Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 1, pp. 243–261
Online ISSN: 1735-8515

VARIOUS KINDS OF REGULAR INJECTIVITY FOR S-POSETS

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(Communicated by Ali Reza Ashrafi)

Dedicated to Professor M. Mehdi Ebrahimi on his 65th Birthday

Abstract. In this paper some properties of weak regular injectivity for S-posets, where S is a pomonoid, are studied. The behaviour of different kinds of weak regular injectivity with products, coproducts and direct sums is considered. Also, some characterizations of pomonoids over which all S-posets are of some kind of weakly regular injective are obtained. Further, we give some Baer conditions which state the relation among some kinds of weak regular injectivity.
Keywords: S-poset, regular injective, weakly regular injective.
MSC(2010): Primary: 06F05; Secondary: 18G05, 20M30.

1. Introduction and Preliminaries

The properties of S-posets, posets with an action of a pomonoid S on them, have been studied in many papers, for example see [2–5, 7, 8, 11]. In [13], various kinds of weak regular injectivity of S-posets have been studied. In [13] the authors used the terms 'regularly (principally) weakly injectivity' for what is called here '(principally) weak regular injectivity'. This is because, these injectivity properties are weaker than the notion of 'regular injectivity' which is used for injective S-posets with respect to regular monomorphisms. In [12] regular injectivity of S-posets over Clifford pomonoids equipped with 'natural order' has been investigated, and in [7] regular injectivity has been considered in general.

Article electronically published on February 25, 2014.
Received: 18 April 2012, Accepted: 15 January 2013.
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In this paper the properties of weak regular injectivity in the category **Pos-S** of S-posets for a pomonoid S are studied. We study the behaviour of different kinds of weak regular injectivity with respect to products, coproducts and direct sums. Some characterizations of pomonoids over which all S-posets are of a kind of weakly regular injective are obtained. As a result, we give some Baer conditions which show how some kinds of injectivity are related to each other. In the rest of this section, we give some preliminaries about S-acts and S-posets needed in the sequel.

Let S be a monoid with identity 1. Recall that a (*right*) *S-act* A is a set together with a map $\lambda : A \times S \to A$, called its *action*, such that, denoting $\lambda(a, s)$ by as, we have $a1 = a$ and $a(st) = (as)t$ for all $a \in A$ and $s, t \in S$. Notice that the action $\lambda : A \times S \to A$ is completely determined by the family $\lambda_a : S \to A,\ s \mapsto as,\ a \in A$, of maps from S to A. The category of all S-acts, with the action-preserving (S-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$) between them, is denoted by **Act-S**. Clearly S is itself an S-act with its operation as the action. An element θ of an S-act is called a *zero* or a *fixed element* if $\theta s = \theta$ for all $s \in S$. For more information about S-acts see [6, 9].

A monoid (semigroup) S is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with the binary operation of S (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$). A *right poideal* of a pomonoid S is a (possibly empty) subset I of S which is both a monoid right ideal ($IS \subseteq I$) and a poset ideal (that is, a down subset of S: $a \leq b, b \in I$ imply $a \in I$). For every subset X of a pomonoid S,

$$\downarrow XS = \{t \in S : \exists x \in X, \exists s \in S, t \leq xs\}$$

is the smallest right poideal of S which contains X and is called the right poideal of S generated by X. One can easily prove that $\downarrow(\bigcup_{x\in X} xS) = \bigcup_{x\in X} \downarrow(xS)$.

Recall from [13] that a pomonoid S is called *right Noetherian* if it satisfies the ascending chain condition on right ideals. This is equivalent to all right ideals of S being finitely generated. Also, recall from [3] that a pomonoid S is called *weakly left reversible* if $sS \cap \downarrow(tS) \neq \emptyset$, for all $s, t \in S$.

A (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order.

An *S-poset map* (or *morphism*) is an action preserving monotone map between S-posets. The category of all right S-posets with S-poset maps between them is denoted by **Pos-S**.

An S-poset *congruence* on an S-poset A is an S-act congruence θ with the property that the quotient S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map.

Recall from [3] and [5] that, for a binary relation H on an S-poset A, the *S-poset congruence on A generated by H* is $\theta(H) = \nu(H \cup H^{-1})$, where $\nu(H)$, called the *S-poset congruence induced by H*, is given by:

$$a\nu(H)a' \text{ if and only if } a \leq_{\alpha(H)} a' \leq_{\alpha(H)} a$$

where $a\alpha(H)a'$ if and only if

$$a = a' \text{ or } a = x_1 s_1,\ x'_1 s_1 = x_2 s_2, \ldots,\ x'_n s_n = a'$$

for some $s_i \in S$ and $(x_i, x'_i) \in H$, $i = 1, \ldots, n$, and $\leq_{\alpha(H)}$ is defined by

$$a \leq_{\alpha(H)} a' \quad \text{if and only if} \quad a \leq a_1 \alpha(H) a'_1 \leq \ldots \leq a_n \alpha(H) a'_n \leq a'$$

for some $a_1, a'_1, \ldots, a_n, a'_n \in A$. Also, recall that an S-act congruence θ on an S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by

$$[a] \leq [b] \text{ if and only if } a \leq_\theta b.$$

Further, recall from [5] that the product of a family $\{A_i\}_{i \in I}$ of S-posets in the category **Pos-S** is the cartesian product $\prod_{i \in I} A_i$ with the componentwise order and action. Also the coproduct is the disjoint union $\dot{\bigcup}_{i \in I} A_i$ with the order given by $x \leq y$ if and only if $x, y \in A_i$ and $x \leq y$ in A_i, for some $i \in I$; and with the action of $s \in S$ on $a \in A_i$ being as defined in A_i.

Also, similar to the direct sum of S-acts (see for example [9]), we define the direct sum for a family $\{A_i : i \in I\}$ of S-posets each with a zero element 0, to be the sub S-poset of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number; and is denoted by $\bigoplus_{i \in I} A_i$. Notice that it is usual to take a unique zero element in the definition of direct sum, but since regular injective S-posets have at least two zero elements, we have ignored the uniqueness condition. In this paper, although we mostly need to assume that our S-posets have a zero element, but in some results we do not need this assumption.

Recall from [5] that in the categories **Pos** and **Pos-S**, the monomorphisms are exactly the injective morphisms and the epimorphisms and the surjective morphisms coincide. Also, regular monomorphisms (morphisms which are equalizers) are exactly the *order-embedding* S-poset maps, that is S-poset maps $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$.

We close this preliminaries by recalling from [13] the definitions of some types of injectivity and some results about S-posets (see also [1] for quasi injectivity).

Definition 1.1. An S-poset A is called:

(1) *regular injective* if it is injective with respect to regular monomorphisms;

(2) *weakly regular injective* (*fg-weakly regular injective, principally weakly regular injective*) if every S-poset map $f : I \to A$ from a (finitely generated, principal) right ideal I of S can be extended to an S-poset map $\overline{f} : S \to A$;

(3) *finitely regular injective* (*cyclicly regular injective*) if it is injective with respect to regular monomorphisms $h : F \to B$ from a finitely generated (cyclic) S-poset F;

(4) *quasi regular injective* if it is injective with respect to regular monomorphisms $h : B \to A$.

The following are Lemma 2.6, Theorem 2.11, and Theorem 2.12 of [7].

Lemma 1.2. *Every non-trivial regular injective S-poset is bounded by two zero elements.*

Theorem 1.3. *The category* **Pos-S** *has enough regular injectives in the sense that each S-poset can be regularly embedded into a regular injective S-poset.*

Theorem 1.4. *An S-poset A is regular injective if and only if every regular monomorphism $A \to B$ of S-posets has a left inverse.*

Corollaries 5.4, 5.11, and 5.12 of [13] can be restated as follows.

Corollary 1.5. *For a pomonoid S, the following conditions are equivalent:*

(i) *All right S-posets are principally weakly regular injective.*

(ii) *All right ideals of S are principally weakly regular injective.*

(iii) *All finitely generated right ideals of S are principally weakly regular injective.*

(iv) *All principal right ideals of* S *are principally weakly regular injective.*

(v) S *is a regular pomonoid.*

Corollary 1.6. *All* S*-posets are fg-weakly regular injective if and only if* S *is a regular pomonoid all of whose finitely generated right ideals are principal.*

Corollary 1.7. *All* S*-posets are weakly regular injective if and only if* S *is a regular principal right ideal pomonoid.*

The following is lemma 1 of [12].

Lemma 1.8. *An* S*-poset* A *is regular injective if and only if for every* S*-poset* C*, sub* S*-poset* $B \subseteq C$*, and* S*-poset morphism* $f : B \to A$*, there exists an* S*-poset morphism* $g : C \to A$ *such that* $g|_B = f$*.*

In view of the above lemma we mostly check in this paper regular injectivity by considering sub S-posets instead of regular monomorphisms (recall that by a sub S-poset of an S-poset A, we mean a subset of A which is closed under the action, and has the same order as A).

The following is an easy to prove fact about injective objects in any category, for example find it in Proposition I.7.30 of [9].

Remark 1.9. Retracts of injective objects in a category are injective.

2. Some Kinds of Weak Regular Injectivity for S-Posets

In this section we consider some kinds of weak regular injectivity for S-posets.

Proposition 2.1. *An* S*-poset* A *is (fg-, principally) weakly regular injective if and only if for every* S*-poset map* $f : K \to A$*, where* $K \subseteq S$ *is a (finitely generated, principal) right ideal, there exists an element* $a \in A$ *such that* $f = \lambda_a$*.*

Proof. We prove the result for weak regular injectivity, the proof for fg-(principally) weak regular injectivity is similar. For necessity, let K be a right ideal of S, and $f : K \to A$ an S-poset map. Then there exists an S-poset map $\overline{f} : S \to A$ such that $\overline{f}|_K = f$, since A is weakly regular injective. Put $\overline{f}(1) = a$. Then for every $k \in K$, $f(k) = \overline{f}(k) = \overline{f}(1k) = \overline{f}(1)k = ak$. For sufficiency, let $f : K \to A$ be an S-poset map. Then, by hypothesis, there exists $a \in A$ such that $f(k) = ak$, for all $k \in K$.

Define $\overline{f} : S \to A$, by $\overline{f}(s) = as$, for all $s \in S$. Then $\overline{f}$ is an S-poset map which extends f. □

Corollary 2.2. *If a (finitely generated, principal) right ideal K of S is (fg-, principally) weakly regular injective then K is principal and generated by an idempotent element.*

Proof. Let K be a weakly regular injective right ideal of S. Then by Proposition 2.1, there exists an element $e \in K$ such that $id_K = \lambda_e$. Therefore, we have $k = ek$ for every $k \in K$ and thus e is an idempotent and $K = eS$. □

Proposition 2.3. *Every finitely (cyclicly) regular injective S-poset is bounded by two zero elements.*

Proof. The proof is similar to the proof of Lemma 1.2. □

Proposition 2.4. *Every finitely generated (cyclic) S-poset which is finitely (cyclicly) regular injective, is regular injective.*

Proof. Let A be a finitely generated S-poset which is finitely regular injective. Let A be a sub S-poset of B. Since A is finitely generated, and finitely regular injective, there exists an S-poset map $g : B \to A$ such that $g|_A = id_A$. Then, A is regular injective, by Theorem 1.4. □

Definition 2.5. An S-poset A is said to be *P-quasi regular injective*, or briefly PQRI, if for every principally weakly regular injective sub S-poset B of A, every S-poset map of B into A can be extended to A.

The following theorem gives a more explicit characterization of regular injective S-posets.

Theorem 2.6. *Let A be an S-poset with a zero and E be a regular injective S-poset into which A can be regularly embedded, which exists according to Theorem* 1.3. *Then the following conditions are equivalent:*
(i) *A is regular injective.*
(ii) *A is a retract of E.*
(iii) *A is a principally weakly regular injective S-poset and $E \oplus A$ is PQRI.*

Proof. (i) $\Leftrightarrow$ (ii) is clear by Theorem 1.4 and Remark 1.9.

(iii) $\Rightarrow$ (ii) Let $D = E \oplus A$. Let $i : A \to D$ and $k : E \to D$ be the usual injection maps which are given by $a \mapsto (0, a)$ and $e \mapsto (e, 0)$, respectively. Let $j : A \to E$ be the regular monomorphism which exists by Theorem 1.3. Then $kj : A \to D$ is a regular monomorphism. Since

$D = E \oplus A$ is PQRI and A is principally weakly regular injective, there exists an S-poset map $h : D \to D$ such that $i = hkj$. Let $p : D \to A$ be the projection map to A. Then $pi = id_A$. Define $\varphi = phk$. Then, we have $\varphi j = phkj = pi = id_A$. Therefore, A is a retract of E.

(i) $\Rightarrow$ (iii) Because A is regular injective, it is obviously principally weakly regular injective. Also, since $E \oplus A$ is the product $E \times A$ and the product of injective objects in any category is clearly injective (using the universal property of products), $E \oplus A$ is regular injective. So $D = E \oplus A$ is clearly PQRI. □

The following result shows that weak left reversiblity of a pomonoid S is a necessary condition for regular injectivity of the two element chain $\mathbf{2} = \{0, 1\}$.

Proposition 2.7. *If the two element chain $\mathbf{2} = \{0, 1\}$ with the trivial action is regular injective, then S is weakly left reversible.*

Proof. Assume that S is not weakly left reversible. So there are $s, t \in S$ such that $sS \cap \downarrow(tS) = \emptyset$. Define a mapping $f : sS \cup \downarrow(tS) \to \mathbf{2}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in sS \\ 0 & \text{if } x \in \downarrow(tS). \end{cases}$$

Since sS and $\downarrow(tS)$ are both closed under the action, we get that f is an S-act map. To show that f is order preserving, let $x_1 \leq x_2, x_1, x_2 \in S$. If $x_1,\ x_2 \in sS$, then $f(x_1) = b \leq f(x_2) = b$. If $x_1, x_2 \in \downarrow(tS)$ then $f(x_1) = a \leq f(x_2) = a$. And if $x_1 \in \downarrow(tS), x_2 \in sS$, then $f(x_1) = 0 \leq f(x_2) = 1$. Therefore f is an S-poset map. Now, because every S-poset map $g : S \to \mathbf{2}$ is of the form $\lambda_{g(1)}$ (which is not an extension of f), f can not be extended to S which is a contradiction. □

3. Products, Coproducts and Direct Sums of Various Kinds of Weakly Regular Injective S-Posets

In this section we consider the behaviour of various kinds of weakly regular injective S-posets with respect to products, coproducts and direct sums. We show that, as is usual for injectivity, all kinds of weak regular injectivity is preserved by taking products, and if a product is of a kind of weak regular injectivity then so are all its components. But this is not the case for coproducts and direct sums.

Theorem 3.1. *Let $\{A_i : i \in I\}$ be a family of S-posets. Then the product $\prod_{i \in I} A_i$ is (finitely, cyclicly, weakly, fg-weakly, principally weakly)* *regular injective if and only if each A_i is (finitely, cyclicly) regular injective.*

Proof. The proof of the if part is routine in any category, using the universal property of products. To prove the converse, let $A = \prod_{i\in I} A_i$ be regular injective and $k \in I$. To prove that A_k is regular injective, consider the diagram

$$\begin{array}{rcl} B & \hookrightarrow & C \\ f\downarrow & & \\ A_k & & \\ p_k\uparrow & & \\ A & & \end{array}$$

where B is a sub S-poset of C and f is an S-poset map. Define $\overline{f} : B \to A$ by

$$\overline{f}(b)(i) = \begin{cases} f(b) & \text{if } i = k \\ \theta_i & \text{if } i \neq k \end{cases}$$

where for $i \in I$, θ_i is a zero element of A_i which exists since $A = \prod_{i\in I} A_i$ has a zero element by Lemma 1.2. The i-th component of that zero element is a zero element of A_i, $i \in I$. Then since f is an S-poset map, so is $\overline{f}$. Now by regular injectivity of A, $\overline{f}$ can be extended to an S-poset map $\overline{\overline{f}} : C \to A$. Then, $p_k\overline{\overline{f}} : C \to A_k$ extends f, where $p_k : \prod_{i\in I} A_i \to A_k$ is the k-th projection map. Therefore A_k is regular injective. The above proof works for finitely and cyclicly regular injectivity. □

The above theorem is equally true for weakly, fg-weakly, principally weakly regular injective S-posets if the S-posets involved have zero elements.

For coproducts, first we note that the following theorem is trivially true.

Theorem 3.2. *Let $\{A_i : i \in I\}$ be a family of S-posets. If the coproduct $\coprod_{i\in I} A_i$ is (quasi, weakly, fg-weakly, principally weakly) regular injective then each A_i is (quasi, finitely, cyclicly, weakly, fg-weakly, principally weakly) regular injective.*

For principally weakly regular injective S-posets, the converse is also true.

Theorem 3.3. *All coproducts of principally weakly regular injective S-posets are principally weakly regular injective.*

Proof. Let $\{A_i : i \in I\}$ be a family of principally weakly regular injective S-posets. Notice that for any principal right ideal sS of S, and any S-poset map $f : sS \to \coprod_{i\in I} A_i$, we have $f(s) \in A_i$ for some $i \in I$, and so $Imf \subseteq A_i$. This is because, sS is closed under the action. Now, f can be extended to an S-poset map $\overline{f} : S \to A_i$, since A_i is principally weakly regular injective. □

In the case of (fg-) weakly regular injective S-posets, we have the following theorem.

Theorem 3.4. *All coproducts of (fg-) weakly regular injective S-posets are (fg-) weakly regular injective if the pomonoid S is left reversible.*

Proof. Let $\{A_i : i \in I\}$ be a family of weakly regular injective S-posets. Notice that by hypothesis, for every right ideal K of S, and an S-poset map $f : K \to \coprod A_i$, we have $Imf \subseteq A_i$ for some $i \in I$. This is because, otherwise, $Imf \subseteq A_i \cap A_j$, for some $i \neq j \in I$. And then taking $J = f^{-1}(A_i)$, $L = f^{-1}(A_j)$, we have $K = J \cup L$ and $J \cap L = \emptyset$ which contradicts left reversibility of S. Now, f is of the form λ_a, for some $a \in A_i$, since A_i is weakly regular injective and by Proposition 2.1. □

Definition 3.5. The pomonoid S is called *left po-reversible* if every two right poideals of S have a nonempty intersection.

For the converse of the above theorem we have the following result.

Theorem 3.6. *If all coproducts of (fg-) weakly regular injective S-posets are (fg-) weakly regular injective, then the pomonoid S is left po-reversible.*

Proof. Let I, J be right poideals of S with $I \cap J = \emptyset$. Define the S-poset map $f : I \cup J \to \mathbb{1} \sqcup \mathbb{1}$, where $\mathbb{1} \sqcup \mathbb{1} = \{a, b\}$ is the two elements discrete poset, by

$$f(s) = \begin{cases} a & \text{if } s \in I \\ b & \text{if } s \in J. \end{cases}$$

Notice that $\mathbb{1}$ is clearly weak regular injective, and so by the hypothesis, $\mathbb{1} \sqcup \mathbb{1} = \{a, b\}$ is weak regular injective. Now, we show that $\overline{f}$ can not be extended to S, which contradicts weak regular injectivity of $\mathbb{1} \sqcup \mathbb{1}$. Let $\overline{f}$ be an extension of f. Then $\overline{f}(1) = a$ or b. Let $\overline{f}(1) = a$. Then for $s \in J$, $\overline{f}(s) = \overline{f}(1)s = as = a \neq b = f(s)$, which is a contradiction. Similarly, in the case where $\overline{f}(1) = b$, we get a contradiction. □

For coproducts of finitely (cyclicly) regular injective S-posets, we have the following result.

Proposition 3.7. *Let S be a pomonoid and $\{A_i : i \in I, |I| > 1\}$ be an arbitrary family of S-posets. Then $\coprod_{i \in I} A_i$ is not cyclicly (finitely) regular injective.*

Proof. The definition of the coproduct of S-posets shows that the coproduct of a family of S-posets can not be generally bounded, and so by Proposition 2.3 it is not cyclicly (finitely) regular injective. □

For direct sums of a family of S-posets, we have the following theorem.

Theorem 3.8. *Let $\{A_i : i \in I\}$ be a family of S-posets, each with a zero element. If the direct sum $\bigoplus_{i \in I} A_i$ is finitely (quasi, cyclicly, weakly, fg-weakly, principally weakly) regular injective then each A_i is finitely (quasi, cyclicly, weakly, fg-weakly, principally weakly) regular injective. The converse is true for finitely (cyclicly, fg-weakly, principally weakly) regular injectivity.*

Proof. We prove the result for finite regular injectivity. The proof of other parts are similar. Let the direct sum $\bigoplus_{i \in I} A_i$ be finitely regular injective. Fix $i \in I$. Let F be a finitely generated S-poset which is a sub S-poset of B and $f : F \to A_i$ be an S-poset map. Consider the injection map $\sigma_i : A_i \to \bigoplus_{i \in I} A_i$, given by $a_i \mapsto (\ldots, 0, a_i, 0, \ldots)$, where a_i is the i-th component. By hypothesis, there exists an S-poset map $\overline{\sigma_i f} : B \to \bigoplus_{i \in I} A_i$ which extends $\sigma_i f$. Then $p_i \overline{\sigma_i f} : B \to A_i$ extends f, where p_i is the i-th projection map. For the converse, let each A_i be a finitely regular injective S-poset. Let F be a finitely generated S-poset which is a sub S-poset of B and $f : F \to \bigoplus_{i \in I} A_i$ be an S-poset map. Assume that F is generated by $\{x_1, x_2, \ldots, x_n\}$. Then, since only finitely many components of each $f(x_i)$ are nonzero, Imf is contained in a direct sum of finitely many A_i, say for $i_1, i_2, \ldots, i_m$. Then, since the direct sum $A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m}$, which is in fact a product, is finitely regular injective by Theorem 3.1, there exists $f' : B \to \bigoplus_{j=i_1}^{i_m} A_j$ which extends $f : F \to \bigoplus_{j=i_1}^{i_m} A_j$. Finally, $\sigma f' : B \to \bigoplus_{i \in I} A_i$ extends f, where $\sigma : \bigoplus_{j=i_1}^{i_m} A_j \to \bigoplus_{i \in I} A_i$ is the injection map, given by $(a_j)_{j=i_1}^{i_m} \mapsto (\ldots, a_{i_1}, \ldots, a_{i_2}, \ldots, a_{i_m}, \ldots)$, where each $a_j, j = i_1, \ldots, i_m$ is the j-th component and other components are 0. □

For the direct sum of weakly regular injective S-posets we have the following result.

Theorem 3.9. *Each direct sum of weakly regular injective S-posets is weakly regular injective if the pomonoid S is right Noetherian.*

Proof. If S is right Noetherian, then an argument similar to that of Theorem 3.8 gives the result. □

For the converse, we have a theorem that follows the next definition.

Definition 3.10. A pomonoid S is called *right po-Noetherian* if all of its right poideals are finitely generated.

It is easy to check that a pomonoid S is right po-Noetherian if and only if it satisfies the *ascending chain condition on right poideals*, that is, for every ascending chain

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

of right poideals of S there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \ldots$.

Theorem 3.11. *If each direct sum of weakly regular injective S-posets is weakly regular injective, then S is right po-Noetherian.*

Proof. Let $\{0\} = I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ be an ascending chain of right poideals of S. Consider the Rees factor S-posets S/I_n for $n \in \mathbb{N}$, which means the factor S-posets $S/\nu(I_n \times I_n)$ (see also [3]), and let E_n be the regular injective S-poset in which S/I_n can be regularly embedded (according to Theorem 1.3). Then $E = \bigoplus_{n\in\mathbb{N}} E_n$ is weakly regular injective by hypothesis. Let $I = \bigcup_{n\in\mathbb{N}} I_n$, and consider the natural epimorphisms $f_n : S \to S/I_n$. Then define the S-poset map $f : I \to E$ by $f(s) = (f_1(s), \ldots, f_n(s), \ldots) = (f_n(s))_{n\in\mathbb{N}}$. Notice that for each $s \in S$, only finitely many components of $f(s)$ are nonzero, because we have $s \in I_k$ for some $k \in \mathbb{N}$, and so $f_n(s) = 0$ for all $n \geq k$. Now, since E is weakly regular injective, $f = \lambda_a$ for some $a \in E$, by Proposition 2.1. Let $a = (a_n)_{n\in\mathbb{N}}$, where $a_n \in E_n$ and for some $k \in \mathbb{N}$, $a_n = 0$ for all $n \geq k$. Then for each $s \in I$, since $f(s) = as$, we get $f_k(s) = a_k s = 0$ and so $s \in I_k$. Thus $I \subseteq I_k$. Hence S is right po-Noetherian. □

Definition 3.12. An S-poset A is called *countably $\sum$-weakly regular injective* if every countable direct sum of A with itself is weakly regular injective.

Theorem 3.13. *Let S be a pomonoid. If each weakly regular injective S-poset is countably $\sum$-weakly regular injective, then S is right po-Noetherian.*

Proof. Applying the notations of Theorem 3.11, put $A = \prod_{n\in\mathbb{N}} E_n$. Then by Theorem 3.1, A is weakly regular injective, and so such is $\bigoplus_{n\in\mathbb{N}} A$. But, $A = \prod_{n\in\mathbb{N}} E_n = E_m \oplus \prod_{n\neq m} E_n$, for each $m \in \mathbb{N}$. Thus

$\bigoplus_{m\in\mathbb{N}} A = \bigoplus_{m\in\mathbb{N}} E_m \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n = E \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n$, which means that E is a direct summand of a weakly regular injective S-poset and hence is weakly regular injective, by Theorem 3.8. The rest of the proof is similar to Theorem 3.11. □

4. When All S-Posets Are Some Kind Of Weakly Regular Injective

In this section we characterize some pomonoids over which all S-posets are of a kind of weakly regular injective.

Definition 4.1. [10] A pomonoid S is called right PP (principally projective) if for every $s \in S$ there exists an idempotent $e \in S$ such that $s = se$ and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$.

The following proposition shows that principal weak regular injectivity of a right PP pomonoid S is a necessary and sufficient condition for principal weak regular injectivity of all S-posets.

Proposition 4.2. *The following statements are equivalent for a right PP pomonoid S:*

(i) *S is principally weakly regular injective.*

(ii) *For every $s \in S$ and S-poset map $f : sS \to S$ there exists $z \in S$ such that $f = \lambda_z$.*

(iii) *S is a regular pomonoid.*

(iv) *All right S-posets are principally weakly regular injective.*

(v) *All right ideals of S are principally weakly regular injective.*

(vi) *All finitely generated right ideals of S are principally weakly regular injective.*

(vii) *All principal right ideals of S are principally weakly regular injective.*

Proof. $(i) \Leftrightarrow (ii)$ Follows from Proposition 2.1.

$(ii) \Rightarrow (iii)$ Let $s \in S$. Since S is right PP, there exists an idempotent $e \in S$ such that $s = se$ and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$. Now, define an S-homomorphism $f : sS \to S$ by $f(st) = et$ for every $t \in S$. Then f is an S-poset map, because $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$. By (ii), there exists $z \in S$ such that $f(x) = zx$ for all $x \in sS$. In particular, $f(s) = e = zs$, and so $s = se = szs$. Thus s is a regular element.

$(iii) \Rightarrow (ii)$ Let $f : sS \to S$ be an S-poset map from a principal right ideal sS of S into S. By (iii), there exists $z \in S$ such that $s = szs$. Hence $f(st) = f(szst) = f(sz)st = \lambda_{f(sz)}(st)$ for all $t \in S$.

The equivalence of (iii), (iv), (v), (vi), and (vii) is clear by Corollary 1.5. □

Corollary 4.3. *Every principally weakly regular injective right PP pomonoid is regular.*

Lemma 4.4. *Let A, B be S-posets with a zero element. If $f : A \to B$ is a regular monomorphism and $A \oplus B$ is quasi regular injective, then A is a retract of B.*

Proof. Consider the injection and projection maps,

$$i : A \to A \oplus B, p : A \oplus B \to A, i' : B \to A \oplus B, \text{ and } p' : A \oplus B \to B,$$

where $pi = id_A$ and $p'i' = id_B$, and the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{i'} & A \oplus B \\ i \downarrow & & & & \\ A \oplus B. & & & & \end{array}$$

Since f and i' are regular monomorphisms, such is $i'f$. Now there exists an S-poset map $g : A \oplus B \to A \oplus B$ such that $gi'f = i$ by quasi regular injectivity of $A \oplus B$. Hence $id_A = pi = p(gi'f) = (pgi')f$. Thus A is a retract of B. □

Proposition 4.5. *Let S be a pomonoid with a zero element. If each finitely generated S-poset is quasi regular injective, then S is a regular pomonoid such that each finitely generated right ideal of which is principal and generated by an idempotent.*

Proof. Since S is itself a finitely generated S-poset, S is quasi regular injective, by hypothesis. Let I be a finitely generated right ideal of S. Then $I \oplus S$ is a finitely generated S-poset. Thus $I \oplus S$ is also quasi regular injective, by hypothesis. Therefore, by Lemma 4.4, I is a retract of S. Hence, there exists an S-poset map $k : S \to I$ such that $k|_I = id_I$. Using this, we show that I is weakly regular injective. Then by Corollary 2.2, I is principal, generated by an idempotent, and consequently S is regular, by Corollary 1.5. Consider an S-poset map $f : J \to I$ from a right ideal J of S. Take the composition of f, and the inclusion map $I \hookrightarrow S$. Since S is quasi regular injective, we get an S-poset map $g : S \to S$ such that $g|_J = f$, and hence $kg : S \to I$ is an S-poset map with $(kg)|_J = f$ as required. □

Corollary 4.6. *Let S be a pomonoid with a zero element. If each finitely generated S-poset is quasi regular injective, then all S-posets are fg-weakly regular injective.*

Proof. It is clear by the above proposition and Corollary 1.6. □

Using the following lemma, the next theorem presents a homological classification of regular pomonoids which are regular injective by the properties of cyclicly regular injective right ideals.

Lemma 4.7. *All principal right ideals of a pomonoid S are regular injective if and only if S is a regular pomonoid which is regular injective.*

Proof. Let S be a pomonoid with all of its principal right ideals being regular injective. Then S is regular injective, since it is itself a principal ideal. Also, since regular injectivity implies principal weak regular injectivity, S is regular by Corollary 1.5.

Conversely, let S be a regular injective S-poset and a regular pomonoid. We show that all principal right ideals of S are regular injective. Let I be a principal right ideal of S, $g : B \to C$ a regular monomorphism and $f : B \to I$ an S-poset map. Consider the inclusion map $\iota : I \to S$. Since S is regular injective, there exists an S-poset map $f' : C \to S$ which extends ιf. Since S is regular, every principal right ideal of S is generated by an idempotent and so is a retract of S. Therefore, there exists an S-poset map $g : S \to I$ such that $g\iota = id_I$. Now, gf' is an S-poset map which extends f. Hence I is regular injective. □

Theorem 4.8. *For a pomonoid S, the following conditions are equivalent:*

(i) *All right ideals of S are cyclicly regular injective.*
(ii) *All finitely generated right ideals of S are cyclicly regular injective.*
(iii) *All principal right ideals of S are cyclicly regular injective.*
(iv) *All principal right ideals of S are regular injective.*
(v) *All principal right ideals of S are finitely regular injective.*
(vi) *S is a regular injective, regular pomonoid.*

Proof. (i)⇒(ii)⇒(iii) are obvious.

(iii)⇒(i) Let I be a right ideal of S, cS a cyclic sub S-poset, C an arbitrary S-poset, $\iota : cS \to C$ the inclusion, and $f : cS \to I$ an S-poset map. Then $f(cS) = f(c)S \subseteq I$ is a principal right ideal of S. By (iii), $f(cS)$ is cyclicly regular injective and so there exists an S-poset map $g : C \to f(cS)$ which extends f. Now, g can be considered as an S-poset map from C into I.

(iii)⇒(iv) By Proposition 2.4, a cyclic, cyclicly regular injective S-poset is regular injective. So a principal right ideal of S, which is cyclicly regular injective, is regular injective.

(iv)⇒(v)⇒(iii) are obvious.

(iv)⇔(vi) follows from Lemma 4.7. □

Theorem 4.9. *Given a pomonoid S, all S-posets are finitely (cyclicly) regular injective if and only if all finitely generated (cyclic) S-posets are regular injective.*

Proof. (⇒) Let A be a finitely generated S-poset. Then every regular embedding from A has a left inverse, by Proposition 2.4. Now, by Theorem 1.4, A is regular injective.

(⇐) Let A be an S-poset and $h : F \to B$ be a regular monomorphism from a finitely generated S-poset F, and $f : F \to A$ be an S-poset map. By hypothesis, F is regular injective. Then there exists an S-poset map $g : B \to F$ such that $gh = id_F$. Thus the composite $fg : B \to A$ is an S-poset map with $(fg)h = f$. So, A is finitely regular injective. The case for cyclic S-posets is similar. □

Theorem 4.10. *Let S be a pomonoid with a zero element. If S is a Noetherian pomonoid for which all finitely generated S-posets are quasi regular injective, then all S-posets are weakly regular injective.*

Proof. Since S is Noetherian, every right ideal of S is finitely generated. Then every right ideal of S is generated by an idempotent, by Proposition 4.5. Thus all S-posets are weakly regular injective, by Corollary 1.7. □

Theorem 4.11. *Let S be a pomonoid with a zero element. Then the following statements are equivalent:*

(i) *All S-posets are quasi regular injective.*

(ii) *All S-posets are regular injective.*

Proof. (i)⇒(ii) Let all S-posets are quasi regular injective and A be an arbitrary S-poset. Then $A \oplus E$, where E is a regular injective S-poset in which A can be regularly embedded (according to Theorem 1.3), is quasi regular injective by hypothesis. Hence A is a retract of E by Lemma 4.4, and so it is regular injective, by Proposition 2.6.

(ii)⇒(i) is obvious. □

The following remark follows by the above theorem and Remark 3.5 of [7].

Remark 4.12. There exists no pomonoid S over which all S-posets are quasi regular injective.

5. **Some Baer Conditions**

The condition that weak injectivity implies injectivity is known as the *Baer Criterion* for injectivity. It is known that although this condition is true for injectivity of R-modules, for every ring R with unit, it is not true for injectivity of S-posets, for an arbitrary pomonoid S (for example, see Example 5.1 (4)).

In this view, here we study some Baer type conditions, in the sense that we find conditions under which a special kind of regular injectivity implies regular injectivity.

First, we give some examples showing the relations between some kinds of regular injectivity.

Example 5.1. (1) It is clear that fg-weak regular injectivity implies principal weak regular injectivity, but the converse is not true in general. For example, let $S = \{1, 0, e, f\}$ be the commutative idempotent pomonoid with 1 as the top identity element, 0 a bottom zero element, $ef = fe = 0$. Take $K = \{0, e, f\}$. Then K is an ideal of S which is principally weakly regular injective. This is because, the principal right ideals of S are S, $\{0\}$, $eS = \{0, e\}$, $fS = \{0, f\}$ and S-poset maps from each to K are clearly of the form λ_a, for some $a \in K$. Therefore by Proposition 2.1, K is principally weakly regular injective. But K can not be fg-weakly regular injective, by Corollary 2.2.

(2) Cyclic regular injectivity implies principal weak regular injectivity, but the converse is not true in general. To see this, let $T = \{e, f\}$ be a right zero semigroup, $R = T^1$ with the order $e, f \leq 1$. Then R is a regular pomonoid. By Corollary 1.5, R is principally weakly regular injective. But, R is not cyclicly regular injective by Proposition 2.3 because it has no zeros.

(3) Weak regular injectivity implies fg-weak regular injectivity, but the converse is not true in general. Take the pomonoid $S = (\mathbb{N}^\infty, \min)$ where ∞ denotes the externally adjoint identity top element, and with the ordinary order of natural numbers. Take $K = \mathbb{N}$ which is a right ideal of S. Since K is not generated by an idempotent it can not be weakly regular injective, by Corollary 2.2. On the other hand, each finitely generated right ideals of S is generated by an idempotent, because it

is of the form $\{1, 2, \ldots, n\} = n\mathbb{N}$, for some $n \in \mathbb{N}$. Therefore, K is fg-weakly regular injective.

(4) Regular injectivity implies weak regular injectivity, but the converse is not true in general. Consider the pomonoid $S = (\mathbb{N}, \max)$ with the ordinary order. Then S is weakly regular injective, by Corollary 1.7 and since S is regular principal right ideal pomonoid. But, S is not regular injective, since it is not bounded by two zeros.

Definition 5.2. An S-poset A is called $\sum$*-regular injective* ($\sum$*-quasi regular injective*) if every direct sum of A with itself is regular injective (quasi regular injective).

Theorem 5.3. *The following are equivalent for a pomonoid S:*

(i) *Each principally weakly regular injective S-poset is regular injective.*

(ii) *Each principally weakly regular injective S-poset is $\sum$- regular injective and has a zero element.*

(iii) *Each principally weakly regular injective S-poset is $\sum$-quasi regular injective and has a zero element.*

(iv) *Each principally weakly regular injective S-poset is quasi regular injective and has a zero element.*

(v) *Each principally weakly regular injective S-poset is a PQRI S-poset and has a zero element.*

Proof. (i)⇒(ii) is true by Theorem 3.8. Also, the second part is clear by Lemma 1.2.

(ii)⇒(iii)⇒(iv)⇒(v) are clear.

(v)⇒(i) Let A be a principally weakly regular injective S-poset with a zero element, and E be a regular injective S-poset in which A can be regularly embedded. Then $A \oplus E$ is principally weakly regular injective, and hence a PQRI by hypothesis. Therefore, by Theorem 2.6, A is regular injective. □

The following result follows by Corollary 1.5, and Theorem 2.6 (or Theorem 5.3).

Proposition 5.4. *Let S be a regular pomonoid. Then the following are equivalent:*

(i) *Every S-poset with a zero element is regular injective.*

(ii) *Every S-poset with a zero element is quasi regular injective and S is po-Noetherian.*

(iii) *Every S-poset with a zero element is PQRI.*

The following remark follows by the above proposition and Remark 3.5 of [7].

Remark 5.5. There exists no pomonoid S over which all S-posets are PQRI.

Theorem 5.6. *For a Noetherian pomonoid S, the following are equivalent:*

(i) *Each weakly regular injective S-poset is regular injective.*

(ii) *Each weakly regular injective S-poset is $\sum$- regular injective and has a zero element.*

(iii) *Each weakly regular injective S-poset is $\sum$-quasi regular injective and has a zero element.*

(iv) *Each weakly regular injective S-poset is quasi regular injective and has a zero element.*

(v) *Each fg-weakly regular injective S-poset is regular injective.*

(vi) *Each fg-weakly regular injective S-poset is quasi regular injective and has a zero element.*

Proof. (i)⇒(ii) is true using Theorem 3.9, and Lemma 1.2.

(ii)⇒(iii)⇒(iv) are clear.

(iv)⇒(i) Let A be weakly regular injective and E be a regular injective S-poset into which A can be regularly embedded. Then $A \oplus E$ is weakly regular injective by Theorem 3.9, and hence is quasi regular injective, by (iv). Thus by Lemma 4.4, A is a retract of E, and so is regular injective by Proposition 2.6.

(i)⇔(v) and (iv)⇔(vi) Since S is Noetherian, each right ideal of S is finitely generated. So an S-poset A is weakly regular injective if and only if it is fg-weakly regular injective. Now the result follows by the assumption. □

Acknowledgments

The authors would like to thank Professor M. Mehdi Ebrahimi for his very useful comments and helpful conversations during this research. We also thank the referees for their valuable comments which improved the paper.

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