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STABILITY OF ESSENTIAL SPECTRA OF BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we show the stability of Gustafson, Weidmann, Kato, Wolf, Schechter and Browder essential spectrum of bounded linear operators on Banach spaces which remain invariant under additive perturbations belonging to a broad classes of operators U such that $\gamma(U^m) < 1$ where $\gamma(.)$ is a measure of non-compactness.
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1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X into Y. The subspace of all compact (respectively, finite rank) operators of $\mathcal{L}(X,Y)$ is denoted by $\mathcal{K}(X,Y)$ $\big($respectively, $\mathfrak{F}_0(X,Y)\big)$. For $U \in \mathcal{L}(X,Y)$, we write $\mathcal{D}(U) \subset X$ for the domain, $\mathcal{N}(U) \subset X$ for the null space and $\mathcal{R}(U) \subset Y$ for the range of U. The nullity, $\alpha(U)$, of U is defined as the dimension of $\mathcal{N}(U)$ and the deficiency, $\beta(U)$, of U is defined as the codimension of $\mathcal{R}(U)$ in Y. The spectrum of U will be denoted by $\sigma(U)$. The resolvent set $\rho(U)$ of U is the complement of $\sigma(U)$ in the complex plane.

An operator $U \in \mathcal{L}(X,Y)$ is called an upper (respectively, a lower) semi-Fredholm operator, if the range $\mathcal{R}(U)$ of U is closed and $\alpha(U) <$

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∞ (respectively, $\beta(U) < \infty$). We denote by $\Phi_+(X,Y)$ (respectively, $\Phi_-(X,Y)$) the set of upper (respectively, lower) semi-Fredholm operators. The set of semi-Fredholm operators is define by $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ and $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is called a set of Fredholm operators. A complex number λ is in Φ_U, Φ_{+U}, Φ_{-U} or $\Phi_{\pm U}$, if $\lambda - U$ is in $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ or $\Phi_\pm(X,Y)$, respectively. For an operator $U \in \Phi_\pm(X,Y)$ we define the index of U by $i(U) = \alpha(U) - \beta(U)$. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathfrak{F}_0(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathfrak{F}_0(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$.

In this paper, we are concerned with the following essential spectra

$$\sigma_{e1}(U) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+U},$$
$$\sigma_{e2}(U) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_-(X)\} := \mathbb{C}\backslash\Phi_{-U},$$
$$\sigma_{e3}(U) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi_\pm(X)\} := \mathbb{C}\backslash\Phi_{\pm U},$$
$$\sigma_{e4}(U) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - U \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_U,$$
$$\sigma_{e5}(U) := \bigcap_{K \in \mathcal{K}(X)} \sigma(U+K),$$
$$\sigma_{e6}(U) := \bigcap_{\substack{UK=KU \\ K \in \mathcal{K}(X)}} \sigma(U+K).$$

They can be ordered as

$$\sigma_{e3}(U) := \sigma_{e1}(U) \cap \sigma_{e2}(U) \subseteq \sigma_{e4}(U) \subseteq \sigma_{e5}(U) \subseteq \sigma_{e6}(U).$$

The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidmann essential spectra [5], $\sigma_{e3}(.)$ is the Kato essential spectrum [10], $\sigma_{e4}(.)$ is the Wolf essential spectrum [5, 16], $\sigma_{e5}(.)$ is the Schechter essential spectrum [6, 7, 13, 15], and $\sigma_{e6}(.)$ denotes the Browder essential spectrum [9, 12]. Note that all these sets are closed and if X is a Hilbert space and U is a self-adjoint operator on X, then all these sets coincide.

Definition 1.1. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.*

(i) The operator F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. The set of Fredholm perturbations denote by $\mathcal{F}(X,Y)$.

(ii) The operator F is called an upper semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ whenever $U \in \Phi_+(X,Y)$. The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}_+(X,Y)$.

(*iii*) The operator F is called an lower semi-Fredholm perturbation if $U + F \in \Phi_-(X,Y)$ whenever $U \in \Phi_-(X,Y)$. The set of lower semi-Fredholm perturbations is denoted by $\mathcal{F}_-(X,Y)$.

In general we have

$\mathcal{K}(X,Y) \subseteq \mathcal{F}_+(X,Y) \subseteq \mathcal{F}(X,Y)$ and $\mathcal{K}(X,Y) \subseteq \mathcal{F}_-(X,Y) \subseteq \mathcal{F}(X,Y)$.

These classes of operators are introduced and investigated in [4]. In particular, it is shown that $\mathcal{F}(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$. If $X = Y$, we write $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X,X)$, $\mathcal{F}_+(X,X)$ and $\mathcal{F}_-(X,X)$ respectively.

We recall the following results established in [8].

Lemma 1.2. *(*[8, Lemma 2.1]*) Let $U \in \mathcal{L}(X,Y)$ and $F \in \mathcal{L}(X,Y)$. Then*

(i) If $U \in \Phi(X,Y)$ and $F \in \mathcal{F}(X,Y)$, then $U + F \in \Phi(X,Y)$ and $i(U + F) = i(U)$.

(ii) If $U \in \Phi_+(X,Y)$ and $F \in \mathcal{F}_+(X,Y)$, then $U + F \in \Phi_+(X,Y)$ and $i(U + F) = i(U)$.

(iii) If $U \in \Phi_-(X,Y)$ and $F \in \mathcal{F}_-(X,Y)$, then $U + F \in \Phi_-(X,Y)$ and $i(U + F) = i(U)$.

The following proposition provides a characterization of the Schechter by means of Fredholm operators.

Proposition 1.3. *(*[15, Theorem 7.27, p. 172]*) Let X be a Banach space and let $U \in \mathcal{L}(X)$. Then*

$$\lambda \notin \sigma_{e5}(U) \textit{ if and only if } \lambda \in \Phi_U^0,$$

where $\Phi_U^0 = \{\lambda \in \Phi_U$ such that $i(\lambda - U) = 0\}$.

The notion of a measure of noncompactness is used in some problems of topology, functional analysis, and operator theory (see [3]). To recall the measure of noncompactness, we denote by M_X the family of all nonempty and bounded subsets of X while N_X denotes its subfamily consisting of all relatively compact sets. Moreover, let conv(A) denote the convex hull of a set $A \subset X$. In [3], a mapping $\gamma : M_X \longrightarrow [0, +\infty[$ is said to be a measure of noncompactness in the space X, if it satisfies the following conditions:

(1) The family $\mathcal{N}(\gamma) = \{D \in M_X;\ \gamma(D) = 0\}$ is nonempty and $\mathcal{N}(\gamma) \subset N_X$. The family $\mathcal{N}(\gamma)$ is called the kernel of the measure of noncompactness γ.

For $S,\ T \in M_X$, we have the following:

(2) $\gamma\big(\lambda S + (1-\lambda)T\big) \leq \lambda\gamma(S) + (1-\lambda)\gamma(T)$, for all $\lambda \in [0,1]$.

(3) If $S \subset T$ then $\gamma(S) \leq \gamma(T)$.

(4) $\gamma(\overline{S}) = \gamma(S)$.

(5) $\gamma(\overline{\text{conv}(S)}) = \gamma(S)$.

(6) If $(S_n)_{n\in\mathbb{N}}$ is a sequence of sets from M_X such that $S_{n+1} \subset S_n$, $\overline{S_n} = S_n,\ n \in \{1,2,...\}$ and $\lim\limits_{n\to+\infty} \gamma(S_n) = 0$, then $S_\infty = \bigcap\limits_{n=1}^{\infty} S_n \neq \emptyset$ and $S_\infty \in \mathcal{N}(\gamma)$.

Definition 1.4. (i) *A measure of noncompactness γ is said to be sublinear if for all $S,\ T \in M_X$, it satisfies the following two conditions:*

(1) $\gamma(\lambda S) = |\lambda|\gamma(S)$ for $\lambda \in \mathbb{R}$ (γ is said to be homogenous).

(2) $\gamma(S+T) \leq \gamma(S) + \gamma(T)$ (γ is said to be subadditive).

(ii) A measure of noncompactness γ is referred to as a measure with maximum property if $\max(\gamma(S), \gamma(T)) = \gamma(S \cup T)$.

(iii) A measure of noncompactness γ is said to be regular if $\mathcal{N}(\gamma) = N_X$, it is sublinear and has maximum property.

For $S \in M_X$, the most important examples of measures of noncompactness [14] are:

• Kuratowski measure of noncompactness

$\gamma(S) = \inf\{\varepsilon > 0 : S \text{may be covered by finitely many of sets of diameter} \leq \varepsilon\}$.

Remark 1.5. *The Kuratowski measure of noncompactness $\gamma(.)$ is regular.*

Let $U \in \mathcal{L}(X)$. We say that U is k-set-contraction if for every set $S \in M_X$, we have $\gamma(U(S)) \leq k\gamma(S)$. We define $\gamma(U)$ by

$$\gamma(U) := \inf\{k : U \text{ is } k\text{-set-contraction}\}.$$

We use the following proposition

Proposition 1.6. *(*[1, Corollary 2.3]*) Let X be a Banach space and $U \in \mathcal{L}(X)$. If $\gamma(U^m) < 1$ for some $m > 0$ then $I + U$ is a Fredholm operator with $i(I + U) = 0$.*

We denote by $\mathcal{P}_\gamma(.)$ the set defined by

$$\mathcal{P}_\gamma(X) = \{U \in \mathcal{L}(X) \text{ such that } \gamma(U^m) < 1, \text{ for some } m > 0\}.$$

Definition 1.7. *Let X and Y be two Banach spaces.*

(i) An operator $U \in \mathcal{L}(X, Y)$ is said to have a left Fredholm inverse if there are maps $R_l \in \mathcal{L}(Y, X)$ and $K \in \mathcal{K}(X)$ such that $I_X + K$ extends R_lU. The operator R_l is called a left Fredholm inverse of U.

(ii) An operator $U \in \mathcal{L}(X, Y)$ is said to have a right Fredholm inverse if there is a map $R_r \in \mathcal{L}(Y, X)$ such that $R_r(Y) \subset \mathcal{D}(U)$ and $UR_r - I_Y \in \mathcal{K}(Y)$. The operator R_r is called a right Fredholm inverse of U.

(iii) An operator $U \in \mathcal{L}(X, Y)$ is said to have a Fredholm inverse if we shall refer to a map which is both a left and a right Fredholm inverse of U.

Definition 1.8. *Let U and V be two operators in $\mathcal{L}(X, Y)$. We denote by $F^{\pm}_{UV}(Y, X)$ the set of left or right inverses $R_\pm$ of U satisfying $VR_\pm \in \mathcal{P}_\gamma(X)$ or $R_\pm V \in \mathcal{P}_\gamma(X)$ following that $U \in \Phi_+(X, Y)$ or $U \in \Phi_-(X, Y)$.*

The purpose of this work is to extend the main result of Theorem 5.1 in [2] to Gustafson, Weidmann, Kato, Wolf, Schechter and Browder essential spectra of bounded linear operators on Banach spaces by means of Kuratowski measure of noncompactness where we use the set $\mathcal{P}_\gamma(.)$ as the set of perturbation operators. More precisely, let U and V be two operators in $\mathcal{L}(X, Y)$. If $U \in \Phi(X, Y)$ and R is a Fredholm inverse of U such that $RV \in \mathcal{P}_\gamma(X)$, then $U + V \in \Phi(X, Y)$ and $i(U + V) = i(U)$. In the same way, if $U \in \Phi_+(X, Y)$ $\big($respectively, $\Phi_-(X, Y)\big)$ and R_l (respectively, R_r) is a left (respectively, right) Fredholm inverse of U such that $VR_l \in \mathcal{P}_\gamma(Y)$ $\big($respectively, $R_rV \in \mathcal{P}_\gamma(X)\big)$, then $U + V \in \Phi_+(X, Y)$ $\big($respectively, $\Phi_-(X, Y)\big)$ and $i(U + V) = i(U)$. Moreover, we prove $\sigma_{ei}(U+V) \subset \sigma_{ei}(U)$, for all $i = 1, 2, 3, 4, 5, 6$ under conditions $U_\lambda V$, $VU_{\lambda l}$ and $U_{\lambda r}V \in \mathcal{P}_\gamma(X)$ where U_λ, $U_{\lambda l}$ and $U_{\lambda r}$ are inverse Fredholm, left Fredholm inverse and right Fredholm inverse of $\lambda - U$ respectively.

The organization of the paper is as follows: In Section 2, we present the main results.

2. Mains results

Theorem 2.1. *Let X and Y be two Banach spaces and let U and V be two operators in $\mathcal{L}(X,Y)$. Then*

(i) If $U \in \Phi(X,Y)$ and $R \in \mathcal{L}(Y,X)$ is a Fredholm inverse of U, such that $RV \in \mathcal{P}_\gamma(X)$, then $U+V \in \Phi(X,Y)$ and $i(U+V) = i(U)$.

(ii) If $U \in \Phi_+(X,Y)$ and $R_l \in \mathcal{L}(Y,X)$ is a left Fredholm inverse of U, such that $VR_l \in \mathcal{P}_\gamma(Y)$, then $U+V \in \Phi_+(X,Y)$ and $i(U+V) = i(U)$.

(iii) If $U \in \Phi_-(X,Y)$ and $R_r \in \mathcal{L}(Y,X)$ is a right Fredholm inverse of U, such that $R_rV \in \mathcal{P}_\gamma(X)$, then $U+V \in \Phi_-(X,Y)$ and $i(U+V) = i(U)$.

(iv) If $U \in \Phi_\pm(X,Y)$ and $F^{\pm}_{UV}(Y,X) \neq \emptyset$, then $U+V \in \Phi_\pm(X,Y)$.

Proof. (i) Since R is a Fredholm inverse of U, there exists $K \in \mathcal{K}(Y)$ such that

$$UR = I - K \text{ on } Y. \tag{2.1}$$

It follows from Eq. (2.1) that the operator $U+V$ can be written in the form

$$U+V = U + (UR+K)V = U(I_X + RV) + KV. \tag{2.2}$$

Using the fact that $RV \in \mathcal{P}_\gamma(X)$ together with Proposition 1.6 one gets

$$I_X + RV \in \Phi(X) \text{ and } i(I_X+RV) = 0. \tag{2.3}$$

Since $U \in \Phi(X,Y)$, applying [11, Theorem 5 (i), p. 159], we have $U(I_X+RV) \in \Phi(X,Y)$. Moreover, since $KV \in \mathcal{K}(X,Y)$, using Lemma 1.2 (i) and Eq. (2.2) , we infer

$$U+V \in \Phi(X,Y) \text{ and } i(U+V) = i\big(U(I_X+RV)\big) = i(U).$$

(ii) If R_l is a left Fredholm inverse of U, then there exists $K \in \mathcal{K}(X)$ such that

$$R_lU = I - K \text{ on } X. \tag{2.4}$$

It follows from Eq. (2.4) that the operator $U+V$ can be written in the form

$$U+V = U + V(R_lU+K) = (VR_l + I_Y)U + VK. \tag{2.5}$$

Using the fact that $VR_l \in \mathcal{P}_\gamma(Y)$, and applying Proposition 1.6 we have $VR_l + I_Y \in \Phi(Y)$ and $i(VR_l+I_Y) = 0$. Moreover, $U \in \Phi_+(X,Y)$, using

[11, Theorem 5 (ii), p. 156], we obtain $(VR_l + I_Y)U \in \Phi_+(X,Y)$. Since $VK \in \mathcal{K}(X,Y)$, applying Lemma 1.2 (ii) and Eq. (2.5), we get

$$U + V \in \Phi_+(X,Y) \text{ and } i(U+V) = i(U).$$

(iii) If R_r is a right Fredholm inverse of U, then there exists $K \in \mathcal{K}(Y)$ such that

$$UR_r = I - K \text{ on } Y,$$

and consequently,

$$U + V = U + V(UR_r + K) = U(UR_r + I_X) + KV.$$

Now, arguing as in (ii) we get

$$U + V \in \Phi_-(X,Y) \text{ and } i(U+V) = i(U).$$

(iv) The statement (iv) is an immediate consequence of the items (ii) and (iii). □

Theorem 2.2. *Let X be a Banach space and let U and V be two operators in $\mathcal{L}(X)$. Then the following statements hold:*

(i) *Assume that for each $\lambda \in \Phi_U$, there exists a Fredholm inverse U_λ of $\lambda - U$ such that $U_\lambda V \in \mathcal{P}_\gamma(X)$, then*

$$\sigma_{e4}(U+V) \subseteq \sigma_{e4}(U) \text{ and } \sigma_{e5}(U+V) \subseteq \sigma_{e5}(U).$$

(ii) *If the hypotheses of (i) is satisfied and if $C\sigma_{e5}(U)$ and $C\sigma_{e5}(U+V)$ are connected, then*

$$\sigma_{e6}(U+V) \subseteq \sigma_{e6}(U).$$

(iii) *Assume that for each $\lambda \in \Phi_{+U}$, there exists a left Fredholm inverse $U_{\lambda l}$ of $\lambda - U$ such that $VU_{\lambda l} \in \mathcal{P}_\gamma(X)$, then*

$$\sigma_{e1}(U+V) \subseteq \sigma_{e1}(U).$$

(iv) *Assume that for each $\lambda \in \Phi_{-U}$, there exists a right Fredholm inverse $U_{\lambda r}$ of $\lambda - U$ such that $U_{\lambda r}V \in \mathcal{P}_\gamma(X)$, then*

$$\sigma_{e2}(U+V) \subseteq \sigma_{e2}(U).$$

(v) *Assume that for each $\lambda \in \Phi_{\pm U}$, the set $F^{\pm}_{(\lambda-U)V}(X) \neq \emptyset$, then*

$$\sigma_{e3}(U+V) \subseteq \sigma_{e3}(U).$$

Proof. (i) Suppose that $\lambda \notin \sigma_{e4}(U)$ $\big($respectively, $\lambda \notin \sigma_{e5}(U)\big)$, then $\lambda \in \Phi_U$ $\big($respectively, by Proposition 1.3, we have $\lambda \in \Phi_U$ and $i(\lambda - U) = 0\big)$. Applying Theorem 2.1 (i) to the operators $\lambda - U$ and $-V$, we prove that $\lambda \in \Phi_{U+V}$ and $i(\lambda - U) = i(\lambda - U - V)$. Therefore $\lambda \notin \sigma_{e4}U + V)$ $\big($respectively, $\lambda \notin \sigma_{e5}(U + V)\big)$. We obtain

$$\sigma_{e4}U + V) \subseteq \sigma_{e4}(U)$$

and

$$\sigma_{e5}(U + V) \subseteq \sigma_{e5}(U). \tag{2.6}$$

(ii) The sets $C\sigma_{e5}(U + V)$ and $C\sigma_{e5}(U)$ are connected. Since U and V are bounded operators, we have $\rho(U)$ and $\rho(U + V)$ are not empty sets. So, using [7, Lemma 3.1], we deduce that

$$\sigma_{e5}(U + V) = \sigma_{e6}(U + V) \text{ and } \sigma_{e5}(U) = \sigma_{e6}(U).$$

So, Eq. (2.6) gives

$$\sigma_{e6}(U + V) \subseteq \sigma_{e6}(U).$$

(iii) Suppose that $\lambda \notin \sigma_{e1}(U)$ then $\lambda \in \Phi_{+U}$. Using Theorem 2.1 (ii) to the operators $\lambda - U$ and $-V$, we prove that $\lambda \in \Phi_{+(U+V)}$. This proves that $\lambda \notin \sigma_{e1}(U + V)$. We find

$$\sigma_{e1}(U + V) \subseteq \sigma_{e1}(U).$$

(iv) By a similar proof as in (iii), we replace $\sigma_{e1}(.)$ and $\Phi_+(.)$ by $\sigma_{e2}(.)$ and $\Phi_-(.)$ respectively, and using Theorem 2.1 (iii) we obtain

$$\sigma_{e2}(U + V) \subseteq \sigma_{e2}(U).$$

(v) Let $\lambda \notin \sigma_{e3}(U)$ then $\lambda \in \Phi_{\pm U}$. Since $F^{\pm}_{(\lambda - U)V}(X) \neq \emptyset$, applying Theorem 2.1 (iv) to the operators $\lambda - U$ and $-V$ we have $\lambda \in \Phi_{\pm(U+V)}$. Therefore

$$\sigma_{e3}(U + V) \subseteq \sigma_{e3}(U).$$

□

Remark 2.3. (i) *The results of the Theorem 2.1 remains valid if we suppose that $U \in \mathcal{C}(X)$ and V is a U-bounded operator on X. Clearly, applying Theorem 2.1, we prove the statements for $\widehat{U} \in \mathcal{L}(X_U, X)$ and $\widehat{V} \in \mathcal{L}(X_U, X)$ and applying Eq. (2.7) we conclude the desired results.*

Let V be an arbitrary U-bounded operator, hence we can regard U and V as operators from X_U into X, denoted by $\hat{U}$ and $\hat{V}$ respectively, that belong to $\mathcal{L}(X_U, X)$. Furthermore, we have the obvious relations

$$(2.7)\qquad \begin{cases} \alpha(\hat{U}) = \alpha(U),\ \beta(\hat{U}) = \beta(U),\ R(\hat{U}) = R(U), \\ \alpha(\hat{U}+\hat{V}) = \alpha(U+V), \\ \beta(\hat{U}+\hat{V}) = \beta(U+V) \text{ and } R(\hat{U}+\hat{V}) = R(U+V). \end{cases}$$

(ii) similarly, as in Theorem 2.2, one may show that the same results hold if we suppose that $U \in \mathcal{C}(X)$ and assume that V is an U-bounded operator on X.

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