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IMPLICIT ITERATION APPROXIMATION FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-PSEUDOCONTRACTIVE TYPE MAPPINGS

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Abstract. In this paper, strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces are established by using a new analytical method, which essentially improve and extend some recent results obtained by Yang [Convergence theorems of implicit iteration process for asymptotically pseudocontractive mappings, Bulletin of the Iranian Mathematical Society, Available Online from 12 April 2011] and others.
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1. Introduction

Let E be an arbitrary real normed linear space with norm $\|\cdot\|$ and E^* be the duality space of E. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between E and E^*. For $1 < p < \infty$, the mapping $J_p : E \to 2^{E^*}$ defined by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|^{p-1}\}$$

is called the duality mapping with gauge function $\varphi(t) = t^{p-1}$. In particular, for $p = 2$, the duality mapping J_2 with gauge function $\varphi(t) = t$ is called the normalized duality mapping. It is well known that the duality

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mapping J_p has the following properties:
(i) $J_p(x) = \|x\|^{p-2}J_2(x)$ for all $x \in E(x \neq 0)$,
(ii) $J_p(\alpha x) = \alpha^{p-1}J_p(x)$ for all $\alpha \geq 0$,
(iii) J_p can be equivalently defined as the subdifferential of the functional $\psi(x) = p^{-1}\|x\|^p$, i.e., $J_p(x) = \partial\psi(x) = \{f \in E^* : \psi(y) - \psi(x) \geq \langle y - x, f\rangle, \forall y \in E\}$(Asplund [1]).

Definition 1.1. *Let K be a nonempty subset of E. A mapping $T : K \to K$ is said to be*
(i) asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in K, n \geq 1,$$

(ii) asymptotically pseudo-contractive, if for all $x, y \in K$, there exist $j(x - y) \in J(x - y)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n\|x - y\|^2, \; n \geq 1,$$

(iii) asymptotically quasi-pseudocontractive type if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\limsup_{n\to\infty}\Big\{\sup_{x\in K}\inf_{j_p(x-x^*)\in J_p(x-x^*)}\langle T^n x - x^*, j_p(x - x^*)\rangle - k_n\|x - x^*\|^p\Big\} \leq 0,$$

(iv) asymptotically nonexpansive in the intermediat sense if

$$\limsup_{n\to\infty}\left\{\sup_{x,y\in K}(\|T^n x - T^n y\| - \|x - y\|)\right\} \leq 0.$$

It is easy to see that an asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense if the domain of T is bounded. Every asymptotically nonexpansive mapping is asymptotically pseudocontractive, and every asymptotically pseudocontractive mapping is asymptotically quasi-pseudocontractive type mapping. But the inverse is not true, in general.

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5], while the concept of asymptotically pseudo-contractive mapping was introduced by Schu [12] in 1991. The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by

Schu [12], Chang [3], Khan et al. [7], Ofoedu [8], Plubtieng et al [10], Xu and Ori [14], Zhou [19], Sun [13], Yang and Hu [15] and Yang [16] in the setting of Hilbert spaces or Banach spaces.

Let K be a nonempty closed convex subset of E and $\{T_i\}_{i=1}^m$ be a finite family of nonexpansive mappings from K into itself (i.e., $\|T_ix - T_iy\| \leq \|x-y\|$ for $x, y \in K$ and $i = 1, 2, ..., m$). In 2001, Xu and Ori [14] introduced the following implicit iteration process. For an arbitrary $x_0 \in K$ and $\alpha_n \in [0,1]$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_1 = (1-\alpha_1)\,x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1-\alpha_2)\,x_1 + \alpha_2 T_2 x_2, \\ \quad \vdots \\ x_N = (1-\alpha_N)\,x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} = (1-\alpha_{N+1})\,x_N + \alpha_{N+1} T_{N+1} x_{N+1}, \\ \quad \vdots \end{cases}$$

The scheme is expressed in its compact form by

$$x_n = (1-\alpha_n)x_n + \alpha_n T_{n(modN)} x_n, n \geq 1.$$

Using this iteration, they proved that the sequence $\{x_n\}$ converges weakly to a common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in a Hilbert space under certain conditions.

In 2006, Chang et al.[3] introduced another implicit iteration process with error. In the sense of [3], the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$ is generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n + u_n, n \geq 1,$$

where $n = (k-1)m + i, i = i(n) \in \{1, 2, \ldots, m\}, k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$. $\{\alpha_n\}$ is a suitable sequence in [0, 1] and $\{u_n\} \subset K$ is such that $\sum_{n=0}^{\infty} \|u_n\| < \infty$. They extended the results of [14] from Hilbert spaces to more general uniformly convex Banach spaces and from nonexpansive mappings to asymptotically nonexpansive mappings.

Yang and Hu [15] proposed another implicit iteration process which appears to be more satisfactory as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n, n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{u_n\}$ is bounded in K.

Since for each $n \geq 1$, it can be written as $n = (k-1)m + i$, where $i = i(n) \in \{1, 2, ..., m\}, k = k(n) - 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence, (1.1) can be expressed in the following form:

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T^n_{i(n)} x_n + \gamma_n u_n, n \geq 1, \tag{1.2}$$

where $\alpha_n + \gamma_n \leq 1$, and $\{u_n\}$ is bounded in K.

Very recently, Yang [16] proved the following result.

Theorem 1.2. *([16]). Let E be a real normed linear space, K be a nonempty convex subset of E, $T_i : K \to K, i = 1, 2, \ldots, m$ be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mappings with $\{k_{in}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = max_{1 \leq i \leq m}\{k_{in}\}$. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by (1.2). Suppose that $\{u_n\}$ is bounded in K and that $\{\alpha_n\}$, $\{\gamma_n\}$ are sequences in [0, 1] satisfying the following conditions:*
(i) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.
Assume that there exists a strict increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\limsup_{n\to\infty} \Big\{ \langle T_i^n x_n - x^*, j(x_n - x^*) \rangle - k_n \|x_n - x^*\|^2 + \varphi(\|x_n - x^*\|) \Big\} \leq 0$$

for $x^* \in F$ and $i = 1, 2, \cdots, m$. Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^m$.

Remark 1.3. *We point out here that the conditions (i) is not always true.*

Example 1.4. *Let $\alpha_n = \frac{1}{\sqrt{n+1}}$ and $k_n = 1 + \frac{1}{\sqrt{n+1}}$, then $\sum\limits_{n=0}^{\infty} \alpha_n(k_n - 1) = \sum\limits_{n=0}^{\infty} \frac{1}{n+1} = \infty$, which show that conditions (i) in Theorem 1.2 is not satisfied. Hence Theorem 1.2 need to be improved.*

The purpose of this paper is, under the condition of removing the restriction $\sum\limits_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$, to prove strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces by using a new analytical method. Our results essentially extend and improve some recent results obtained by Yang [16] and others.

Now we consider Ishikawa type implicit iteration process with errors for a finite family of asymptotically quasi-pseudocontractive type mappings as follows:

$$(1.3)\quad \begin{cases} x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T^n_{i(n)} y_n + \gamma_n u_n \\ y_n = (1-\beta_n-\mu_n)x_n + \beta_n T^n_{i(n)} x_n + \mu_n v_n, \quad (n \geq 0), \end{cases}$$

where $n = (k-1)m + i, i = i(n) \in \{1, 2, \cdots, m\}, k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$. $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ are four suitable sequences in [0, 1] with $\alpha_n + \gamma_n \leq 1, \beta_n + \mu_n \leq 1$ and $\{u_n\}, \{v_n\}$ are bounded sequences in K.

The following lemmas plays an important role in this paper.

Lemma 1.5 ([17]**.** *] Let E be a real normed linear space and $J_p : E \to 2^{E^*}$ a duality mapping. Then*

$$\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y)\rangle$$

for all $x, y \in E, 1 < p < \infty$ and $j_p(x+y) \in J_p(x+y)$.

Lemma 1.6. *Let $\varphi_i(i = 1, 2, \ldots, m) : [0, \infty) \to [0, \infty)$ be strictly increasing functions with $\varphi_i(0) = 0$ and let $\{a_n\}, \{b_n\}, \{c_n\}, \{\delta_n\}$ be nonnegative real sequences such that $\sum_{n=1}^{\infty} \delta_n = \infty$, $\lim\limits_{n\to\infty} \frac{b_n}{\delta_n} = 0, \sum_{n=1}^{\infty} c_n < \infty$. Suppose that*

$$(1.4)\qquad a^p_{n+1} \leq a^p_n - \delta_n \varphi_i(a_{n+1}) + b_n + c_n,\ n \geq n_0,$$

where n_0 is some nonnegative integer and $p \in (1, \infty)$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Setting $\liminf\limits_{n\to\infty} a_n = \tau$, then $\tau \geq 0$. Now we prove $\tau = 0$. If $\tau > 0$, then there exists a positive integer $N_1 > 0$ such that $a_n \geq \frac{\tau}{2}$ for all $n \geq N_1$. By the strictly increasing property of φ_i, we have $\varphi_i(a_{n+1}) > \varphi_i(\frac{\tau}{2}) \geq \min\limits_{1\leq i\leq m} \varphi_i(\frac{\tau}{2}) =: \sigma$. Since $\lim\limits_{n\to\infty} \frac{b_n}{\delta_n} = 0$, there exists a positive integer $N_2 > N_1$ such that $\frac{b_n}{\delta_n} \leq \frac{1}{2}\sigma$ for all $n \geq N_2$. Taking $N_3 = \max\{N_2, n_0\}$, then from (1.4), we have

$$a^p_{n+1} \leq a^p_n - \delta_n\sigma + \delta_n\frac{\sigma}{2} + c_n = a^p_n - \delta_n\frac{\sigma}{2} + c_n$$

for all $n \geq N_3$, which means that $\delta_n \frac{\sigma}{2} \leq a_n^p - a_{n+1}^p + c_n$. Hence for any positive integer $h \geq N_3$, we obtain

$$\frac{\sigma}{2} \sum_{n=N_3}^{h} \delta_n \leq a_{N_3}^p - a_{h+1}^p + \sum_{n=N_3}^{h} c_n \leq a_{N_3}^p + \sum_{n=N_3}^{h} c_n,$$

and so

$$\infty = \frac{\sigma}{2} \sum_{n=N_3}^{\infty} \delta_n \leq a_{N_3}^p + \sum_{n=N_3}^{\infty} c_n,$$

a contradition. This implies that $\tau > 0$ is impossible. Therefore $\tau = 0$, which there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $a_{n_j} \to 0 (j \to \infty)$. Since $\lim_{n\to\infty} \frac{b_n}{\delta_n} = 0, \sum_{n=1}^{\infty} c_n < \infty$, for any given $\varepsilon > 0$, there exist two positive integers $j_0 > 0$ and $N_4 > 0$, such that for all $n \geq N_4, \sum_{n=N_4}^{\infty} c_n < \varepsilon^p$, $\frac{b_n}{\delta_n} < \frac{1}{2}\omega$ and $a_{n_j} < \varepsilon$ for all $j \geq j_0$, where $\omega = \min\{\varphi_1(\varepsilon), \varphi_2(\varepsilon), \cdots, \varphi_m(\varepsilon)\}$. Let $N_5 = \max\{j_0, N_4\}$. For fixed $j_* > N_5$ and all $k \geq 0$, we now want to show that $a_{n_{j_*}+k} < 2\varepsilon$. To see this consider two possible cases.

Case I: $a_{n_{j_*}+1} < \varepsilon$.

In this case, $a_{n_{j_*}+1}^p < \varepsilon^p + c_{n_{j_*}} + c_{n_{j_*}+1}$ and so we have the desired result.

Case II: $a_{n_{j_*}+1} \geq \varepsilon$.

In this case, $\varphi_i(a_{n_{j_*}+1}) \geq \varphi_i(\varepsilon) \geq \omega > 0$ since for each $i = 1, 2, \cdots, m$, φ_i is a strictly increasing function. From (1.4), we also have

$$\begin{aligned} a_{n_{j_*}+1}^p &\leq a_{n_{j_*}}^p - \delta_{n_{j_*}} \varphi_i\big(a_{n_{j_*}+1}\big) + b_{n_{j_*}} + c_{n_{j_*}} \\ &\leq a_{n_{j_*}}^p - \delta_{n_{j_*}} \left(\omega - \frac{\omega}{2}\right) + c_{n_{j_*}} \\ &< \varepsilon^p + c_{n_{j_*}} + c_{n_{j_*}+1}. \end{aligned}$$

By using induction, we have

$$a_{n_{j_*}+k}^p < \varepsilon^p + \sum_{i=n_{j_*}}^{n_{j_*}+k} c_i < \varepsilon^p + \varepsilon^p = 2\varepsilon^p < (2\varepsilon)^p$$

for all $k \geq 0$. This shows $a_n \to 0$ as $n \to \infty$. The proof of Lemma 1.6 is completed.

2. Main results

Theorem 2.1. *Let E be a real normed linear space, K a nonempty convex subset of E and $T_i : K \to K(i = 1, 2, \dots, m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\le i\le m}\{k_n^{(i)}\}$. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by (1.3). Suppose that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and that $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ are sequences in [0, 1] satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$,

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\mu_n \to 0 (n \to \infty)$.

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ with $\varphi_i(0) = 0$ such that

$$\limsup_{n\to\infty} \Big\{ \inf_{j_p(x_n-x^*)\in J_p(x_n-x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*)\rangle - k_n\|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \Big\} \le 0 \tag{2.1}$$

for $x^* \in F$ and $i = 1, 2, \dots, m$. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

Proof. For $i = 1, 2, \dots, m$, let

$$\sigma_n^{(i)} = \inf_{j_p(x_n-x^*)\in J_p(x_n-x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*)\rangle - k_n\|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|),$$

then there exist $j_p^{(i)}(x_n - x^*) \in J_p(x_n - x^*)$, such that

$$\langle T_i^n x_n - x^*, j_p^{(i)}(x_n - x^*)\rangle - k_n\|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) < \sigma_n^{(i)} + \varepsilon_n^{(i)} \le \xi_n, \tag{2.2}$$

where $\varepsilon_n^{(i)} \in (0, 1)$ with $\varepsilon_n^{(i)} \to 0(n \to \infty)$, and $\xi_n = \max_{1\le i\le m}\{\sigma_n^{(i)},\ 0\} + \max_{1\le i\le m}\{\varepsilon_n^{(i)}\}$. It is easy see (using (2.1)) that $\lim_{n\to\infty} \xi_n = 0$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, $M = Sup_{n\ge 0}\{\, \|u_n - x^*\| + \|v_n - x^*\|\,\} < \infty$. Also, since for each $i = 1, 2, \dots, m$, $T_i : K \to K$ is an asymptotically nonexpansive in the intermediate sense, there exists $n_0 \geq 1$ such that $\sup_{x,y\in K}(\|T_i^n x - T_i^n y\| - \|x-y\|) \leq 1$ for all $n \geq n_0, i = 1,2,\ldots,m$.

It follows from (1.3) that

$$\begin{aligned}
& \frac{\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
= & \frac{\|(1-\alpha_n-\gamma_n)(x_{n-1}-x^*) + \alpha_n(T_i^n y_n - x^*) + \gamma_n(u_n - x^*)\|}{1+\|x_{n-1}-x^*\|} \\
\leq & \frac{\|x_{n-1}-x^*\| + \alpha_n\|T_i^n y_n - x^*\| + \|u_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
\leq & \frac{\|x_{n-1}-x^*\| + \sup_{x,y\in K}(\|T_i^n x - T_i^n y\| - \|x-y\|)}{1+\|x_{n-1}-x^*\|} \\
+ & \frac{\alpha_n\|y_n - x^*\| + \|u_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
\leq & \frac{\|x_{n-1}-x^*\| + 1 + \alpha_n\big[\|x_n - x^*\| + \|T_i^n x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
+ & \frac{\|v_n - x^*\|\big] + \|u_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
\leq & \frac{\|x_{n-1}-x^*\| + 1 + 2\alpha_n\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \\
+ & \frac{\sup_{x,y\in K}(\|T_i^n x - T_i^n y\| - \|x-y\|) + 2M}{1+\|x_{n-1}-x^*\|} \\
\text{(2.3)} \quad \leq & \; 3 + 2M + \frac{2\alpha_n\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|}
\end{aligned}$$

for all $n \geq n_0$.

Since $1-2\alpha_n \to 1$ $(n\to\infty)$, there exists $n_1 \geq n_0$ such that $1-2\alpha_n > \frac{1}{2} > 0$ for all $n \geq n_1$, which together with (2.3) gives that

$$\text{(2.4)} \qquad \frac{\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \leq \frac{3+2M}{1-2\alpha_n} \leq 6 + 4M.$$

Let $c_n^{(i)} = \sup_{x,y\in K}(\|T_i^n x - T_i^n y\| - \|x-y\|)$, $d_n = \max\left\{0\,,\ \max_{1\leq i\leq m} c_n^{(i)}\right\}$, then $\lim_{n\to\infty} d_n = 0$.

By (1.3), we have

$$\begin{aligned}
\|x_n - y_n\| &\leq \beta_n \|T_i^n x_n - x_n\| + \mu_n \|v_n - x_n\| \\
&\leq \beta_n (\|T_i^n x_n - x^*\| - \|x_n - x^*\|) \\
&+ (2\beta_n + \mu_n)\|x_n - x^*\| + \mu_n \|v_n - x^*\| \\
&\leq \beta_n \sup_{x,y\in K} (\|T_i^n x - T_i^n y\| - \|x - y\|) \\
&+ (2\beta_n + \mu_n)\|x_n - x^*\| + \mu_n M \\
&\leq (2\beta_n + \mu_n)\|x_n - x^*\| + \beta_n d_n + \mu_n M \qquad (2.5)
\end{aligned}$$

for all $n \geq n_1$.

For $j_p^{(i)}(x_n - x^*) \in J_p(x_n - x^*)$, $\forall n \geq 0$, we have from (1.3) and Lemma 1.5 that

$$\begin{aligned}
&\left(\frac{\|x_n - x^*\|}{1 + \|x_{n-1} - x^*\|}\right)^p \\
= &\frac{\|(1 - \alpha_n - \gamma_n)(x_{n-1} - x^*) + \alpha_n (T_i^n y_n - x^*) + \gamma_n (u_n - x^*)\|^p}{(1 + \|x_{n-1} - x^*\|)^p} \\
\leq &\frac{(1 - \alpha_n)^p \|x_{n-1} - x^*\|^p + p\alpha_n \left\langle T_i^n x_n - x^*, j_p^{(i)}(x_n - x^*)\right\rangle}{(1 + \|x_{n-1} - x^*\|)^p} \\
+ &\frac{p\alpha_n \left\langle T_i^n y_n - T_i^n x_n, j_p^{(i)}(x_n - x^*)\right\rangle}{(1 + \|x_{n-1} - x^*\|)^p} \\
+ &\frac{p\gamma_n \left\langle u_n - x^*, j_p^{(i)}(x_n - x^*)\right\rangle}{(1 + \|x_{n-1} - x^*\|)^p} \qquad (2.6)
\end{aligned}$$

for all $n \geq n_1, i = 1, 2, \ldots, m$.

Next we consider the second and third term on the right side of (2.6). From (2.4) and (2.5), we obtain that

$$\begin{aligned}
&\frac{p\alpha_n \left\langle T_i^n y_n - T_i^n x_n, j_p^{(i)}(x_n - x^*)\right\rangle}{(1 + \|x_{n-1} - x^*\|)^p} \\
\leq\ &p\alpha_n \frac{\|T_i^n y_n - T_i^n x_n\| \|x_n - x^*\|^{p-1}}{(1 + \|x_{n-1} - x^*\|)^p} \\
=\ &p\alpha_n \frac{(\|T_i^n x_n - T_i^n y_n\| - \|x_n - y_n\|) + \|x_n - y_n\|}{1 + \|x_{n-1} - x^*\|} \\
\cdot\ &\left(\frac{\|x_n - x^*\|}{1 + \|x_{n-1} - x^*\|}\right)^{p-1}
\end{aligned}$$

$$\leq p\alpha_n \left(\frac{d_n}{1+\|x_{n-1}-x^*\|} + \frac{\|x_n - y_n\|}{1+\|x_{n-1}-x^*\|} \right) \cdot \left(\frac{\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \right)^{p-1}$$

$$\leq p\alpha_n \left(d_n + \frac{(2\beta_n+\mu_n)\|x_n - x^*\| + \beta_n d_n + \mu_n M}{1+\|x_{n-1}-x^*\|} \right) \cdot \left(\frac{\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \right)^{p-1}$$

$$\leq p\alpha_n (6+4M)^{p-1}[d_n + (2\beta_n+\mu_n)(6+4M) + \beta_n d_n + \mu_n M] \tag{2.7}$$

for all $n \geq n_1$.

In view of (2.4), we deduce that

$$\frac{p\gamma_n \left\langle u_n - x^*, j_p^{(i)}(x_n - x^*) \right\rangle}{(1+\|x_{n-1}-x^*\|)^p} \leq \frac{p\gamma_n \|u_n - x^*\| \|x_n - x^*\|^{p-1}}{(1+\|x_{n-1}-x^*\|)^p} \leq p\gamma_n M(6+4M)^{p-1}. \tag{2.8}$$

Substituting (2.2), (2.7) and (2.8) into (2.6) yields that

$$\begin{aligned} \left(\frac{\|x_n - x^*\|}{1+\|x_{n-1}-x^*\|} \right)^p \leq\ & \frac{(1-\alpha_n)^p \|x_{n-1}-x^*\|^p + p\alpha_n \xi_n}{(1+\|x_{n-1}-x^*\|)^p} \\ & + \frac{p\alpha_n \Big(k_n \|x_n - x^*\|^p - \varphi_i(\|x_n - x^*\|) \Big)}{(1+\|x_{n-1}-x^*\|)^p} \\ & + p\alpha_n (6+4M)^{p-1} \big[d_n + (2\beta_n+\mu_n)(6+4M) \\ & + \beta_n d_n + \mu_n M \big] + p\gamma_n M (6+4M)^{p-1} \end{aligned} \tag{2.9}$$

for all $n \geq n_1, i = 1, 2, \ldots, m$.

Since $1 - p\alpha_n k_n \to 1$ $(n \to \infty)$, there exists $n_2 \geq n_1$ such that $0 < \frac{1}{2} < 1 - p\alpha_n k_n < 1$ for all $n \geq n_2$. It follows from (2.9) that

$$\begin{aligned} \|x_n - x^*\|^p \leq\ & \frac{(1-\alpha_n)^p \|x_{n-1}-x^*\|^p + p\alpha_n \xi_n - p\alpha_n \varphi_i(\|x_n - x^*\|)}{1 - p\alpha_n k_n} \\ & + \frac{\left(p\alpha_n A_n + p\gamma_n M (6+4M)^{p-1} \right) (1+\|x_{n-1}-x^*\|)^p}{1 - p\alpha_n k_n} \end{aligned} \tag{2.10}$$

for all $n \geq n_2, i = 1, 2, \ldots, m$,
where $A_n = (6+4M)^{p-1}[d_n + (2\beta_n+\mu_n)(6+4M) + \beta_n d_n + \mu_n M] \to$

$0(n \to \infty)$. Note that $(1 + \|x_{n-1} - x^*\|)^p \le 2^{p-1}(1 + \|x_{n-1} - x^*\|^p)$,

$$\begin{aligned}(1-\alpha_n)^p &= 1 - p\alpha_n + \frac{p(p-1)\alpha_n^2}{2!} - \frac{p(p-1)(p-2)\alpha_n^3}{3!} + \cdots + (-\alpha_n)^p \\ &= 1 - p\alpha_n + \alpha_n B_n,\end{aligned}$$

where

$$B_n = \frac{p(p-1)\alpha_n}{2!} - \frac{p(p-1)(p-2)\alpha_n^2}{3!} + \cdots + (-\alpha_n)^{p-1} \to 0(n \to \infty).$$

In virtue of (2.10), we conclude that

$$\begin{aligned}\|x_n - x^*\|^p &\le \Big[1 + \frac{p\alpha_n(k_n-1) + \alpha_n B_n + p2^{p-1}\alpha_n A_n}{1 - p\alpha_n k_n} \\ &+ \frac{pM2^{p-1}(6+4M)^{p-1}\gamma_n}{1 - p\alpha_n k_n}\Big]\|x_{n-1} - x^*\|^p \\ &+ \frac{p\alpha_n(\xi_n + 2^{p-1}A_n) + pM2^{p-1}(6+4M)^{p-1}\gamma_n}{1 - p\alpha_n k_n} \\ &- \frac{p\alpha_n\varphi_i\big(\|x_n - x^*\|\big)}{1 - p\alpha_n k_n} \\ &\le \big[1 + 2p\alpha_n(k_n-1) + 2\alpha_n B_n + p2^p\alpha_n A_n \\ &+ pM2^p(6+4M)^{p-1}\gamma_n\big]\|x_{n-1} - x^*\|^p \\ &+ 2p\alpha_n(\xi_n + 2^{p-1}A_n) + pM2^p(6+4M)^{p-1}\gamma_n \\ &- p\alpha_n\varphi_i\big(\|x_n - x^*\|\big) \end{aligned} \tag{2.11}$$

for all $n \ge n_2, i = 1, 2, \ldots, m$.

Now we take a nonnegative integer $n_3 \ge n_2$ such that $x_{n_3} \ne x^*$(if not, $x_n = x^*$ for all $n \ge n_2$, then $x_n \to x^*(n \to \infty)$, and so we have done). Since $k_n \to 1, \xi_n \to 0,\ B_n \to 0,\ A_n \to 0\ (n \to \infty),\ \sum_{n=0}^{\infty}\gamma_n < \infty$, there exists a positive integer $N > n_3$ such that, for all $n \ge N$, $(k_n - 1 + \frac{1}{p}B_n)(2G)^p + \big(\xi_n + 2^{p-1}A_n(1 + (2G)^p)\big) < \frac{\min_{1\le i\le m}\{\varphi_i(G)\}}{4}$ and $\sum_{n=N}^{\infty}\gamma_n < \frac{G^p}{pM2^p(6+4M)^{p-1}(1+(2G)^p)}$, where $G = \max\{\|x_{n_3} - x^*\|, \|x_{n_3+1} - x^*\|, \ldots, \|x_{N-1} - x^*\|, \|x_N - x^*\|\}$, and obviously $0 < G < \infty$.

Next we proceed by induction to show $\|x_{N+k} - x^*\| \le 2G$ for all $k \ge 1$. To see this consider two possible cases.

Case III: $\|x_{N+1} - x^*\| \le G$.

In this case, $\|x_{N+1}-x^*\|^p \le G^2+pM2^p\,(6+4M)^{p-1}\,(1+(2G)^p)\gamma_{N+1}$ and so we have the desired result.

Case IV: $\|x_{N+1}-x^*\| > G$.

In this case, $\varphi_i(\|x_{N+1}-x^*\|) > \varphi_i(G) \ge \min\limits_{1\le i\le m}\{\varphi_i(G)\} > 0$ since for each $i=1,2,\cdots,m$, φ_i is a strictly increasing function. From (2.11), we also have

$$\begin{aligned}
&\|x_{N+1}-x^*\|^2 \le \|x_N-x^*\|^p \\
&+[2p\alpha_{N+1}(k_{N+1}-1)+2\alpha_{N+1}B_{N+1}+pM2^p(6+4M)^{p-1}\gamma_{N+1} \\
&+p2^p\alpha_{N+1}A_{N+1}](2G)^p+2p\alpha_{N+1}(\xi_{N+1}+2^{p-1}A_{N+1}) \\
&+pM2^p(6+4M)^{p-1}\gamma_{N+1}-p\alpha_{N+1}\min_{1\le i\le m}\{\varphi_i(G)\} \\
&=\|x_N-x^*\|^p \\
&-p\alpha_{N+1}\Bigg[\min_{1\le i\le m}\{\varphi_i(G)\}-2\left(\xi_{N+1}+2^{p-1}A_{N+1}(1+(2G)^p)\right) \\
&-2\Big(k_{N+1}-1+\frac{1}{p}B_{N+1}\Big)(2G)^p\Bigg]+pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_{N+1} \\
&\le\|x_N-x^*\|^p-p\alpha_{N+1}\left(\min_{1\le i\le m}\{\varphi_i(G)\}-\frac{\min\limits_{1\le i\le m}\{\varphi_i(G)\}}{4}\right) \\
&+pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_{N+1} \\
&\le\|x_N-x^*\|^p+pM2^p(6+4M)(1+(2G)^p)\gamma_{N+1} \\
&\le G^p+pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_{N+1}.
\end{aligned}$$

By using induction, we get that

$$\begin{aligned}
\|x_{N+k}-x^*\|^2 &\le G^p+pM2^p(6+4M)^{p-1}(1+(2G)^p)\sum_{i=N+1}^{N+k}\gamma_i \\
&\le G^p+G^p=2G^p\le(2G)^p
\end{aligned}$$

for all $k\ge 1$.

This shows $\|x_n-x^*\|\le 2G$ for all $n\ge N$. Therefore, it follows from (2,11) that

$$\|x_n-x^*\|^p \quad\le\quad \|x_{n-1}-x^*\|^p+2p\alpha_n\Bigg[\left(k_n-1+\frac{B_n}{p}+2^{p-1}A_n\right)(2G)^p$$

$$+ \quad \xi_n + 2^{p-1}A_n\Big] + pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_n$$

$$(2.12) \qquad - \quad p\alpha_n\varphi_i\big(\|x_n - x^*\|\big)$$

for all $n \geq N, i = 1, 2, \cdots, m$.

Taking $\delta_n = p\alpha_n$, $c_n = pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_n$, $a_n = \|x_n - x^*\|$ and $b_n = 2p\alpha_n\left[\left(k_n - 1 + \frac{B_n}{p} + 2^{p-1}A_n\right)(2G)^p + \xi_n + 2^{p-1}A_n\right]$ for all $n \geq N$. By (2.12) and Lemma 1.6 ensures that $\|x_n - x^*\| \to 0$ as $n \to \infty$, that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Remark 2.2. *Theorem 2.1 improves and extends Theorem 1.2 (i.e., Theorem 2.1 of Yang [16]) in the following aspects:*
(1) Extend asymptotically pseudocontractive mapping to asymptotically quasi-pseudocontractive type mappings.
(2) It abolishes the condition that $\sum_{n=0}^{\infty}\alpha_n(k_n - 1) < \infty$.
(3) The proof of sequence $\{x_n\}$ *boundedness is entirely different from what it was before.*
(4) Extend implicit iterative scheme (1.2) to Ishikawa type implicit iteration process (1.3).
(5) Condition

$$\limsup_{n\to\infty}\Big\{\langle T_i^n x_n - x^*, j(x_n - x^*)\rangle - k_n\|x_n - x^*\|^2 + \varphi(\|x_n - x^*\|)\Big\} \leq 0$$

is replaced by the condition

$$\limsup_{n\to\infty}\Big\{\inf_{j_p(x_n-x^*)\in J_p(x_n-x^*)}\langle T_i^n x_n - x^*, j_p(x_n - x^*)\rangle - k_n\|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|)\Big\} \leq 0.$$

From Theorem 2.1, we obtain the following result immediately.

Theorem 2.3. *Let E be a real normed linear space, K a nonempty bounded convex subset of E and $T_i : K \to K (i = 1, 2, \cdots, m)$ a finite family of asymptotically nonexpansive mappings with $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq m}\{k_n^{(i)}\}$. Assume that $F =$*

$\bigcap_{i=1}^m F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by (1.3). Suppose that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and that $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ are sequences in [0, 1] satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$,

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\mu_n \to 0 (n \to \infty)$.

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ with $\varphi_i(0) = 0$ such that

$$\limsup_{n\to\infty} \Big\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \Big\} \leq 0$$

for $x^* \in F$ and $i = 1, 2, \cdots, m$. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

Proof. Since T_i is an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, we have

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in K} \left(\|T_i^n x - T_i^n y\| - \|x - y\| \right) \right\} \leq \limsup_{n\to\infty} \left[(k_n - 1) \operatorname{diam}(K) \right] = 0,$$

where $\operatorname{diam}(K) = \sup_{x,y\in K} \|x - y\|$. This implies that every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense. Also since every asymptotically nonexpansive mapping is asymptotically pseudo-contractive mapping. The conclusion now follows easily from Theorem 2.1.

If $\gamma_n = \mu_n = 0 (\forall n \geq 1)$ in Theorem 2.1 and Theorem 2.3, then we have the following results.

Theorem 2.4. *Let E be a real normed linear space, K a nonempty convex subset of E and $T_i : K \to K (i = 1, 2, \cdots, m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq m}\{k_n^{(i)}\}$. Assume that $F =$*

$\bigcap_{i=1}^m F(T_i) \neq \emptyset$ *denotes the set of common fixed points of* $\{T_i\}_{i=1}^m$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^n y_n \\ y_n = (1-\beta)x_n + \beta_n T_{i(n)}^n x_n, \quad (n \geq 0). \end{cases}$$

Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in [0, 1] satisfying the following conditions:*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$.

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ *with* $\varphi_i(0) = 0$ *such that*

$$\limsup_{n\to\infty} \Big\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \Big\} \leq 0$$

for $x^* \in F$ *and* $i = 1, 2, \cdots, m$. *Then* $\{x_n\}$ *converges strongly to a common fixed point* x^* *of* $\{T_i\}_{i=1}^m$.

Theorem 2.5. *Let* E *be a real normed linear space,* K *a nonempty bounded convex subset of* E *and* $T_i : K \to K (i = 1, 2, \cdots, m)$ *a finite family of asymptotically nonexpansive mappings with* $\{k_n^{(i)}\} \subset [1, \infty)$ *such that* $\lim\limits_{n\to\infty} k_n = 1$, *where* $k_n = \max_{1 \leq i \leq m}\{k_n^{(i)}\}$. *Assume that* $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ *denotes the set of common fixed points of* $\{T_i\}_{i=1}^m$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^n y_n \\ y_n = (1-\beta)x_n + \beta_n T_{i(n)}^n x_n, \quad (n \geq 0). \end{cases}$$

Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in [0, 1] satisfying the following conditions:*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$.

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ *with* $\varphi_i(0) = 0$ *such that*

$$\limsup_{n\to\infty} \Big\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \Big\} \leq 0$$

for $x^* \in F$ *and* $i = 1, 2, \cdots, m$. *Then* $\{x_n\}$ *converges strongly to a common fixed point* x^* *of* $\{T_i\}_{i=1}^m$.

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