4.1) and Lemma 4.1 implies that

$$\frac{\mathrm{vol}_{\widetilde{g}}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))} = \frac{\int_0^R dr\int_{\mathbf{D}_p(r)}\widetilde{\sigma}(r,\theta)d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\int_0^R \mathrm{s}_c(r)^{n-1}dr}$$

is monotone decreasing for any $R > 0$. Now the theorem follows similarly as Theorem 4.2. □

5. Gromov pre-compactness theorem

The notion of Hausdorff distance between metrics spaces was generalized by M. Gromov, and the corresponding pre-compactness theorem for Riemannian manifolds was proved in [4]. Gromov pre-compactness property has been generalized to Finsler manifolds by Shen [8] in reversible case and by Shen and Zhao [10] in non-reversible case. To state our result let us first recall some notations related to Gromov pre-compactness, for details one is referred to see [10]. As is known in [10], any Finsler manifold (M, F) induces a general metric space (M, d). Let $(\mathcal{M}^\delta, d^\delta_{GH})$ denote the collection of compact general metric space with δ-Gromov-Hausdorff distance d^δ_{GH} whose reversibilities are not large than $\delta < \infty$, and $\mathrm{Cap}_M(\epsilon)$ be the maximal number of disjoint forward geodesic ball of radius ϵ in M. Also, let $(\mathcal{M}^\delta_*, d^\delta_{GH})$ be the collection of proper pointed general metric space whose reversibilities are not large than $\delta < \infty$.

Lemma 5.1. *[10] (1) Let $\mathcal{C} \subset (\mathcal{M}^\delta, d^\delta_{GH})$ be a class satisfying the following conditions:*
(a) There is a constant D such that $\mathrm{Diam}M \leq D$ for all $M \in \mathcal{C}$.
(b) For each $\epsilon > 0$ there exists $N = N(\epsilon) < \infty$ such that $\mathrm{Cap}_M(\epsilon) \leq N(\epsilon)$ for all $M \in \mathcal{C}$.
Then $\mathcal{C}$ is pre-compact in the δ-Gromov-Hausdorff topology.
(2) A class $\mathcal{C} \subset (\mathcal{M}^\delta_*, d^\delta_{GH})$ is pre-compact if for each $r > 0$ and $\epsilon > 0$, there exists a number $N = N(r, \epsilon) < \infty$ such that for every $\overline{B_x(r)} \subset (M, x) \in \mathcal{C}$, one has $\mathrm{Cap}_{\overline{B_x(r)}}(\epsilon) \leq N(r, \epsilon)$.

The following Gromov pre-compactness theorem removes the additional restriction on S-curvature.

Theorem 5.2. *For any integer $n \geq 2$, $c \in \mathbb{R}$, and $D > 0$, the following classes are pre-compact in the (pointed) δ-Gromov-Hausdorff topology:*
(1) The collection $\{(M_i, F_i)\}$ of compact Finsler n-manifolds satisfying conditions

$$\mathrm{Diam}(M_i) \leq D, \quad \mathbf{Ric}_{M_i} \geq (n-1)c,$$

uniformity constant $\mu_{F_i} \leq \delta^2 < \infty$, for all i.
(2) The collection $\{(M_i, x_i, F_i)\}$ of pointed forward complete Finsler n-manifolds satisfying conditions

$$\mathbf{Ric}_{M_i} \geq (n-1)c,$$

uniformity constant $\mu_{F_i} \leq \delta^2 < \infty$, for all i.

Proof. Note that $\lambda^2_{F_i} \leq \mu_{F_i}$, one has $\{(M_i, F_i)\} \subset (\mathcal{M}^\delta, d^\delta_{GH})$. For each (M_i, F_i), note that $\mathrm{Diam}(M_i) \leq D$, one has $M_i = \overline{B_{x_i}(D)}$ for any $x_i \in M_i$. Since M_i is compact, there are finite disjoint forward geodesic balls $B_{x_1}(\epsilon), \cdots, B_{x_l}(\epsilon)$ of radius ϵ in M_i. Let $B_{x_{l_0}}(\epsilon)$ be the forward geodesic ball with the smallest minimal volume. Then we have, by Theorem 4.3,

$$l \leq \frac{\mathrm{vol}_{min}(M_i)}{\mathrm{vol}_{min}(B_{x_{l_0}}(\epsilon))} = \frac{\mathrm{vol}_{min}(B_{x_{l_0}}(D))}{\mathrm{vol}_{min}(B_{x_{l_0}}(\epsilon))} \leq \frac{\mathrm{vol}(\mathbb{B}^n_c(D))}{\mathrm{vol}(\mathbb{B}^n_c(\epsilon))} \cdot \delta^n.$$

Now (1) is easily followed by (1) of Lemma 5.1. By (2) of Lemma 5.1, we can prove (2) similarly. □

6. The first Betti number

Let (M, F) be a compact Finsler n-manifold of reversibility λ_F and $\widetilde{M}$ be its universal covering space. By the Hurewicz theorem,

$$H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)].$$

Then $H_1(M, \mathbb{Z})$ acts by deck transformation on the covering space

$$\bar{M} := \widetilde{M}/[\pi_1(M), \pi_1(M)]$$

with quotient M. Denote by $\bar{F}$ the pulled-back Finsler metric of F on $\bar{M}$. Since $H_1(M, \mathbb{Z})$ is a finitely generated Abelian group, the rank of $H_1(M, \mathbb{Z})$ is equal to $b_1(M)$ =dim$H_1(M, \mathbb{R})$. For simplicity, set $\Gamma := H_1(M, \mathbb{Z})$. Recall that any finite-index subgroup of Γ has the same rank as Γ. The following lemma is proved by [12] which is the Finsler version of Lemma 37 in [6], page 274.

Lemma 6.1. *Given any fixed point $p \in \bar{M}$, there exists a finite-index subgroup $\Gamma' \subset \Gamma$ that is generated by elements $\gamma_1, \cdots, \gamma_{b_1}$ such that*

$$d(p, \gamma_i(p)) \leq (1 + \lambda_{\bar{F}})\mathrm{Diam}(M).$$

Furthermore, for each $\gamma \in \Gamma' - \{1\}$, we have

$$d(p, \gamma(p)) > \mathrm{Diam}(M).$$

Now we prove

Theorem 6.2. *There exists a finite constant $C = C(n, D, \Lambda, k)$ such that for any compact n-dimensional Finsler manifold (M, F) with diameter D, uniformity constant $\mu_F \leq \Lambda$ and $\mathbf{Ric}_M \geq -(n-1)k^2 (k \geq 0)$ one has $b_1 \leq C(n, D, \Lambda, k)$. Moreover, $b_1 \leq n$ when Dk is sufficiently small (depends on n and Λ).*

Proof. By Lemma 6.1, we can choose a covering $\bar{M}$ of M with free Abelian group of deck transformation $\Gamma = \langle \gamma_1, \cdots, \gamma_{b_1} \rangle$ such that for some point $p \in \bar{M}$, $d(p, \gamma_i(p)) \leq (1 + \lambda_F)D$ and $d(p, \gamma(p)) > D, \forall \gamma \in \Gamma - \{1\}$. Note that $\lambda_{\bar{F}} = \lambda_F$, it is clear that

$$B_{\gamma_i(p)}\left(\frac{D}{2\lambda_F}\right) \subset B_p\left((1 + \lambda_F)D + \frac{D}{2\lambda_F}\right), \quad \forall 1 \leq i \leq b_1,$$

and all these forward balls are mutually disjoint and have the same minimal volume, as γ_i acts isometrically. Therefore,

$$b_1 \leq \frac{\mathrm{vol}_{min}\left(B_p\left((1+\lambda_F)D + \frac{D}{2\lambda_F}\right)\right)}{\mathrm{vol}_{min}\left(B_p\left(\frac{D}{2\lambda_F}\right)\right)}.$$

Notice that $\mu_{\bar{F}} = \mu_F$ and $1 \leq \lambda_F^2 \leq \mu_F$, which together with Theorem 4.3 yields

$$b_1 \leq \Lambda^{\frac{n}{2}} \cdot \frac{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left((1+\sqrt{\Lambda})D + \frac{D}{2}\right)\right)}{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left(\frac{D}{2\sqrt{\Lambda}}\right)\right)} =: C(n, D, \Lambda, k).$$

To prove the second result, it suffices to show the case when $k > 0$. Suppose on the contrary that $b_1 \geq n+1$, and define $I_r \subset \Gamma$ by

$$I_r = \left\{\gamma \in \Gamma : \gamma = \sum_{i=1}^{n+1} k_i \cdot \gamma_i, \sum_{i=1}^{n+1} |k_i| \leq r\right\}.$$

For $\gamma = \sum_{i=1}^{n+1} k_i \cdot \gamma_i \in I_r$, we have

$$d(p, \gamma(p)) \leq \sum_{i=1}^{n+1} |k_i| d(p, \gamma_i(p)) \leq r(1+\lambda_F)D,$$

which implies that

$$B_{\gamma(p)}\left(\frac{D}{2\lambda_F}\right) \subset B_p\left(r(1+\lambda_F)D + \frac{D}{2\lambda_F}\right), \quad \forall \gamma \in I_r.$$

On the other hand, all these forward balls are mutually disjoint and have the same minimal volume, as γ acts isometrically. We can use Theorem 4.3 to conclude that the cardinality $\sharp I_r$ of I_r is bounded from above by

$$\frac{\mathrm{vol}_{min}\left(B_p\left(r(1+\lambda_F)D + \frac{D}{2\lambda_F}\right)\right)}{\mathrm{vol}_{min}\left(B_p\left(\frac{D}{2\lambda_F}\right)\right)} \leq \Lambda^{\frac{n}{2}} \cdot \frac{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left(r(1+\sqrt{\Lambda})D + \frac{D}{2}\right)\right)}{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left(\frac{D}{2\sqrt{\Lambda}}\right)\right)}.$$

If $r \in \mathbb{N}$, then it is clear that

$$(2r+1)^{n+1} \leq \sharp I_{(n+1)r} \leq \Lambda^{\frac{n}{2}} \cdot \frac{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left(\left((n+1)r(1+\sqrt{\Lambda}) + \frac{1}{2}\right)D\right)\right)}{\mathrm{vol}\left(\mathbb{B}^n_{-k^2}\left(\frac{D}{2\sqrt{\Lambda}}\right)\right)}$$

$$= \Lambda^{\frac{n}{2}} \cdot \frac{\displaystyle\int_0^{((n+1)r(1+\sqrt{\Lambda})+\frac{1}{2})D} \left(\frac{\sinh kt}{k}\right)^{n-1} dt}{\displaystyle\int_0^{\frac{D}{2\sqrt{\Lambda}}} \left(\frac{\sinh kt}{k}\right)^{n-1} dt}$$

$$= \Lambda^{\frac{n}{2}} \cdot \frac{\displaystyle\int_0^{((n+1)r(1+\sqrt{\Lambda})+\frac{1}{2})Dk} \sinh^{n-1}(t)dt}{\displaystyle\int_0^{\frac{Dk}{2\sqrt{\Lambda}}} \sinh^{n-1}(t)dt}.$$

Since $\lim_{t\to 0} \frac{\sinh t}{t} = 1$, there exists $\eta > 0$ such that

$$\frac{1}{2} < \frac{\sinh t}{t} < 2, \quad \forall 0 < t < \eta.$$

Take $Dk < \eta/((n+1)r(1+\sqrt{\Lambda})+\frac{1}{2})$, then

$$(2r+1)^{n+1} \leq \Lambda^{\frac{n}{2}} \cdot \frac{\displaystyle\int_0^{((n+1)r(1+\sqrt{\Lambda})+\frac{1}{2})Dk} 2^{n-1}t^{n-1}dt}{\displaystyle\int_0^{\frac{Dk}{2\sqrt{\Lambda}}} 2^{-(n-1)}t^{n-1}dt}$$

$$= 2^{3n-2} \cdot \Lambda^n \cdot \left((n+1)r(1+\sqrt{\Lambda})+\frac{1}{2}\right)^n.$$

The last formula is false when r is sufficiently large (depends on n and Λ), or equivalently, when Dk is sufficiently small, and so we are done. $\Box$

Corollary 6.3. *For any compact n-dimensional Finsler manifold (M,F) with nonnegative Ricci curvature one has $b_1 \leq n$.*

7. Finiteness of fundamental group

In this last section we shall use Theorem 4.3 to obtain some finiteness results of fundamental groups for Finsler manifolds. For given Finsler manifold (M,F), let $f : (\widetilde{M},\widetilde{F}) \to (M,F)$ be the universal covering with pulled-back metric, then it is known that the fundamental group is isomorphic to the deck transformation group and each deck transformation is an isometry of $(\widetilde{M},\widetilde{F})$ (see [11] for details). Recall that the first systole of a compact Finsler manifold (M,F), say $\mathrm{sys}_1(M)$, is defined to be the length of shortest closed, non-contractible curve in M. Let $\mathfrak{R}(n,\delta)$ be a pre-compact family of forward complete Finsler n-manifolds of reversibility $\leq \delta$ with respect to δ-Gromov-Hausdorff distance, and $\mathfrak{R}(n, \delta, \sigma) = \{(M, F) \in \mathfrak{R}(n, \delta) : \mathrm{sys}_1(M) \geq \sigma\}$.

Lemma 7.1. *[12] There are only finitely many isomorphic classes of fundamental groups in $\mathfrak{R}(n, \delta, \sigma)$.*

Theorem 7.2. *Let $\mathfrak{R}(n, c, \delta, \sigma, D)$ be the class of compact Finsler n-manifolds $\{(M_i, F_i)\}$ with*

$$\mathbf{Ric}_{M_i} \geq (n-1)c, \quad \mu_{F_i} \leq \delta^2, \quad \mathrm{sys}_1(M_i) \geq \sigma, \quad \mathrm{Diam}(M_i) \leq D.$$

Then there are only finitely many isomorphic classes of fundamental groups in $\mathfrak{R}(n, c, \delta, \sigma, D)$ for fixed n, c, δ, σ, D.

Proof. By Theorem 5.2 it is clear that $\mathfrak{R}(n, c, \delta, \sigma, D) \subset \mathfrak{R}(n, \delta, \sigma)$, the conclusion follows from Lemma 7.1 directly. □

Lemma 7.3. *[12] Let (M, F) be a compact Finsler n-manifold of reversibility λ and $\widetilde{M}$ be its universal covering space. For each $p \in M$, there always exists a generating set $\{\gamma_1, \cdots, \gamma_m\}$ for the fundamental group $\Gamma = \pi_1(M, p)$ such that $d(\widetilde{p}, \gamma_i(\widetilde{p})) \leq (1 + \lambda)\mathrm{Diam}(M)$ (where $\widetilde{p} \in f^{-1}(p)$ is in the fiber over $p \in M$) and such that all relations for Γ in these generators are of form $\gamma_i \gamma_j \gamma_k^{-1} = 1$.*

Given $n \in \mathbb{N}, c \in \mathbb{R}, \delta \in [1, \infty)$, and $v, D \in (0, \infty)$, let $\mathfrak{M}(n, c, \delta, v, D)$ be the class of compact Finsler n-manifolds $\{(M_i, F_i)\}$ with

$$\mathbf{Ric}_{M_i} \geq (n-1)c, \quad \mu_{F_i} \leq \delta^2, \quad \mathrm{vol}_{min}(M_i) \geq v \quad \mathrm{Diam}(M_i) \leq D.$$

Anderson [1] obtained two results concerning the finiteness of fundamental group of Riemannian manifolds, and his results have been extended to Finsler manifolds recently [12]. We have the following two results which remove the additional condition on S-curvature in [12].

Theorem 7.4. *There are only finitely many isomorphic classes of fundamental groups in $\mathfrak{M}(n, c, \delta, v, D)$ for fixed n, c, δ, v, D.*

Proof. For each $(M, F) \in \mathfrak{M}(n, c, \delta, v, D)$, choose the generating set $\{\gamma_1, \cdots, \gamma_m\}$ of $\pi_1(M, p)$ as in Lemma 7.3. Since the number of possible relations is bounded by 2^{m^3}, it suffices to show m is bounded. Let $\Omega_p \subset \widetilde{M}$ be a fundamental domain constructed as [11]. The sets $\gamma_i(\Omega_p), 1 \leq i \leq m$ are mutually disjoint and have same minimal volume as M. Since $d(\widetilde{p}, \gamma_i(\widetilde{p})) \leq (1+\delta)D, \gamma_i(\Omega_p) \subset B_{\widetilde{p}}(2(1+\delta)D), \forall 1 \leq i \leq m$. Since $\mathbf{Ric}_M \geq (n-1)c$ and $\mu_F \leq \delta^2$, one has $\mathbf{Ric}_{\widetilde{M}} \geq (n-1)c$ and $\mu_{\widetilde{F}} \leq \delta^2$, and thus by Theorem 5.5 of [14], $\mathrm{vol}_{min}(B_{\widetilde{p}}(R)) \le \delta^n \cdot \mathrm{vol}(\mathbb{B}_c^n(R)), \forall R > 0$. Hence,

$$m \le \frac{\mathrm{vol}_{min}(B_{\widetilde{p}}(2(1+\delta)D))}{\mathrm{vol}_{min}(\Omega_p)} \le \frac{\delta^n \cdot \mathrm{vol}(\mathbb{B}_c^n(2(1+\delta)D))}{v} < \infty. \tag{7.1}$$

□

Theorem 7.5. *For fixed n, c, δ, v, D there exist*

$$L = \frac{vD}{\delta^n \mathrm{vol}(\mathbb{B}_c^n(2D))}, \quad N = \frac{\delta^n \mathrm{vol}(\mathbb{B}_c^n(2D))}{v}$$

such that for each $(M, F) \in \mathfrak{M}(n, c, \delta, v, D)$, if $\alpha \in \pi_1(M)$ with $\alpha^t \neq 1$ for all $t \le N$, then $\|\alpha\|_{geo} \ge L$, here $\|\alpha\|_{geo}$ is the geometric norm of α which is the length of a shortest loop representing α.

Proof. Let $\alpha \in \pi_1(M)$ with $\alpha^t \neq 1$ for all $t \le N$. Define $U(r) = \{\alpha^s : 0 \le s \le r\}$. For $p \in M$ fix a $\widetilde{p} \in f^{-1}(p)$ and let Ω_p be a fundamental domain as before. Suppose on the contrary that $\|\alpha\|_{geo} < L$, then $d(\widetilde{p}, \gamma(\widetilde{p})) < rL$ for each $\gamma \in U(r)$, which implies $\gamma(\Omega_p) \subsetneq B_{\widetilde{p}}(rL + D)$. Since $\gamma(\Omega_p)$ are mutually disjoint and has same minimal volume as M, one has

$$\sharp U(r) < \frac{\mathrm{vol}_{min}(B_{\widetilde{p}}(rL + D))}{\mathrm{vol}_{min}(\Omega_p)} \le \frac{\delta^n \mathrm{vol}(\mathbb{B}_c^n(rL + D))}{v}.$$

By assumption, $\sharp U(N) \ge N$, and by taking $r = N$ we have

$$\frac{\delta^n \mathrm{vol}(\mathbb{B}_c^n(2D))}{v} = N \le \sharp U(N) < \frac{\delta^n \mathrm{vol}(\mathbb{B}_c^n(2D))}{v},$$

which is a contradiction. □

Let G be a finitely generated group and $S = \{g_i\}$ be a generating set for G. For each $g \in G$, define $\|g\|_{alg}$ be the smallest length of the word in terms of g_i and their inverse that represents g. We call $\| \cdot \|_{alg}$ the *algebraic norm* associated with the generating set S. One is referred to see [4, 5, 11] for more details of algebraic norm.

The following theorem is the Finsler version of Wei's result [13].

Theorem 7.6. *Given any constants $\delta \ge 1$ and $v > 0$, there exists $\epsilon = \epsilon(n, \delta, v) > 0$ such that if a compact n-manifold M admits a Finsler metric F satisfying the conditions $\mathbf{Ric}_M \ge -(n-1)\epsilon, \mathrm{Diam}(M) = 1, \mu_F \le \delta^2$ and $\mathrm{vol}_{min}(M) \ge v$, then $\pi_1(M)$ is of polynomial growth of order $\le n$.*

Proof. Choose a base point $\widetilde{p}$ in the universal covering $(\widetilde{M}, \widetilde{F}) \xrightarrow{f} (M, F)$, and let $p = f(\widetilde{p})$ and $\{\gamma_1, \cdots, \gamma_k\}$ be a set of generators of $\pi_1(M)$ viewed as deck transformation in the isometry group of $(\widetilde{M}, \widetilde{F})$. Define $\Gamma(s) = \{\gamma \in \pi_1(M) : \|\gamma\|_{alg} \leq s\}$, and $l = \max_{1 \leq i \leq k}\{d(\widetilde{p}, \gamma_i(\widetilde{p}))\}$. Choose the fundamental domain Ω_p as before, one has $\gamma(\Omega_p) \subset B_{\widetilde{p}}(sl + \mathrm{Diam}(M))$ for each $\gamma \in \Gamma(s)$, and consequently,

$$\sharp\Gamma(s) \leq \frac{\mathrm{vol}_{min}(B_{\widetilde{p}}(sl + \mathrm{Diam}(M))}{\mathrm{vol}_{min}(M)}. \tag{7.2}$$

Suppose that for any $1 > \epsilon > 0$, there exists a Finsler metric F satisfying $\mathbf{Ric}_M \geq -(n-1)\epsilon, \mathrm{Diam}(M) = 1, \mu_F \leq \delta^2$ and $\mathrm{vol}_{min}(M) \geq v$ such that $\pi_1(M)$ is not of polynomial growth of order $\leq n$. Clearly, $(M, F) \in \mathfrak{M}(n, -1, \delta, v, 1)$. By Lemma 7.3 and (7.1), one can choose a finite generating set $\{\gamma_1, \cdots, \gamma_m\}$ of $\pi_1(M)$ such that

(i) $m \leq \frac{\delta^n \cdot \mathrm{vol}(\mathbb{B}^n_{-1}(2(1+\delta)))}{v} := N(n, \delta, v)$.
(ii) $d(\widetilde{p}, \gamma_i(\widetilde{p})) \leq 1 + \delta$, for each $1 \leq i \leq m$.
(iii) every relation is of form $\gamma_i\gamma_j\gamma_k^{-1} = 1$.

Since $\pi_1(M)$ is not of polynomial growth of order $\leq n$, for each $j \in \mathbb{N}$, there exists $s_j \in \mathbb{N}$ such that

$$\sharp\Gamma(s_j) > j \cdot (s_j)^n. \tag{7.3}$$

It is crucial that this relation is independent of ϵ, as follows from (i) and (iii).

Now by (7.2) and Theorem 5.5 of [14] we have

$$\sharp\Gamma(s) \leq \frac{\delta^n \mathrm{vol}(\mathbb{B}^n_{-\epsilon}((1+\delta)s + 1))}{v}$$

$$= \frac{\delta^n \mathrm{vol}(\mathbb{S}^{n-1}(1))}{v} \int_0^{(1+\delta)s+1} \left(\frac{\sinh\sqrt{\epsilon}t}{\sqrt{\epsilon}}\right)^{n-1} dt.$$

Choose $\eta > 0$ such that $\sinh t/t < 2$ for any $0 < t < \eta$. If $\sqrt{\epsilon}((1+\delta)s + 1) < \eta$, then

$$\sharp\Gamma(s) \leq \frac{\delta^n \mathrm{vol}(\mathbb{S}^{n-1}(1))}{v} \int_0^{(1+\delta)s+1} (2t)^{n-1} dt$$

$$\leq \frac{2^{n-1}\delta^n \mathrm{vol}(\mathbb{S}^{n-1}(1))(2(1+\delta))^n}{nv} s^n.$$

In summary, for any fixed, sufficiently large s_0, there is $\epsilon_0 = \epsilon_0(s_0, \delta) := \eta^2/((1+\delta)s_0+1)^2$ such that for each $s \leq s_0$ and $\epsilon \leq \epsilon_0$,

$$(7.4) \quad \sharp\Gamma(s) \leq C(n,\delta,v)s^n, C(n,\delta,v) := \frac{2^{2n-1}\delta^n \mathrm{vol}(\mathbb{S}^{n-1}(1))(1+\delta)^n}{nv}.$$

Now let $j_0 > C(n,\delta,v)$, by (7.3), there exists s_{j_0} such that

$$\sharp\Gamma(s_{j_0}) > C(n,\delta,v)(s_{j_0})^n.$$

But we get a contradiction by taking $\epsilon \leq \epsilon_0(s_{j_0}, \delta)$ and (7.4). □

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(B. Y. Wu) Department of Mathematics, Minjiang University, Fuzhou, China
E-mail address: `bingyewuyahoo.cn`